Research Project

Damien Mégy Mihai Pavel Matei Toma

May 2023

Variation of moduli spaces of sheaves in higher dimensions

Background

The moduli theory of sheaves has attracted considerable interest over the last decades, due to its broad application in various research fields such as hyperkähler geometry, enumerative geometry, gauge theory, etc. Its main focus is the construction and study of moduli spaces of sheaves over a fixed base space X. In projective geometry, X is generally taken to be a smooth projective variety. In this setting, one has to restrict to a bounded family of sheaves on X in order to construct moduli spaces of finite type. This is usually done by imposing certain stability conditions on sheaves, such as Gieseker stability which we recall below.

In what follows, let us fix a smooth projective variety X of dimension n over $\mathbb{C}$, endowed with an ample class $[H] \in \mathrm{Amp}^1(X)$. By definition, a torsion-free (coherent) sheaf E on X is called *H-Gieseker-semistable* (resp. *H-Gieseker-stable*) if for all $F \subset E$ of rank $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ we have

$$\frac{P_H(F,m)}{\mathrm{rk}(F)} \leq \frac{P_H(E,m)}{\mathrm{rk}(E)} \text{ (resp. } <) \text{ for } m \gg 0,$$

where

$$P_H(E,m) = \chi(E \otimes \mathcal{O}_X(mH)) = \int_X \mathrm{ch}(E) \left(1 + mH + \frac{m^2H^2}{2!} + \cdots + \frac{m^nH^n}{n!}\right) \mathrm{td}_X$$

denotes the Hilbert polynomial of E.

Choose a class $v \in K(X)_{\mathrm{num}}$ of rank $r > 0$, which will fix the numerical type of sheaves. There are two natural questions worth addressing:

(A) Does a moduli space $M_H(v)$ of H-Gieseker-semistable sheaves of class v on X exist? If yes, then is $M_H(v)$ projective?

(B) How does $M_H(v)$ change when H varies inside the ample cone of X?

Both questions have been extensively studied by several authors. Due to the work of Mumford, Gieseker, Maruyama and Simpson, we have a satisfactory answer to (**A**). By employing the GIT methods, they constructed a projective moduli space $M_H(v)$ of H-Gieseker-semistable sheaves. Regarding (**B**), one has to study the variation of M_H with the change of polarization. We briefly review the state-of-the-art below:

Surface case. In the surface case the picture is rather well-understood (see [5, 4]). It is known that the real ample cone $\mathrm{Amp}(X)_{\mathbb{R}}$ is divided up by a locally finite set of rational linear walls into chambers within which M_H does not change. Furthermore, for two polarizations H_1 and H_2 divided by a wall, the corresponding moduli spaces M_{H_1} and M_{H_2} are related by a finite sequence of so-called Thaddeus-flips. When the second Chern class of v is large enough, it has also been shown that M_{H_1} and M_{H_2} are in fact birational (see [3, Sect. 4.C]).

Higher-dimensional case. In higher dimensions, $\mathrm{Amp}(X)_{\mathbb{R}}$ still admits a decomposition into walls and chambers, however its structure is more complicated. If one applies the same construction as in the surface case, then the obtained walls are not linear anymore, and as noted by Qin [5], they are not necessarily of finite type inside the ample cone. Also, Schmitt [6] gave an explicit example of a Calabi-Yau threefold of Picard number two for which there exists at least one wall with no rational classes.

To avoid the difficulties appearing in higher dimensions regarding the change of polarization, we introduce the following notion of $\underline{\omega}$-(semi)stability which gives more flexibility than Gieseker-(semi)stability.

Definition. For $i = 1, \dots, n$, let $\omega_i \in \mathrm{NS}^i(X)_{\mathbb{R}}$ be real numerical cycles of codimension i, and consider the vector $\underline{\omega} = (\omega_1, \dots, \omega_n)$. For any torsion-free sheaf E on X define

$$P_{\underline{\omega}}(E, m) = \int_X \mathrm{ch}(E) \left(1 + m\omega_1 + \frac{m^2}{2!}\omega_2 + \cdots + \frac{m^{n-1}}{(n-1)!}\omega_{n-1} + \frac{m^n}{n!}\omega_n\right) \mathrm{td}_X .$$

We say E is *$\underline{\omega}$-(semi)stable* if for all $F \subset E$ of rank $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ we have

$$\frac{P_{\underline{\omega}}(F, m)}{\mathrm{rk}(F)} \leq \frac{P_{\underline{\omega}}(E, m)}{\mathrm{rk}(E)} \text{ (resp. } <) \text{ for } m \gg 0.$$

As $\mathrm{NS}^n(X) \cong \mathbb{Z}$, we may assume without loss of generality that $w_n = 1$. Note that we recover the notion of H-Gieseker-(semi)stability for $\underline{\omega} = (H, H^2, \dots, H^n)$ with $H \in \mathrm{Amp}(X)$. In order to have a well-behaved family of $\underline{\omega}$-(semi)stable sheaves, for which one can study the moduli problem, we need to impose certain limitations on $\underline{\omega}$. For this reason, we will have to restrict the domain of $\underline{\omega}$ to a certain cone $\mathcal{C} \subset \mathrm{NS}^1(X)_{\mathbb{R}} \times \cdots \times \mathrm{NS}^n(X)_{\mathbb{R}}$. In what follows we let $\underline{\omega}$ vary inside $\mathcal{C}$.

Questions (**A**) and (**B**) of the previous section take the following form:

(A) Does a moduli space $M_{\underline{\omega}}(v)$ of $\underline{\omega}$-semistable sheaves of class v on X exist? If yes, then is $M_{\underline{\omega}}(v)$ projective?

(B) How does $M_{\underline{\omega}}(v)$ change when $\underline{\omega}$ varies inside $\mathcal{C}$?

Main Objectives

One of the goals of this project is to address the above questions.

(1) Regarding (**A**), we want to a construct a *good* moduli space (in the sense of Alper [1]) $M_{\underline{\omega}}(v)$ of $\underline{\omega}$-semistable sheaves. One of the main difficulties here is to prove that the family of $\underline{\omega}$-semistable sheaves of a fixed numerical type is *bounded*. This is a rather technical aspect of the theory, and the answer might depend on the chosen polarization $\underline{\omega} \in \mathcal{C}$.

(2) Study the geometric structure of the moduli space $M_{\underline{\omega}}(v)$. The question on the projectivity of $M_{\underline{\omega}}(v)$ is quite delicate and its answer might depend again on the chosen polarization $\underline{\omega}$. One may also ask here whether there any polarizations $\underline{\omega}$ for which the notion of $\underline{\omega}$-stability is somehow related to that of multi-Gieseker-stability introduced by Greb–Ross–Toma in [2]. This question is inspired by the result of [2, Thm. 11.6], which shows the projectivity of certain moduli spaces of sheaves on projective threefolds endowed with a polarization of the form $\underline{\omega} = (\omega, \omega^2, \dots, \omega^n)$, where $\omega \in \mathrm{NS}^1(X)_{\mathbb{R}}$ is a real ample class.

(3) Regarding (**B**), the notion of $\underline{\omega}$-stability is rather flexible, as one can vary independently the components of $\underline{\omega} = (\omega_1, \dots, \omega_n)$. To ensure that the boundedness assumptions are satisfied, we will most probably have to restrict to a compact subset $K \subset \mathcal{C}$. The objective is to describe the wall and chamber structure of K, with respect to the variation of $\underline{\omega}$-(semi)stability. We expect that K has a nice decomposition into walls and chambers (e.g. it is divided by a locally finite set of linear walls), which is better behaved than that obtained in the Gieseker-semistable case. Given this, we would like to study the variation of $M_{\underline{\omega}}(v)$ with $\underline{\omega} \in K$ and wall-crossing phenomena.

References

[1] Jarod Alper. "Good moduli spaces for Artin stacks". English. In: *Ann. Inst. Fourier* 63.6 (2013), 2349–2402.

[2] Daniel Greb, Julius Ross, and Matei Toma. "Variation of Gieseker moduli spaces via quiver GIT". English. In: *Geom. Topol.* 20.3 (2016), 1539–1610.

[3] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves. 2nd ed.* English. Cambridge University Press, 2010, xviii + 325.

[4] Kenji Matsuki and Richard Wentworth. “Mumford-Thaddeus principle on the moduli space of vector bundles on an algebraic surface”. English. In: *Int. J. Math.* 8.1 (1997), 97–148.

[5] Zhenbo Qin. “Equivalence classes of polarizations and moduli spaces of sheaves”. English. In: *J. Differ. Geom.* 37.2 (1993), 397–415.

[6] Alexander Schmitt. “Walls of Gieseker semistability and the Mumford-Thaddeus principle for moduli spaces of sheaves over higher dimensional bases”. English. In: *Comment. Math. Helv.* 75.2 (2000), 216–231.