Title: State-Constrained Controlled Porous Media Systems and Spatial-Temporal Epidemics Models
Area: Stochastic analysis; Stochastic control; Stochastic Partial Differential Equations; Epidemics

Participants

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Overview of the project

The aim of this collaboration is to provide a rigorous mathematical study of some control issues for stochastic systems with porous media characteristics i.e. governed by a coupled system of stochastic partial differential equations of type

$$(1)\quad \begin{cases} dX^{t,\xi,\eta,u}(s) & = \left(\Delta\theta_1\left(X^{t,\xi,\eta,u}(s)\right) + f_1\left(X^{t,\xi,\eta,u}(s), Y^{t,\xi,\eta,u}(s), u(s)\right)\right) ds \\ & +\sigma_1\left(X^{t,\xi,\eta,u}(s), Y^{t,\xi,\eta,u}(s), u(s)\right) dW(s), \\ dY^{t,\xi,\eta,u}(s) & = \left(\Delta\theta_2\left(Y^{t,\xi,\eta,u}(s)\right) + f_2\left(X^{t,\xi,\eta,u}(s), Y^{t,\xi,\eta,u}(s), u(s)\right)\right) ds \\ & +\sigma_2\left(X^{t,\xi,\eta,u}(s), Y^{t,\xi,\eta,u}(s), u(s)\right) dW(s),\ s \geq t; \\ X^{t,\xi\eta,u}(t) & = \xi \in \mathbb{L}^2\left(\Omega; H^{-1}\left(\mathcal{O}_1\right)\right),\ Y^{t,\xi\eta,u}(t) = \eta \in \mathbb{L}^2\left(\Omega; H^{-1}\left(\mathcal{O}_2\right)\right) \end{cases}$$

We are eluding the specific assumptions, but θ are monotonous functions and $\Delta\theta$ are obtained via a diffusive (but non-mixing) interaction. The drift and noise coefficients are still regular but mixing, while u is an internal control parameter. The domains $\mathcal{O}$ are bounded and present smooth boundaries. W is a Brownian motion with adequate covariance and set on a filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.
The contributions are intended for a **two-years project** and the developments envisaged are as follows.
1. Provide a theoretical characterization of the ability to maintain the system within a given set of constraints $K \subset H^{-1}(\mathcal{O}_1) \times H^{-1}(\mathcal{O}_2)$. This property will be referred to as *viability*.
2. As a consequence, by using one of the components (say Y) of (1) as an asymptotic supervisor, derive explicit conditions allowing to *exponentially stabilize* X or, at least, a suitable finite projection of this component.
3. Among the stabilizing (or viable) controls, pick the ones minimizing a suitable cost (e.g. quadratic ones). Write down the associated *Hamilton-Jacobi-Bellman* equation and derive the viscosity properties of the value function (at least in the particular case of finite projection stabilization).
4. Develop *numerical methods* to compute near-optimal controls; one can envisage new-generation algorithms such as dual linear programming ones or gradient descent via neural networks optimization.
[+] Extend these methods to other types of noise (e.g. systems driven by trajectory-depending intensity random point processes).
5. Apply the method to the study of a space-time epidemics (SIR with demography and vaccination) model: deduce the *safety zones* and the *minimal intervention policies* allowing to stabilize the infections.

Methodology and qualifications

In the uncoupled framework, without control, existence and uniqueness of solutions for (1) have been obtained in several papers (see [3] and references therein). Coupled equations (with a different aim and coupling mechanism) have been recently envisaged in [12]. The literature on the subject is growing intensively and an exhaustive overview exceeds the scope of this short presentation. On the other hand, the literature on state-constrained systems has known important developments in the past 30 years with the development of set-valued analysis and the extension to stochastic finite-dimensional systems by [1]. Adapting the notion of tangency (in the Bouligand sense), [4] offers an important semi-group-based method to deal with deterministic PDEs.
We regroup the methodological items in correspondence with the aforementioned developments intended. The participants have extensive experience in the state-constrained control in either finite-dimensional settings with various stochasticity cf. [5], or semilinear SPDEs, cf. [10]. The quasi-tangential method in [10] (adapting, to the stochastic case the results in [4]) is tailor-made to cope with the SPDEs via a sewing-type lemma. Although approximation schemes for porous media systems (let alone coupled systems) are less ubiquitous than their semilinear or finite-dimensional counterpart, the experience I. Ciotir has in this area of porous media equations (e. g., [6]) provides encouraging premises. I. Munteanu is highly experienced in

stabilization issues, having singly authored a book on the subject [13]. A previous collaboration between I. Munteanu and D. Goreac (cf. [11]) has allowed them to tackle the stabilization results for semilinear Brownian-perturbed systems, albeit from a different controllability perspective. On the other hand, viability of multi-dimensional (or coupled) systems is known to provide level-set/epigraphical methods particularly useful for comparison results (cf. [8]). This is a natural idea in the context of stabilization where a time exponentially-shrinking ball can be considered, in the spirit of the applications in [10]. Nagumo-type results based on the previous step should provide us with a well-fitting method.
The paper [9] authored by one of the participants proposes an explicit dual LP algorithm, in the spirit of [14]. Since the linear programming method can be extended to various settings (based on Krylov's shaking of coefficients in relation to the HJB system), it is expected that the aforementioned framework (at least from the projection point of view) fits the same kind of reasoning.
Finally, we mention the very recent paper [7] explaining, via the notion of micro and macrosites, how a Markov process with piecewise PDE dynamics can be associated to a reaction or interaction network. With this method and exploiting a random walk involving θ weights, one can infer a space-time SIR-like model staring from a reaction network. The safety zones are then deduced using viability in the spirit of [2] and the developments follow as in [9].

Financing issues 4 (four) weeks of research visits out of which two for I. Munteanu in either INSA Rouen or Univ. Gustave Eiffel and one visit to "Al. I. Cuza" University, Iasi for each of the French partners (I. Ciotir and D. Goreac). This includes transportation and usual local expenses.

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