On generalized NS-algebras, twisted Rota-Baxter operators and Nijenhuis operators

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Project presentation

The purpose of this project is to introduce and study generalized structure of NS-algebras using homomorphisms. We discuss their relationships with twisted Rota-Baxter operators and Nijenhuis operators.

NS-algebras (corresponding to associative algebras) have been introduced by Leroux ([6]) and independently by Uchino ([15]), as algebras with three operations $\prec$, $\succ$ and $\vee$ satisfying certain axioms that imply that the new operation $* = \prec + \succ + \vee$ is associative. NS-algebras generalize both dendriform ([11]) and tridendriform ([12]) algebras. Examples are obtained via so-called twisted Rota-Baxter operators (see [15]), which are a generalization of $\mathcal{O}$-operators involving a Hochschild 2-cocycle, and via Nijenhuis operators (see [6]). We recall from [1] that a Nijenhuis operator $N : A \to A$ on an associative algebra (A, μ) with multiplication denoted by $\mu(x \otimes y) = xy$, for $x, y \in A$, is a linear map satisfying

$$N(x)N(y) = N(N(x)y + xN(y) - N(xy)), \quad \forall\, x, y \in A. \tag{0.1}$$

By [6], if one defines $x \prec y = xN(y)$, $x \succ y = N(x)y$ and $x \vee y = -N(xy)$, then $(A, \prec, \succ, \vee)$ is an NS-algebra, and in particular the new multiplication defined on A by $x * y = xN(y) + N(x)y - N(xy)$ is associative. Basic examples (see [1]) are obtained by taking a fixed element $a \in A$ and defining $N_1, N_2 : A \to A$ by $N_1(x) = ax$ and $N_2(x) = xa$, for all $x \in A$; it turns out that N_1, N_2 are Nijenhuis operators and in each case the new multiplication $*$ as above boils down to $x * y = xay$, for all $x, y \in A$. This property can be regarded also in the following (converse) way: the fact that the new multiplication on A given by $x * y = xay$ is associative (usually, this new operation $*$ is said to be a "perturbation" of the old multiplication of A via the element a) can be obtained as a consequence of a property of Nijenhuis operators (or, alternatively, that it can be given a Nijenhuis operator interpretation).

Let us mention that one can define NS-algebras corresponding to other classes of algebras than associative, for instance corresponding to Lie or Leibniz algebras (see [2], [3]) and, much more generally, to any class of algebras defined by multilinear relations (see [14]).

Hom-type and BiHom-type algebras are certain algebraic structures (of growing interest in recent years) whose study began in some early papers such as [5], [13] and more recently [4], and can be roughly described as being defined by some identities obtained by twisting a classical algebraic identity (such as associativity) by one or two maps. For instance, a BiHom-associative algebra (A, μ, α, β) is an algebra (A, μ), with notation $\mu : A \otimes A \rightarrow A$, $\mu(x \otimes y) = xy$, together with two (multiplicative with respect to μ) commuting linear maps (called structure maps) $\alpha, \beta : A \rightarrow A$ such that $\alpha(x)(yz) = (xy)\beta(z)$ for all $x, y, z \in A$. There exist BiHom analogues of many types of algebras, in particular of (tri)dendriform algebras, infinitesimal bialgebras etc (see for instance [8], [9], [10] and references therein). Examples of (Bi)Hom-type algebras can be obtained from classical types of algebras by a procedure called "Yau twisting".

The BiHom analogue of the "perturbations" mentioned above has been introduced in [10] as follows. Let (A, μ, α, β) be a BiHom-associative algebra and let $a \in A$ be such that $\alpha^2(a) = \beta^2(a) = a$. Define a new operation on A by $x * y = \alpha(x)(\alpha(a)y)$; then $(A, *, \alpha^2, \beta^2)$ is a BiHom-associative algebra.

The starting point of this project is to look for a "Nijenhuis operator interpretation" of this fact. One can notice that Nijenhuis operators defined by the relation (0.1) can be considered on any type of algebra (not necessarily associative), so we aim to find a Nijenhuis operator on (A, μ, α, β) depending on the given element a and that would lead to the operation $*$. It turns out (just as it happened before with a certain context in which one was forced to consider a generalized version of Rota-Baxter operators, see [7]) that the solution to this problem is to consider a generalized version of Nijenhuis operators on BiHom-associative algebras. And indeed, the operators $N_1, N_2 : A \rightarrow A$, $N_1(x) = \alpha(a)x$ and $N_2(x) = x\alpha(a)$, for $x \in A$, are such generalized Nijenhuis operators from which one can obtain the multiplication $*$ in a certain way.

We are led to introduce the concept of BiHom-NS-algebra, the BiHom analogue of Leroux's and Uchino's NS-algebras, generalizing BiHom-(tri)dendriform algebras, and to show that relevant generalized Nijenhuis operators lead to BiHom-NS-algebras. Moreover, we aim to consider a categorical viewpoint and discuss an adjunction between BiHom-NS-algebras and twisted Rota-Baxter operators.

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