Dedicated to Dr. TOADER MOROZAN
on the occasion of his 70th birthday

LYAPUNOV ITERATIONS FOR COUPLED RICCATI DIFFERENTIAL EQUATIONS ARISING IN CONNECTION WITH NASH DIFFERENTIAL GAMES

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We provide iterative procedures for numerical computation of some stabilizing solutions of two type of coupled matrix Riccati differential equations arising in connection with Nash differential games. The procedures proposed are based on solutions of uncoupled symmetric or nonsymmetric Lyapunov equations.

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1. INTRODUCTION AND PROBLEM FORMULATION

In this paper we study the existence of stabilizing solutions of two pairs of coupled matrix Riccati differential equations associated with linear-quadratic games of the form

$$\dot{x} = A(t)x(t) + B_1(t)u_1(t) + B_2(t)u_2(t); \quad x(0) = x_0,$$

where $x \in \mathbf{R}^n$, $u_i \in \mathbf{R}^{r_i}$, $i = 1, 2$, and the cost functionals associated with the two players are

$$J_1 = \frac{1}{2}x_f^{\mathrm{T}} X_{1f} x_f + \frac{1}{2}\int_{t_0}^{t_f} (x^{\mathrm{T}} Q_1(t)x + u_1^{\mathrm{T}} R_{11}(t)u_1 + u_2^{\mathrm{T}} R_{12}(t)u_2)\mathrm{d}t$$

and

$$J_2 = \frac{1}{2}x_f^{\mathrm{T}} X_{2f} x_f + \frac{1}{2}\int_{t_0}^{t_f} (x^{\mathrm{T}} Q_2(t)x + u_1^{\mathrm{T}} R_{21}(t)u_1 + u_2^{\mathrm{T}} R_{22}(t)u_2)\mathrm{d}t,$$

with $x_f = x(t_f)$. All weighting matrices are assumed to be real and symmetric with Q_i non-negative definite and R_{ii}, $i = 1, 2$, positive definite.

The Riccati equations examined in this paper are associated with two types of strategies of the two players: the feedback Nash strategies and the open-loop Nash strategies. It is known (see [1, 3, 5] for precise definitions and further details on this topic) that the optimal feedback and open-loop Nash strategies have the form

$$u_1(t) = -R_{11}^{-1}(t)B_1^{\mathrm{T}}(t)X_1(t)\,x(t), \quad u_2(t) = -R_{22}^{-1}(t)B_2^{\mathrm{T}}(t)X_2(t)\,x(t),$$

where $x(t)$ can be determined from the initial value problem

$$\dot{x} = [A(t) - S_1(t)X_1(t) - S_2(t)X_2(t)]x(t), \quad x(0) = x_0,$$

provided it is possible to determine for all $t \in [t_0, t_f]$ the solutions $(X_1(t), X_2(t))$ of the coupled matrix Riccati differential equations (1) and (2), respectively, with terminal values $X_i(t_f) = X_{if},\ i = 1, 2$. Using the notation

$$S_i(t) = B_i(t)R_{ii}^{-1}(t)B_i^{\mathrm{T}}(t), \quad 1 \le i \le 2,$$

and

$$S_{ij}(t) = B_j(t)R_{jj}^{-1}(t)R_{ij}(t)R_{jj}^{-1}(t)B_j^{\mathrm{T}}(t), \quad 1 \le i, j \le 2,$$

in the case of feedback Nash strategies we have to determine the solution (X_1, X_2) of the system

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}X_1 + A^{\mathrm{T}}(t)X_1 + X_1A(t) - X_1S_1(t)X_1 - X_1S_2(t)X_2 - \\ &\qquad - X_2S_2(t)X_1 + X_2S_{12}(t)X_2 + Q_1(t) = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}X_2 + A^{\mathrm{T}}(t)X_2 + X_2A(t) - X_2S_2(t)X_2 - X_2S_1(t)X_1 - \\ &\qquad - X_1S_1(t)X_2 + X_1S_{21}(t)X_1 + Q_2(t) = 0, \end{aligned} \tag{1}$$

and in the case of open-loop Nash strategies the solution (X_1, X_2) of the system

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}X_1 + A^{\mathrm{T}}(t)X_1 + X_1A^{\mathrm{T}}(t) - X_1S_1(t)X_1 - X_1S_2(t)X_2 + Q_1(t) = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}X_2 + A^{\mathrm{T}}(t)X_2 + X_2A(t) - X_2S_1(t)X_1 - X_2S_2(t)X_2 + Q_2(t) = 0, \end{aligned} \tag{2}$$

where we assume for convenience that $A : \mathbf{R} \to \mathbf{R}^{n\times n}$; $Q_i, S_i : \mathbf{R} \to \mathcal{S}_n$, $i = 1, 2$; $S_{ij} : \mathbf{R} \to \mathcal{S}_n$, $(ij) \in \{(1, 2), (2, 1)\}$ are bounded and continuous matrix valued functions; here, as usual, $\mathcal{S}_n \subset \mathbf{R}^{n\times n}$ is the linear subspace of all symmetric $n \times n$ matrices.

If the differential game is considered on an infinite time horizon (i.e., $t_f = +\infty$), then the optimal strategy is constructed using a special global solution of equations (1) and (2), respectively. Such solutions have to achieve the exponentially stable behavior of the trajectories of the closed-loop system. In this paper we are interested in deriving procedures for numerical computation of such global solutions of (1) and (2), respectively.

Systems (1) and (2) were investigated either as mathematical objects with interest in themselves in [1], Chapter 6, or in connection with several aspects of two-player Nash differential games (see [2, 3, 5, 10, 11, 12] and references therein).

We mention that system (2) can be rewritten as a non-symmetric (rectangular) matrix Riccati differential equation for the block matrix $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$. Therefore we can use for its solution all results and methods known for this type of equations (see [1], Chapter 6, [7] and [9]) – it is known that the global existence of the solutions of such differential equations is only guaranteed under rather restrictive conditions.

Existence results for the nonlinear system (1) are also rare; although the solutions X_j, $1 \leq j \leq 2$, of (1) are symmetric if the terminal (or initial) values X_{jf}, $1 \leq j \leq 2$, are symmetric, the existence of the corresponding solutions can frequently only be guaranteed locally (see [8]).

The situation is better if one confines to differential systems (1) or (2) under assumptions leading to positive systems; in particular, (1) and (2) were studied under these restrictions in [2], [4] and in [10], respectively.

In the present paper we assume that (1), (2) are also in the case of positive systems. Therefore, according to the assumptions from [10, 2, 4] we make the following hypothesis concerning the coefficients of (1) and (2):

$\mathbf{H_1}$) (i) For each $t \in \mathbf{R}$, $A(t) = (a_{ij}(t))$ is a Metzler matrix, i.e., $a_{ij}(t) \geq 0$ for $i \neq j$.
(ii) $S_i(t) \preceq 0, i = 1, 2, \ \forall\, t \in \mathbf{R}$.
(iii) $S_{ij}(t) \succeq 0, (i, j) \in \{(1, 2), (2, 1)\}, \ \forall\, t \in \mathbf{R}$.
(iv) $Q_l(t) \succeq 0, \ t \in \mathbf{R}, \ l = 1, 2$.
Here and below $\preceq$ and $\succeq$ denote the corresponding componentwise ordering.

Our aim is to construct sequences of iterates which converge towards the stabilizing solution of (1) and (2), respectively.

At each step we will have to solve two uncoupled symmetric Lyapunov differential equations or uncoupled nonsymmetric Lyapunov equations (Sylvester equations), respectively.

2. STABILIZING SOLUTIONS

Since (1) and (2) are nonstandard (coupled) Riccati differential equations, we consider that the results obtained could be useful to clarify the concept of stabilizing solutions of such equations.

To this end, we regard these equations as nonlinear differential equations on a Hilbert space $\mathcal{X}$. For equation (1) we take $\mathcal{X} = \mathcal{S}_n \oplus \mathcal{S}_n$ while for equation (2) we take $\mathcal{X} = \mathbf{R}^{n\times n} \oplus \mathbf{R}^{n\times n}$. The usual inner product is given by

$$\langle X, Y \rangle = \operatorname{Tr}\left[\ Y_1^{\mathrm{T}} \quad X_1\ \right] + \operatorname{Tr}\left[\ Y_2^{\mathrm{T}} \quad X_2\ \right] \tag{3}$$

for all $X = (X_1, X_2), Y = (Y_1, Y_2)$ in $\mathcal{X}$.

On $\mathcal{X}$, (1) and (2) may be written in a compact form as

$$\frac{\mathrm{d}}{\mathrm{d}t}X + \mathcal{R}(t, X) + Q(t) = 0, \tag{4}$$

where $Q(t) = \begin{pmatrix} Q_1(t) & Q_2(t) \end{pmatrix}$ and $\mathcal{R}(t, X) = \begin{pmatrix} \mathcal{R}_1(t, X) & \mathcal{R}_2(t, X) \end{pmatrix}$, with

$$\mathcal{R}_1(t, X) = A^{\mathrm{T}}(t)X_1 + X_1A(t) - X_1S_1(t)X_1 - X_1S_2(t)X_2 - X_2S_2(t)X_1 + X_2S_{12}(t)X_2,$$

$$\mathcal{R}_2(t, X) = A^{\mathrm{T}}(t)X_2 + X_2A(t) - X_2S_2(t)X_2 - X_2S_1(t)X_1 - X_1S_1(t)X_2 + X_1S_{21}(t)X_1,$$

in case of (1), and

$$\mathcal{R}_1(t, X) = A^{\mathrm{T}}(t)X_1 + X_1A(t) - X_1S_1(t)X_1 - X_1S_2(t)X_2,$$

$$\mathcal{R}_2(t, X) = A^{\mathrm{T}}(t)X_2 + X_2A(t) - X_2S_1(t)X_1 - X_2S_2(t)X_2,$$

in case of (2).

For each solution $X(t) = (X_1(t), X_2(t))$ of equation (4) we may construct the operator valued function $\mathcal{L}_X : \mathbf{R} \to \mathcal{B}[X]$ defined by $\mathcal{L}_X(t)U = (\mathcal{L}_{1X}(t)U, \mathcal{L}_{2X}(t)U)$ where

$$\begin{aligned}
\mathcal{L}_{1X}(t)U = {} & (A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))U_1 + U_1(A(t) - \\
& - S_1(t)X_1(t) - S_2(t)X_2(t))^{\mathrm{T}} - (S_1(t)X_2(t) - S_{21}(t)X_1(t))U_2 - \\
& - U_2(X_2(t)S_1(t) - X_1(t)S_{21}(t)), \\
\mathcal{L}_{2X}(t)U = {} & -(S_2(t)X_1(t) - S_{12}(t)X_2(t))U_1 - U_1(X_1(t)S_2(t) - X_2(t)S_{12}(t)) + \\
& + (A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))U_2 + \\
& + U_2(A(t) - S_1(t)X_1(t) - S_2(t)X_2(t))^{\mathrm{T}}
\end{aligned} \tag{5}$$

in case of (1) and

$$\begin{aligned}
\mathcal{L}_{1X}(t)U &= (A(t) - S_1(t)X_1^{\mathrm{T}}(t))U_1 + U_1(A(t) - S_1(t)X_1(t) - \\
&\quad - S_2(t)X_2(t))^{\mathrm{T}} - S_1(t)X_2^{\mathrm{T}}(t)U_2, \\
\mathcal{L}_{2X}(t)U &= (A(t) - S_2(t)X_2^{\mathrm{T}}(t))U_2 + U_2(A(t) - S_1(t)X_1(t) - \\
&\quad - S_2(t)X_2(t))^{\mathrm{T}} - S_2(t)X_1^{\mathrm{T}}(t)U_1,
\end{aligned} \tag{6}$$

in case of (2). It is easy to see that

$$\mathcal{R}'(t, X(t)) = \mathcal{L}_X^*(t), \tag{7}$$

where $\mathcal{R}'(t, \cdot)$ is the Fréchet derivative of the function $X \to \mathcal{R}(t, X)$ while $\mathcal{L}_X^*(t)$ is the adjoint operator of $\mathcal{L}_X(t)$ with respect to the inner product (3).

Definition 2.1. We say that a solution $\tilde{X}(t) = (\tilde{X}_1(t), \tilde{X}_2(t))$ of (4) is

a) *a stabilizing solution* if the zero state equilibrium of the linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Z = \mathcal{L}_{\tilde{X}}(t)Z \tag{8}$$

on $\mathcal{X}$ is exponentially stable;

b) *a closed-loop stabilizing solution* if the zero state equilibrium of the linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x = A_{cl}(t)x \tag{9}$$

on $\mathbf{R}^n$ is exponentially stable, where $A_{cl}(t) = A(t) - S_1(t)\tilde{X}_1(t) - S_2(t)\tilde{X}_2(t)$.

Remark 2.2. a) On account of (7), in the time invariant case, the concept of a stabilizing solution introduced above can be characterized by the fact that the eigenvalues of the operator $\mathcal{R}'(X)$ are located in the open left half-plane $\mathrm{Re}\lambda < 0$.

b) It was shown in [4] that if $\tilde{X}(t) \succeq 0$ is a stabilizing solution of (1), then it also is a closed-loop stabilizing solution of the same equation.

Reasoning as in Lemma 8.1 (ii), (iii) in [4], we deduce that if $\tilde{X}(t) \succeq 0$ is a stabilizing solution of (2), then the solution $Z_k = 0$ of the linear differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}Z_k = \Lambda_{k,\tilde{X}}(t)Z_k, \quad k = 1, 2, \tag{10}$$

is exponentially stable, where

$$\Lambda_{k,\tilde{X}}(t)Z_k = (A(t) - S_k(t)\tilde{X}_k^{\mathrm{T}}(t))Z_k + Z_k(A(t) - \\ -S_1(t)\tilde{X}_1(t) - S_2(t)\tilde{X}_2(t))^{\mathrm{T}} \tag{11}$$

is a nonsymmetric Lyapunov operator (i.e., a Sylvester operator).

Unfortunately, we are unable to show that the exponential stability of the evolution generated by the Sylvester operator (11) implies the exponential stability of the corresponding closed-loop matrix $A_{cl}(t)$ defined by (9).

c) Necessary and sufficient conditions under which a closed-loop stabilizing solution of (4) is also a stabilizing solution can be derived using the developments from Section 6 in [4].

In [10, 2, 4] sequences of iterates $X^j = (X_1^j, X_2^j)$ converging towards the stabilizing solution were provided. At each step X^j is obtained either as solution of the linear differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}X^j + \mathcal{L}^*_{X^{j-1}}(t)X^j + Q^j(t) = 0 \tag{12}$$

on $\mathcal{X}$ in the time-varying case or as solution of the algebraic linear equations

$$\mathcal{L}^*_{X^{j-1}}X^j + Q^j = 0 \tag{13}$$

on $\mathcal{X}$ in the time invariant case.

In this paper we replace equations (12) and (13) respectively, by uncoupled Lyapunov differential equations or uncoupled algebraic Lyapunov equations, respectively.

At the end of this section we introduce the set of functions

$$\Omega(\mathcal{R},Q)=\Big\{P:\mathbf{R}\to\mathcal{X}\mid P(t)\succeq 0 \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}P(t)+\mathcal{R}(t,P(t))+Q(t)\prec 0\Big\}. \tag{14}$$

related to equation (4).

We recall that for $H:\mathbf{R}\to\mathcal{X}$ we write $H(t)\succ\succ 0$ if there exists a positive constant δ such that $H(t)\succeq\delta\,\mathbf{1}_n\succ 0$, where $\mathbf{1}_n$ is the $n\times n$ matrix with all entries equal to 1 (for details see Ex. 2.5 (ii) in [4]). We shall write $H(t)\prec\prec 0$ if $-H(t)\succ\succ 0$.

Remark 2.3. In (14), the operator $\mathcal{R}(\cdot\,,\cdot)$ takes different forms according to whether the set $\Omega(\mathcal{R},Q)$ is associated either with (1) or with (2).

3. LYAPUNOV TYPE ITERATIONS FOR EQUATION (1)

Let $\{X^j(t)\}_{j\geq 0}$ be the sequence of functions $X^j:\mathbf{R}\to\mathcal{X}$, $X^j(t)=(X_1^j(t),X_2^j(t))$ with $X_l^j(t)$ being the unique solution bounded on $\mathbf{R}$ of the Lyapunov differential equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}X_l^j(t)+[A(t)-S_1(t)X_1^{j-1}(t)-S_2(t)X_2^{j-1}(t)]^{\mathrm{T}}X_l^j(t)+\\ +X_l^j(t)[A(t)-S_1(t)X_1^{j-1}(t)-S_2(t)X_2^{j-1}(t)]+Q_l^{j-1}(t)=0,\end{aligned} \tag{15}$$

with $l=1,2$, $X_l^0(t)=0$, $t\in\mathbf{R}$, where

$$Q_1^{j-1}(t)=Q_1(t)+X_1^{j-1}(t)S_1(t)X_1^{j-1}+X_2^{j-1}(t)S_{12}(t)X_2^{j-1}, \tag{16}$$

$$Q_2^{j-1}(t)=Q_2(t)+X_2^{j-1}(t)S_2(t)X_2^{j-1}(t)+X_1^{j-1}(t)S_{21}(t)X_1^{j-1}(t). \tag{17}$$

Before stating the main result of this section we make the assumption

$\mathbf{H_2}$) (i) The zero state equilibrium of the linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t)=A(t)x(t)$$

on $\mathbf{R}^n$ is exponentially stable.

(ii) The set $\Omega(\mathcal{R},Q)$ is not empty.

Now, we prove:

THEOREM 3.1. *Under assumptions* $\mathbf{H_1}$ *and* $\mathbf{H_2}$, *the sequence* $\{X^j(t)\}_{j\geq 0}$ *defined by* (15)–(17) *is well defined and convergent. If* $\tilde X(t):=\lim_{j\to\infty}X^j(t)$ *then* $\tilde X(t)$ *is the stabilizing solution of* (1). *Moreover,* $\tilde X(t)$ *is the minimal* *solution of* (1) *with respect to the class of global bounded nonnegative solutions of* (1)*.*

Proof. We shall show iteratively the following items:

a_j) $0 \preceq X^j(t) \preceq P(t)$ for all $P(t) \in \Omega(\mathcal{R}, Q)$;

b_j) the zero state equilibrium of the linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = A_j(t) x(t)$$

is exponentially stable, where

$$A_j(t) = A(t) - S_1(t) X_1^j(t) - S_2(t) X_2^j(t); \tag{18}$$

c_j) $X^j(t) \preceq X^{j+1}(t)$ for all $t \in \mathbf{R}$.

From assumption $\mathbf{H_2}$ together with $X_l^0(t) = 0$, we get that items a_j) and b_j) are fulfilled for $j = 0$.

To check that c_0) is also true, let us remark that $X_l^1(t)$ is the unique bounded solution of the Lyapunov differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} X_l^1(t) + A^{\mathrm{T}}(t) X_l^1(t) + X_l^1(t) A(t) + Q_l(t) = 0.$$

Since $Q_l(t) \succeq 0$, by Theorem 4.7 (iv) of [4] we have $X_l^1(t) \succeq 0 = X_l^0(t)$, $t \in \mathbf{R}$. This is just c_0).

Let us assume next that a_i), b_i), c_i) are fulfilled for $0 \le i \le j-1$ and prove that then they also hold for $i = j$.

If b_{j-1}) is fulfilled, then it follows from Theorem 4.7 (i) of [4] that equation (15) has unique bounded solution on $\mathbf{R}$, so that $X^j(t)$ is well defined.

Seting $P(t) = (P_1(t), P_2(t)) \in \Omega(\mathcal{R}, Q)$, one can see that it verifies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P(t) + \tilde{\mathcal{R}}(t, P(t)) + Q(t) + \hat{Q}(t) = 0, \tag{19}$$

where $\hat{Q}(t) = (\hat{Q}_1(t), \hat{Q}_2(t)) \succ\succ 0$. It is easy to check that $P_l(t)$ verifies the Lyapunov equations

$$\frac{\mathrm{d}}{\mathrm{d}t} P_l(t) + A_{j-1}^{\mathrm{T}}(t) P_l(t) + P_l(t) A_{j-1}(t) + H_l^{j-1}(t) = 0, \quad l = 1, 2, \tag{20}$$

where $A_{j-1}(t)$ is as in (18) with $X_l^j(t)$ replaced by $X_l^{j-1}(t)$ and $H^{j-1}(t) = (H_1^{j-1}(t), H_2^{j-1}(t))$, with

$$\begin{aligned} H_1^{j-1}(t) = & -[P_1(t) - X_1^{j-1}(t)] S_1(t) [P_1(t) - X_1^{j-1}(t)] - \\ & -[P_2(t) - X_2^{j-1}(t)] S_2(t) P_1(t) - P_1(t) S_2(t) [P_2(t) - X_2^{j-1}(t)] + \\ & + P_2(t) S_{12}(t) P_2(t) + X_1^{j-1}(t) S_1(t) X_1^{j-1}(t) + Q_1(t) + \hat{Q}_1(t), \end{aligned} \tag{21}$$

$$H_2^{j-1}(t) = -[P_2(t) - X_2^{j-1}(t)]S_2(t)[P_2(t) - X_2^{j-1}(t)] - \\ -[P_1(t) - X_1^{j-1}(t)]S_1(t)P_2(t) - P_2(t)S_1(t)[P_1(t) - X_1^{j-1}(t)] + \\ +P_1(t)S_{21}(t)P_1(t) + X_2^{j-1}(t)S_2(t)X_2^{j-1}(t) + Q_2(t) + \hat{Q}_2(t). \tag{22}$$

From (15) and (20) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(P_l(t) - X_l^j(t)) + A_{j-1}^{\mathrm{T}}(t)(P_l(t) - X_l^j(t)) + \\ +(P_l(t) - X_l^j(t))A_{j-1}(t) + M_l^{j-1}(t) = 0, \tag{23}$$

where $M_l^{j-1}(t) = H_l^{j-1}(t) - Q_l^{j-1}(t)$.

Since a_{j-1}) is fulfilled, it follows from (16), (17) and (21)–(22) that $M_l^{j-1}(t) \succ\succ 0$, $t \in \mathbf{R}$.

Applying Theorem 4.7 (iv) in [4] to equation (23), we conclude that

$$P_l(t) - X_l^j(t) \succ c\,\mathbf{1}_n, \quad t \in \mathbf{R}, \tag{24}$$

where c is a positive constant. Thus, we deduce that a_j) is fulfilled.

To check that b_j) is fulfilled, we rewrite equation (20) in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_l(t) + A_j^{\mathrm{T}}(t)P_l(t) + P_l(t)A_j(t) + H_l^J(t) = 0, \tag{25}$$

where $A_j(t)$ is as in (18) and the matrices $H_l^j(t)$ are as in (21)–(22), with $X_l^{j-1}(t)$ replaced by $X_l^j(t)$.

Equation (20) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}X_l^j(t) + A_j^{\mathrm{T}}(t)X_l^j(t) + X_l^J(t)A_j(t) + G_l^j(t) = 0, \tag{26}$$

where $A_j(t)$ is given by (18) and

$$G_1^j(t) = Q_1(t) + X_1^{j-1}(t)S_1(t)X_1^{j-1}(t) + X_2^{j-1}(t)S_{12}(t)X_2^{j-1}(t) - \\ -(X_1^{j-1}(t) - X_1^j(t))S_1(t)X_1^J(t) - X_1^j(t)S_1(t)(X_1^{j-1}(t) - X_1^j(t)) - \\ -X_2^j(t)X_2(t)(X_2^{j-1}(t) - X_2^j(t)) - (X_2^{j-1}(t) - X_2^j(t))S_2(t)X_2^j(t), \tag{27}$$

$$G_2^j(t) = Q_2(t) + X_1^{j-1}(t)S_{21}(t)X_1^{j-1}(t) + X_2^{j-1}(t)S_2(t)X_2^{j-1}(t) - \\ -X_2^j(t)S_1(t)(X_1^{j-1}(t) - X_1^j(t)) - (X_1^{j-1}(t) - X_1^j(t))S_1(t)X_2^j(t) - \\ -X_2^J(t)S_2(t)(X_2^{j-1}(t) - X_2^j(t)) - (X_2^{j-1}(t) - X_2^j(t))S_2(t)X_2^j(t). \tag{28}$$

Subtracting (26) from (25) and taking into account (24), we obtain that the function $t \to P_l(t) - X_l^j(t)$ is a bounded and uniform positive solution of the Lyapunov equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Y_l(t) + A_j^{\mathrm{T}}(t)Y_l(t) + Y_l(t)A_j(t) + \Theta_l^j(t) = 0 \tag{29}$$

with $\Theta_l^j(t) = H_l^j(t) - G_l^j(t)$. It is easy to see that $\Theta_l^j(t) \succ\succ 0$.

Applying the implication (vi) $\to$ (i) of Theorem 4.5 in [4] to equation (29), we conclude that the zero state equilibrium of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A_j(t)x(t)$$

is exponentially stable. Thus, we proved that item b_j) is fulfilled.

To check the validity of item c_j), we subtract equation (26) from equation (15) written for $X_l^{j+1}(t)$ and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(X_l^j(t) - X_l^{j+1}(t)) = A_j^{\mathrm{T}}(t)(X_l^{j+1}(t) - X_l^j(t))+ \\ +(X_l^{j+1}(t) - X_l^j(t))A_j(t) + \Delta_l^j(t),\end{aligned} \tag{30}$$

where $\Delta_l^j(t) = Q_l^j(t) - G_l^j(t)$, $l = 1, 2$. Combining (16)–(17) written for $j+1$ instead of j with (27)–(28), one can see that $\Delta_l^j(t) \succ\succ 0$. Applying Theorem 4.7 (iv) of [4] to equation (30), we conclude that

$$X_l^{j+1}(t) - X_l^j(t) \geq 0, \quad t \in \mathbf{R}.$$

This shows that c_j) is fulfilled.

It follows from a_j) and c_j), $j \geq 0$, that the sequences $\{X_l^j(t)\}_{j\geq 0}$, $l = 1, 2$, $t \in \mathbf{R}$, are convergent. Set $\tilde{X}_l(t) = \lim_{j\to\infty} X_l^j(t)$, $l = 1, 2$, $t \in \mathbf{R}$. By standard arguments we deduce that $t \to \tilde{X}(t) = (\tilde{X}_1(t), \tilde{X}_2(t))$ is a solution of (1). As in [4], one can prove that $\tilde{X}(t)$ is just the stabilizing solution of (1).

In the same way as in the proof of item a_j), one shows that $X_l^j(t) \preceq Y_l(t)$ for arbitrary $Y(t) = \begin{pmatrix} Y_1(t), & Y_2(t) \end{pmatrix}$ which verifies

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) + \mathcal{R}(t, Y(t)) + Q(t) \preceq 0, Y_l(t) \succeq 0.$$

This allows us to conclude that $\tilde{X}(t)$ is the minimal solution of (1), thus the proof is complete. $\square$

Remark 3.2. a) Using Theorem 4.7 (iii) in [4], we can deduce that in the time invariant case the unique bounded solution of (15) is constant. Therefore, in the time invariant case, for each iteration we have to solve two algebraic Lyapunov equations, namely,

$$A_{j-1}^{\mathrm{T}}X_l^j + X_l^j A_{j-1} + Q_l^j = 0, \quad l = 1, 2,$$

with $A_{j-1} = A - S_1X_1^{j-1} - S_2X_2^{j-1}$ and Q_l^{j-1} as in (16)–(17).

b) If $A(\cdot), S_j(\cdot), S_{kl}(\cdot), Q_l(\cdot)$ are periodic functions with period $\theta > 0$, then the unique bounded solution of (15) is a periodic function with the same period θ.

The initial value $X_l^j(0)$ is obtained as a solution of the Stein equation:

$$X_l^j(0) = \Phi_{j-1}^{\mathrm{T}}(\theta,0)X_l^j(0)\Phi_{j-1}(\theta,0) + \int_0^\theta \Phi_{j-1}^{\mathrm{T}}(s,0)Q_l^{j-1}(s)\Phi_{j-1}(s,0)\mathrm{d}s,$$

where $\Phi_{j-1}(t,\tau)$ is the fundamental matrix solution of $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A_{j-1}(t)x(t)$.

4. LYAPUNOV TYPE ITERATIONS FOR EQUATION (2)

Consider the sequence of functions $\{X^j(t)\}_{j\geq 0}$, $X^j(t) = (X_1^j(t), X_2^j(t))$, where $X_l^j : \mathbf{R} \to \mathbf{R}^{n\times n}$ is the unique bounded solution of the nonsymmetric Lyapunov differential equations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}X_l^j(t) + (A^{\mathrm{T}}(t) - X_l^{j-1}(t)S_l(t))X_l^j(t)+\\
+X_l^j(t)(A(t) - S_1(t)X_1^{j-1}(t) - S_2(t)X_2^{j-1}(t))\\
+Q_l(t) + X_l^{j-1}(t)S_l(t)X_l^{j-1}(t) = 0,\\
l = 1,2,\ j \geq 1,\ X_l^0(t) = 0,\ l = 1,2.
\end{aligned} \tag{31}$$

The main result of this section is

THEOREM 4.1. *Under assumptions* $\mathbf{H_1}$ *and* $\mathbf{H_2}$*, the sequence* $\{X^j(t)\}_{j\geq 0}$ *is well defined and convergent. If*

$$\tilde{X}(t) = \lim_{j\to\infty} X^j(t), \quad t \in \mathbf{R}, \tag{32}$$

then $\tilde{X}(t)$ *is the stabilizing and minimal solution of* (2).

The *proof* follows the same line as in the case of Theorem 3.1, and it is thus omitted. However we remark that instead of the item b$_j$) we should prove the new item

b$_j^*$) The zero state equilibrium of the linear differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Z_l(t) = [A(t) - S_l(t)(X_l^j(t))^{\mathrm{T}}]Z_l(t) + Z_l(t)[A(t) - S_1(t)X_1^j(t) - S_2(t)X_2^j(t)]^{\mathrm{T}}$$

on $\mathbf{R}^{n\times n}$ is exponentially stable.

Remarks 4.2. a) If $A(\cdot)$, $S_j(\cdot)$, $Q_l(\cdot)$ in (2) are constant, then one can deduce inductively that the unique solution of (31) is constant. Therefore, in the time invariant case, at each iteration we have to solve the nonsymmetric algebraic Lyapunov equations

$$(A^{\mathrm{T}} - X_l^{j-1}(t)S_l)X_l^j + X_l^j(A - S_1X_1^{j-1} - S_2X_2^j) + Q_l + X_l^{j-1}S_lX_l^{j-1} = 0.$$

b) If $A(\cdot), S_j(\cdot), Q_l(\cdot)$ in (2) are periodic functions with period $\theta > 0$, then one can deduce via Theorem 4.7 (ii) in [4] that the unique bounded solution of (31) is a periodic function with the same period θ. In this case, the initial values $X_l^j(0)$ are obtained as solutions of the nonsymmetric Stein equations

$$X_l^j(0) = \Theta_{j-1,l}^{\mathrm{T}}(\theta,0)X_l^j(0)\Phi_{j-1}(\theta,0)+$$

$$+\int_0^\theta \Theta_{j-1,l}^{\mathrm{T}}(s,0)[Q_l(s)+X_l^{j-1}(s)S_l(s)X_l^{j-1}(s)]\Phi_{j-1}(s,0)\mathrm{d}s,$$

where $\Theta_{j-1,l}(t,\tau)$ is the fundamental matrix solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = [A(t) - S_l(t)(X_l^{j-1}(t))^{\mathrm{T}}]x(t)$$

while $\Phi_{j-1}(t,\tau)$ is the fundamental matrix solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = [A(t) - S_1(t)X_1^{j-1}(t) - S_2(t)X_2^{j-1}(t)]x(t).$$

At the end of this section we give the time varying counterpart of Corollary 1 from [10].

Corollary 4.3. *If there exists* $P(t) = (P_1(t), P_2(t)) \in \Omega(\mathcal{R}, Q)$ *such that the zero solution of the linear differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = [A(t) - S_1(t)P_1(t) - S_2(t)P_2(t)]z(t)$$

is exponentially stable then $\tilde{X}(t) = (\tilde{X}_1(t), \tilde{X}_2(t))$*defined by* (32) *is a closed-loop stabilizing solution of system* (2).

Proof. From Theorem 4.1 we have $\tilde{X}_l(t) \preceq P_l(t)$.The conclusion follows from Proposition 4.1 (ii) in [4]. $\square$

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