STABILITY OF EQUILIBRIA IN A MODEL FOR ELECTROHYDRAULIC SERVOMECHANISMS

ANDREI HALANAY and IOAN URSU

The mathematical model of an electrohydraulic servomechanism is developed and a theorem on the stability of equilibria is proved.

AMS 2000 Subject Classification: 70K20, 93D05, 93D15.

Key words: Lyapunov stability, critical case, Lyapunov-Malkin Theorem.

1. INTRODUCTION

(Electro)hydraulic servomechanisms (EHS) are used as actuators of modern aircrafts control surfaces. The usual performance specifications for such devices are high accuracy of reference signals tracking and high bandwidth implying small servo time constants. But a property sine qua non of such vital system for aircraft safety is, first of all, the stability of its equilibria; this condition concerns the basis itself of the system approach, study and design. Because of the complexity of EHS analysis and the nonlinearities in its dynamics, both the design and the control of EHS are still difficult and immature, although various methodologies of the automatic control theory were brought to the proof in this field; from the classical linearization [1], to the artificial intelligence paradigm ([17], [18]). In the last twenty years, a large amount of work in EHS control systems has been devoted to problems such as design of observers ([3], [9], [12]), feedback linearization ([5], [13], [19]), feedback stabilization ([14]), high bandwidth control ([2]), robust and antichattering control ([16]).

During the qualification testing of the hydraulic servos for aircraft’s primary flight controls actuation ([15]), a mass overload simulated on the test rig has generated, as a response to an accidentally impulse type disturbance introduced at the mechanical input, a strong self-excited motion regime of the entire test rig, induced, certainly, by the powered hydraulic servo. Thus, our belief is that modern linear control theory cannot explain all strange aircraft incidents or even catastrophes when the actuator is, for sure, involved; these misfortunes can originate in a problem of stability of the equilibrium point for
the loaded hydraulic servomechanism. So, the paper is a new attempt of analysis and confirmation, using the classical approach of Lyapunov-Malkin, that such harmful phenomena in hydraulic servo’s operation can be foreseen and kept under by analytical calculus. Recently, a methodology of control synthesis was described in a series of authors works ([6]–[8]), ensuring the equilibria stability property of hydraulic servo. In fact, the approach can be subsumed to pseudoactive control paradigm ([18]).

* * *

Detailed approaches of different problems emerging in the process of mathematical modeling of hydraulic servos, with emphasis on those used in aircraft control chains, have been presented in classical books ([4], [11]) but also in [6]. Herein, a basic mathematic model of EHS dynamics is succinctly rewritten. EHS is in fact a combination electrohydraulic servovalve (EHSV)-hydrocylinder (HC). The paradigm of EHSV as ideal spool valve ([11]) and having rectangular ports is assumed. An algebraic sign convention is chosen, that of positive incoming in cylinder flow and negative outgoing from cylinder flow. Thus, the first equation of the EHSV-controlled piston is obtained based on the fact that the flow into and out of the cylinder is described by two components, one being due to the movement of the piston and one to compressibility effects: for the case $\sigma > 0$,

$$
(1.1) \quad c_d w \sigma \sqrt{\frac{2(p_s-p_1)}{\rho}} = S\dot{z} + \frac{V+Sz}{B}p_1, \quad -c_d w \sigma \sqrt{\frac{2p_2}{\rho}} = -S\dot{z} + \frac{V-Sz}{B}p_2,
$$

and, similarly, if $\sigma < 0$,

$$
(1.1') \quad c_d w \sigma \sqrt{\frac{2p_1}{\rho}} = S\dot{z} + \frac{V+Sz}{B}p_1, \quad -c_d w \sigma \sqrt{\frac{2(p_s-p_2)}{\rho}} = -S\dot{z} + \frac{V-Sz}{B}p_2.
$$

The parameters are $c_d$ – volumetric flow coefficient of the valve port; $w$ – valve-port width; $p_s$ – supply pressure; $\rho$ – volumetric density of oil; $V$ – cylinder semivolume; $B$ – bulk modulus of the oil. The variables are $p_1$ and $p_2$ – hydraulic pressures in the chambers of the HC; $\sigma$ – relative displacement spool-sleeve; $z$ – load displacement.

The equation of motion of the HC piston assembly is defined as a force balance equation relating to the displacement of the load generated by the hydraulic pressures $p_1$ and $p_2$

$$
(1.2) \quad m\ddot{z} + fz + kz = S(p_1 - p_2).
$$

The components of the load are: $m$ – equivalent inertial load of primary control surface reduced at the EHS’s rod; equivalent friction force, defined
by a coefficient $f_r$: elastic force, defined by an equivalent aerodynamic elastic force coefficient $k$. $S$ is the equivalent active area of the piston.

The linear displacement induced to spool valve by the torque motor of EHSV is described as a first order system

\[ \pi \dot{\sigma} + \sigma = k_v u. \]

The parameters are: $\sigma$ – time constant of the (servo) valve; $k_v$ – proportional valve displacement/voltage coefficient. The relative displacement spool-sleeve $\sigma$ is induced in the presence of the control signal $u$.

For this system, the aim of control synthesis is to have a good tracking by the load displacement $z$ of the specified desired position references signals $x$. The closed loop performance of the system can be measured by the actual (realized) servo time constant. Thus, a good tracking system is characterized by fast (little) servo time constant. In practice, this control objective is performed by implementing a compensator (controller) to generate the control $u$ based on state feedback information given by transducers.

Using the notation

\[ C := c_d w \sqrt{\frac{T}{\rho}} \]

and

\[ x_1 := z; \quad x_2 := \dot{z}; \quad x_3 = p_1; \quad x_4 = p_2; \quad x_5 := \sigma, \]

for the state variables, the canonical first order system derived from equations (1.1)–(1.3) is

\[
\begin{cases}
\dot{x}_1 = x_2, \\
\dot{x}_2 = \frac{k}{m} x_1 - \frac{f_r}{m} x_2 + \frac{S}{m} x_3 - \frac{S}{m} x_4, \\
\dot{x}_3 = \frac{B}{V_0 + S x_1} \left( C x_5 \sqrt{p_s - x_3} - S x_2 \right), \\
\dot{x}_4 = \frac{B}{V_0 - S x_1} \left( -C x_5 \sqrt{x_4} + S x_2 \right), \\
\dot{x}_5 = -\frac{1}{\tau} x_5 + \frac{k_v}{\tau} u(x_1, x_2, x_3, x_4, x_5).
\end{cases}
\]

Let $p \in (0, 1)$ and define, for $p_s (1 - p) - \frac{k_v}{S} > 0$,

\[ \dot{x}_1 = x_0, \quad \dot{x}_2 = 0, \quad \dot{x}_3 = \frac{k}{S} x_0 + p_s p, \quad \dot{x}_4 = p_s p, \quad \dot{x}_5 = 0. \]

If $u(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5) = 0$, (1.7) is an equilibrium point of (1.6).
2. STABILITY ANALYSIS

When the equilibria defined in (1.7) are translated to zero through
\[ y_1 = x_1 - \hat{x}_1, \quad y_2 = x_2, \quad y_3 = x_3 - \hat{x}_3, \quad y_4 = x_4 - \hat{x}_4, \quad y_5 = x_5 \]
and \( \tilde{u} \) is defined by
\[ \tilde{u}(\vec{y}) = \frac{k_v}{\tau} u(\vec{y} + \hat{x}), \]
system (1.1) becomes
\[
\begin{aligned}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -\frac{k}{m} y_1 - \frac{f_r}{m} y_2 + \frac{S}{m} y_3 - \frac{S}{m} y_4, \\
\dot{y}_3 &= \frac{BC}{V_0 + S y_1 + S x_0} y_5 \sqrt{p_s - y_3 - \hat{x}_3} - \frac{BS y_2}{V_0 + S y_1 + S x_0} := f_3(\vec{y}), \\
\dot{y}_4 &= -\frac{V_0 - S y_1 - S x_0}{V_0 - S y_1 - S x_0} y_5 \sqrt{y_4 + \hat{x}_4} + \frac{B S y_2}{V_0 - S y_1 - S x_0} := f_4(\vec{y}), \\
\dot{y}_5 &= -\frac{1}{\tau} y_5 + \tilde{u}(y).
\end{aligned}
\]

Denote by
\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
\frac{k}{m} & -\frac{f_r}{m} & \frac{S}{m} & -\frac{S}{m} & 0 \\
0 & a_{32} & 0 & 0 & a_{35} \\
0 & a_{42} & 0 & 0 & a_{45} \\
a_{51} & a_{52} & a_{53} & a_{54} & -\frac{1}{\tau}
\end{pmatrix}
\]
the Jacobian matrix at zero, where
\[
\begin{align*}
a_{32} &= \frac{\partial f_3}{\partial y_2}(0), & a_{35} &= \frac{\partial f_3}{\partial y_5}(0), & a_{42} &= \frac{\partial f_4}{\partial y_2}(0), & a_{45} &= \frac{\partial f_4}{\partial y_5}(0), \\
a_{51} &= \frac{\partial \tilde{u}}{\partial y_1}(0), & a_{52} &= \frac{\partial \tilde{u}}{\partial y_2}(0), & a_{53} &= \frac{\partial \tilde{u}}{\partial y_3}(0), & a_{54} &= \frac{\partial \tilde{u}}{\partial y_4}(0).
\end{align*}
\]
It is easy to see that the characteristic polynomial \( Q \) of \( A \) has zero as a root, so
\[
Q(\lambda) = \lambda Q_1(\lambda).
\]
This corresponds to a critical case for stability theory that can be approached through the use of a theorem of Lyapunov and Malkin ([10]).

The following theorem settles the problem of stability for the zero solution of the system (2.3).
Theorem 2.1. Suppose \( Q_1 \) in (2.5) is a stable polynomial (all its roots have strictly negative real parts). Then the zero solution of (3.2) is simple stable by Lyapunov. If initial data are small enough, then \( \lim_{t \to \infty} y_1(t) \) can be made arbitrary small, thus \( \lim_{t \to \infty} [x_1(t) - \hat{x}_1] \) is arbitrary small.

Proof. Transform the linear part

\[
\dot{w} = Aw
\]

of (2.3) through \( \eta = \sum_{j=1}^{5} d_j w_j \) such that \( d_1, \ldots, d_5 \) are not all zero and \( \dot{\eta} = 0 \), so \( \sum_{j=1}^{5} d_j \dot{w}_j = 0 \). From (2.4) and (2.6) we deduce that

\[
\begin{align*}
-d_2 \frac{k}{m} + d_5 a_{51} &= 0, \\
d_1 - d_2 \frac{f_r}{m} + d_3 a_{32} + d_4 a_{42} + d_5 a_{52} &= 0, \\
d_2 \frac{S}{m} + d_5 a_{53} &= 0, \\
d_3 \frac{S}{m} + d_5 a_{54} &= 0, \\
d_3 a_{35} + d_4 a_{45} - \frac{1}{\tau} d_5 &= 0.
\end{align*}
\]

If \( a_{54} + a_{53} = 0 \) and \( \frac{k}{m} a_{53} + \frac{S}{m} a_{51} = 0 \), then the last term of \( Q_1 \), namely,

\[
b_5 = -\frac{k}{m} a_{35} a_{53} - \frac{k}{m} a_{45} a_{54} - \frac{S}{m} a_{35} a_{51} + \frac{S}{m} a_{45} a_{51} - \frac{a_{35} a_{42} a_{53} S}{m} + a_{32} a_{45} a_{53} \frac{S}{m} - a_{35} a_{42} a_{54} \frac{S}{m} + a_{32} a_{45} a_{54} \frac{S}{m} = 0,
\]

which contradicts the hypotheses that \( Q_1 \) is a stable polynomial. It follows that \( d_2 = 0, d_5 = 0 \), and we take

\[
d_4 = 1 \quad \text{so} \quad d_1 = \frac{a_{45}}{a_{35}}, \quad d_3 = \frac{a_{32} a_{45} - a_{42} a_{35}}{a_{35}}.
\]

Introduce a new state variable through

\[
y_4 = y - d_1 y_1 - d_3 y_3.
\]
The system that results is

\[
\begin{align*}
\mathbf{y} &= d_1 y_2 + \frac{d_3 B}{V_0 + Sx_0 + Sy_1} (cy_5 \sqrt{p_y - y_3 - \hat{x}_3} - S y_2) + \\
&\quad + \frac{B}{V_0 - Sx_0 - Sy_1} (-C y_5 \sqrt{p_y + y - d_1 y_1 - d_3 y_3 + S y_2}) := Y(y, y_1, y_2, y_3, y_5), \\
y_1 &= y_2, \\
y_2 &= \left( \frac{S d_1}{m} - \frac{k}{m} \right) y_1 - \frac{f_r}{m} y_2 + \frac{S (1 + d_3) y_3}{m} - \frac{S y}{m}, \\
\dot{y}_3 &= \frac{B C}{V_0 + Sx_0 + Sy_1} y_5 \sqrt{p_y - y_3 - \hat{x}_3} - \frac{B S y_2}{V_0 + Sx_0 + Sy_1}, \\
\dot{y}_5 &= -\frac{1}{\tau} y_5 + \hat{u}(y_1, y_2, y_3, y - d_1 y_1 - d_3 y_3, y_5) := Y_5(y, y_1, y_2, y_3, y_5).
\end{align*}
\]

(2.9)

The linear part of the first equation in (2.9) is zero and the characteristic equation of the Jacobian matrix of (2.9) at zero has exactly one zero root, the other ones having strictly negative real parts. This implies that we can solve with respect to \(y_1, y_2, y_3\) and \(y_5\) the algebraic system

\[
\begin{align*}
y_2 &= 0, \\
(Sd_1 - k)y_1 - f_r y_2 + S(1 + d_3)y_3 - S y &= 0, \\
C y_5 \sqrt{p_y - y - \hat{x}_3} - S y_2 &= 0, \\
Y_5(y, y_1, y_2, y_3, y_5) &= 0.
\end{align*}
\]

(2.10)

Remark that \(y = 0, y_1 = y_2 = y_3 = y_5 = 0\) verify (2.10). Thus, the Implicit Function Theorem implies that in a small neighborhood of \(y = 0\) there exists a unique solution

\[
(2.11) \quad y_1 = \varphi_1(y), \quad y_2 = 0, \quad y_3 = \varphi_3(y), \quad y_5 = 0
\]

of (2.10) with \(\varphi_1(0) = \varphi_3(0) = 0\).

The new variables

\[
(2.12) \quad \xi_1 = y_1 - \varphi_1(y), \quad \xi_3 = y_3 - \varphi_3(y)
\]

bring system (2.9) to

\[
\begin{align*}
\dot{y} &= Y(y, \xi_1 + \varphi_1(y), y_2, \xi_3 + \varphi_3(y), y_5) := \tilde{Y}(y, \xi_1, y_2, \xi_3, y_5), \\
\dot{\xi}_1 &= y_2 - \varphi'_1(y) \tilde{Y}, \\
\dot{\xi}_2 &= \left( \frac{S d_1}{m} - \frac{k}{m} \right) \xi_1 - \frac{f_r}{m} y_2 + \frac{S (1 + d_3)}{m} \xi_3, \\
\dot{\xi}_3 &= \frac{B}{V_0 + Sx_0 + S[\xi_1 + \varphi_1(y)]} [C y_5 \sqrt{p_y - \xi_3 - \varphi_3(y) - \hat{x}_3} - S y_2 - \varphi'_3(y) \tilde{Y}, \\
\dot{y}_5 &= Y_5(y, \xi_1 + \varphi_1(y), y_2, \xi_3 + \varphi_3(y), y_5).
\end{align*}
\]

(2.13)
Since \( \tilde{Y}(y, 0, 0, 0, 0) = Y(y, \varphi_1(y), 0, \varphi_3(y), 0) = 0 \), system (2.13) belongs to the singular class covered by Lyapunov-Malkin Theorem in [10], Ch. IV, §34 (see also [7]). As proved in that theorem, (2.13) has the family of equilibria

\[
y = c, \quad \xi_1 = \varphi_1(c), \quad y_2 = 0, \quad \xi_3 = \varphi_3(c), \quad y_5 = 0,
\]

with \( c \) a small constant, that are stable by Lyapunov. By (2.11), \( c = 0 \) in (2.14) gives the zero solution of (2.3). Solutions of (2.13) that start close to equilibria (2.14) satisfy

\[
\lim_{t \to \infty} y(t) = \alpha, \quad \lim_{t \to \infty} \xi_1(t) = \varphi_1(\alpha), \quad \lim_{t \to \infty} y_2(t) = 0, \quad \lim_{t \to \infty} \xi_3(t) = \varphi_3(\alpha), \quad \lim_{t \to \infty} y_5(t) = 0.
\]

Here, \( \alpha = y(0) + \int_0^\infty \tilde{Y}[y(t), \xi_1(t), y_2(t), \xi_3(t), y_5(t)]dt \) the convergence of the integral resulting from the proof of Lyapunov-Malkin Theorem (see p. 116–117 in [10]). It is also proved that \( |\alpha| < |y(0)| + K \gamma \) for some positive \( K \) and that \( \forall \varepsilon > 0, \exists \eta > 0 \) such that if \( |y(0)| < \eta, |\xi_1(0)| < \eta, |y_2(0)| < \eta, |\xi_3(0)| < \eta \) and \( |y_5(0)| < \eta \) then one can choose \( K \) such that \( |y(0)| + K \gamma < \varepsilon \) and so the last statement of Theorem 2.1 is proved.

3. CONCLUDING REMARKS

The methodology for treating the problem of equilibrium stability in (E)HS models developed in our recent works [6]–[8], as applied to the present study, concerns the covering of the following steps:

1) the choice of a feedback control law \( u(x_1, x_2, x_3, x_4, x_5) \) with \( u(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5) = 0 \) (sometimes by a trial and error type methodology);

2) the use of the \( (p, x) \) – parametrization of the coefficients of the polynomial \( Q_1(\lambda) \); the parameter \( p \in (0, 1) \) refers to the initial conditions for state variables \( x_3 \) and \( x_4 \) and the parameter \( x \in (-x_{\text{min}}, x_{\text{max}}) \) is in fact the desired output;

3) the study of the Routh-Hurwitz conditions of stability relatively to polynomial \( Q_1(\lambda) \).

Finally, it should be mentioned that the situation \( \sigma < 0 \) can be approached in a similar way.

REFERENCES


Received 2 October 2006

"POLITEHNICA" University of Bucharest
Department of Mathematics I
Splaiul Independenței 313
060032 Bucharest, Romania
halanay@mathem.pub.ro

and

"Elie Carafoli" National Institute for Aerospace Research
Bd. Iuliu Maniu 220, 061126, Bucharest, Romania
iursu@aero.incas.ro