STATES ON PSEUDO-BCK ALGEBRAS

LAVINIA CORINA CIUNGU

The notion of a state is an analogue of a probability measure and was first introduced by Köpka and Chovanec for MV-algebras and by Riečan for BL-algebras. The states have also been studied for different types of non-commutative fuzzy structures such as pseudo-MV algebras, pseudo-BL algebras, bounded Rℓ-monoids, residuated lattices and pseudo-BCK semilattices. In this paper we investigate the states on pseudo-BCK algebras and show that Georgescu’s original problem in [10] for pseudo-BL algebras has a negative solution for good pseudo-BCK algebras. We prove that every Bosbach state on a pseudo-BCK algebra is a Riečan state and that every Riečan state on a good pseudo-BCK algebra with pseudo-double negation is a Bosbach state. In contrast to the case of pseudo-BL algebras, we show that there exist linearly ordered pseudo-BCK algebras having no Bosbach states.

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Key words: Pseudo-BCK algebra, Bosbach state, Riečan state, state-morphism, deductive system.

1. INTRODUCTION

The notion of a state is an analogue of a probability measure and has a very important role in the theory of quantum structures (see [8]). The state on MV algebras was first introduced by Köpka and Chovanec [16] while the state on BL algebras was introduced by Riečan [19]. In the case of non-commutative fuzzy structures, the states were introduced by Dvurečenskij [5] for pseudo-MV algebras, by Georgescu [10] for pseudo-BL algebras and by Dvurečenskij and Rachůnek [9] for bounded non-commutative Rℓ-monoids. In the case of a pseudo-MV algebra $M$, Dvurečenskij [4] proved that there is an $ℓ$-group $(G, u)$ with strong unit $u$ such that $M$ is isomorphic to $\Gamma(G, u) = \{ g \in G \mid 0 \leq g \leq u \}$. This allowed him to define a partial addition $+$, that is, $x + y$ is defined if $x \leq y^{-} = u - y$ and the state is a mapping $s : M \rightarrow [0, 1]$ which preserves the partial addition $+$ and $s(1) = 1$. We recall that the elements $a$ and $b$ are orthogonal if $a + b$ is defined in $M$. The other non-commutative structures do not have such a group representation and it was more difficult to define the notion of states for these structures. We recall that a state on MV-algebras
always exists in contrast to pseudo-MV algebras (see [5]); on the other hand, in [6] was solved the existence of states for linearly ordered pseudo-BL algebras (see also [7]). In the case of pseudo-BL algebras Georgescu [10] defined the Bosbach and Riečan states and, for a good pseudo-BL algebra, he proved that any Bosbach state is also a Riečan state. He asked to find an example of Riečan state on a good pseudo-BL algebra which is not a Bosbach state.

The notions of Bosbach and Riečan states on good pseudo-BL algebras were generalized by Dvurečenskij and Rachünk [9] for bounded \( R\ell \)-monoids. They also proved that for good bounded \( R\ell \)-monoids the two notions of states coincide. In [1] there were investigated the Bosbach and Riečan states on residuated lattices and was proved that there exist Riečan states on good residuated lattices which are not Bosbach states. In [18] the above results were also proved in the case of pseudo-BCK semilattices. Inspired by the above mentioned results, in this paper we extend the notion of states to pseudo-BCK algebras. One of the main results consists of proving that any Bosbach state on a good pseudo-BCK algebra is a Riečan state, but the converse turns out not to be true. We also prove that every Riečan state on a good pseudo-BCK algebra with pseudo-double negation is a Bosbach state. In contrast to the case of pseudo-BL algebras, we show that there exist linearly ordered pseudo-BCK algebras having no Bosbach states. Additionally, we prove some new properties of pseudo-BCK algebras and give examples of pseudo-BCK algebras and good pseudo-BCK algebras.

2. PSEUDO-BCK ALGEBRAS AND THEIR BASIC PROPERTIES

The pseudo-BCK algebras were introduced by Georgescu and Iorgulescu [11] as generalization of BCK algebras in order to give a structure corresponding to pseudo-MV algebras, as the bounded commutative BCK algebras correspond to MV algebras. Properties of pseudo-BCK algebras and their connection with others fuzzy structures were established by Iorgulescu [12], [13], [14], [15].

Definition 2.1 ([12]). A pseudo-BCK algebra (more precisely, reversed left-pseudo-BCK algebra) is a structure \( A = (A, \leq, \rightarrow, \leadsto, 1) \), where \( \leq \) is a binary relation on \( A \), \( \rightarrow \) and \( \leadsto \) are binary operations on \( A \) and 1 is an element of \( A \) satisfying, for all \( x, y, z \in A \), the axioms below:

\[
\begin{align*}
(A_1) \quad & x \rightarrow y \leq (y \rightarrow z) \leadsto (x \rightarrow z), \quad x \leadsto y \leq (y \leadsto z) \rightarrow (x \leadsto z); \\
(A_2) \quad & x \leq (x \rightarrow y) \leadsto y, \quad x \leq (x \leadsto y) \rightarrow y; \\
(A_3) \quad & x \leq x; \\
(A_4) \quad & x \leq 1; \\
(A_5) \quad & \text{if } x \leq y \text{ and } y \leq x, \text{ then } x = y; \\
(A_6) \quad & x \leq y \text{ iff } x \rightarrow y = 1 \text{ iff } x \leadsto y = 1.
\end{align*}
\]
Remarks 2.2 ([12]). (1) A pseudo-BCK algebra \( A = (A, \le, \to, \cdot, 1) \) is commutative if \( \to = \cdot \).

(2) Any commutative pseudo-BCK algebra is a BCK algebra.

Example 2.3. Let \( A = \{a_1, a_1, b_1, c_1, a_2, b_2, c_2, 1\} \) with \( a_1 < a_1, b_1 < c_1 < 1 \) and \( a_1, b_1 \) incomparable, \( o_2 < o_2, b_2 < c_2 < 1 \) and \( a_2, b_2 \) incomparable. Assume that any element of the set \( \{a_1, a_1, b_1, c_1\} \) is incomparable with any element of the set \( \{o_2, a_2, b_2, c_2\} \). Consider the operations \( \to, \cdot \) given by the tables

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Then \( A = (A, \le, \to, \cdot, 1) \) is a proper pseudo-BCK algebra.

Remark 2.4 ([17]). Let \((A_i, \le, \to, \cdot, 1)_{i \in I}\) be a collection of pseudo-BCK algebras such that

(i) \( I_i = 1 \) for all \( i \in I \),

(ii) \( A_i \cap A_j = \{1\} \) for all \( i, j \in I, i \neq j \).

Denote \( A = \bigcup_{i \in I} A_i \) and define

\[
\begin{align*}
    x \to y &= \begin{cases} 
    x \to_i y & \text{if } x, y \in A_i \text{ for some } i \in I \\
    y & \text{otherwise},
    
    x \cdot y &= \begin{cases} 
    x \cdot_i y & \text{if } x, y \in A_i \text{ for some } i \in I \\
    y & \text{otherwise}.
    
\end{cases}
\end{cases}
\]

Then \((A, \le, \to, \cdot, 1)\) is a pseudo-BCK algebra called the union of the pseudo-BCK algebras \((A_i, \le, \to, \cdot, 1)_{i \in I}\).

Proposition 2.5 ([14], [15]). In any pseudo-BCK algebra the following properties hold:

\( (c_1) \) \( x \le y \) implies \( y \to z \le x \to z \) and \( y \cdot z \le x \cdot z \);

\( (c_2) \) \( x \le y, y \le z \) implies \( x \le z \);

\( (c_3) \) \( x \to (y \cdot z) = y \to (x \rightarrow z) \) and \( x \cdot (y \to z) = x \rightarrow (y \cdot z) \);

\( (c_4) \) \( z \le y \to x \) iff \( y \le z \cdot x \);

\( (c_5) \) \( z \to x \le (y \to z) \to (y \to x), z \cdot x \le (y \cdot z) \cdot (y \cdot z) \);

\( (c_6) \) \( x \le y \to x, x \le y \cdot x \);

\( (c_7) \) \( 1 \to x = x = 1 \cdot x \);

\( (c_8) \) \( x \le y \) implies \( z \to x \le z \to y \) and \( z \cdot x \le z \cdot y \);

\( (c_9) \) \( (y \to x) \cdot x \to x = y \to x, (y \cdot x) \cdot x \cdot x = y \cdot x \).
Definition 2.6 ([12]). If there is an element \(0\) of a pseudo-BCK algebra \(\mathcal{A} = (A, \leq, \rightarrow, \Leftarrow, 1)\) such that \(0 \leq x\) (i.e. \(0 \rightarrow x = 0 \Leftarrow x = 1\)) for all \(x \in A\), then it is called the zero of \(\mathcal{A}\). A pseudo-BCK algebra with zero is called a bounded pseudo-BCK algebra and is denoted by \(\mathcal{A} = (A, \leq, \rightarrow, \Leftarrow, 0, 1)\).

Example 2.7. Let \(A = \{0, a, b, c, 1\}\) with \(0 < a, b < c < 1\) and \(a, b\) incomparable. Consider the operations \(\rightarrow, \Leftarrow\) given by the tables

\[
\begin{array}{c|cccc}
\rightarrow & 0 & a & b & c \\
\hline
0 & 1 & 1 & 1 & 1 \\
a & 0 & 1 & b & 1 \\
b & a & a & 1 & 1 \\
c & 0 & a & b & 1 \\
1 & 0 & a & b & c \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\Leftarrow & 0 & a & b & c \\
\hline
0 & 1 & 1 & 1 & 1 \\
a & b & 1 & b & 1 \\
b & 0 & a & 1 & 1 \\
c & 0 & a & b & 1 \\
1 & 0 & a & b & c \\
\end{array}
\]

Then \(\mathcal{A} = (A, \leq, \rightarrow, \Leftarrow, 0, 1)\) is a bounded pseudo-BCK algebra.

Definition 2.8 ([12]). A pseudo-BCK algebra with \((pP)\) condition (i.e. with pseudo-product condition) or a pseudo-BCK\((pP)\) algebra for short, is a pseudo-BCK algebra \(\mathcal{A} = (A, \leq, \rightarrow, \Leftarrow, 1)\) satisfying the \((pP)\) condition: for all \(x, y \in A\) there exist \(x \odot y = \min\{z \mid x \leq y \rightarrow z\} = \min\{z \mid y \leq x \Leftarrow z\}\).

Definition 2.9 ([12]).

(1) Let \(\mathcal{A} = (A, \leq, \rightarrow, \Leftarrow, 0, 1)\) be a pseudo-BCK algebra. If the poset \((A, \leq)\) is a lattice, then we say that \(\mathcal{A}\) is a pseudo-BCK lattice.

(2) Let \(\mathcal{A} = (A, \leq, \rightarrow, \Leftarrow, 0, 1)\) be a reduct of a residuated lattice, then it is obvious that \(\mathcal{A}\) is a bounded pseudo-BCK\((pP)\) algebra.

Remark 2.10 ([14]). Pseudo-BCK\((pP)\) algebras are categorically isomorphic with left-porims (partially ordered, residuated, integral left-monoids).

Remark 2.11 ([12]). (Bounded) pseudo-BCK\((pP)\) lattices are categorically isomorphic with (bounded) residuated lattices.

Examples 2.12.

(1) If \(\mathcal{A} = (A, \leq, \rightarrow, \Leftarrow, 0, 1)\) is the bounded pseudo-BCK lattice from Example 2.7, then \(\min\{z \mid b \leq a \rightarrow z\} = \min\{a, b, c, 1\}\) and \(\min\{z \mid a \leq b \Leftarrow z\} = \min\{a, b, c, 1\}\) do not hold. Thus, \(b \odot a\) does not exist, so \(\mathcal{A}\) is not a pseudo-BCK\((pP)\) algebra. Moreover, since \((A, \leq)\) is a lattice, \(\mathcal{A}\) is a pseudo-BCK lattice.

(2) If \(\mathcal{A} = (A, \leq, \rightarrow, \Leftarrow, 0, 1)\) is a reduct of a residuated lattice, then it is obvious that \(\mathcal{A}\) is a bounded pseudo-BCK\((pP)\) algebra.

Remark 2.13 ([12]). Any bounded linearly ordered pseudo-BCK algebra is with \((pP)\) condition.
Proposition 2.14 ([15]). In any pseudo-BCK algebra \((pP)\) the following properties hold:

\((c_{10})\) \(x \circ y \leq x, y;\)
\((c_{11})\) \((x \rightarrow y) \circ x \leq x, y;\)
\((c_{12})\) \(y \leq x \rightarrow (y \circ x), y \leq x \rightarrow (x \circ y);\)
\((c_{13})\) \(x \rightarrow y \leq (x \circ z) \rightarrow (y \circ z), x \rightarrow y \leq (z \circ x) \rightarrow (z \circ y);\)
\((c_{14})\) \(x \circ (y \rightarrow z) \leq y \rightarrow (x \circ z), (y \rightarrow z) \circ x \leq y \rightarrow (z \circ x);\)
\((c_{15})\) \((y \rightarrow z) \circ (x \rightarrow y) \leq x \rightarrow z, (x \rightarrow y) \circ (y \rightarrow z) \leq x \rightarrow z;\)
\((c_{16})\) \((x \rightarrow y) = (x \circ y) \rightarrow z, x \rightarrow (y \rightarrow z) = (y \circ x) \rightarrow z;\)
\((c_{17})\) \((x \circ z) \rightarrow (y \circ z) \leq x \rightarrow (z \rightarrow y), (z \circ x) \rightarrow (z \circ y) \leq x \rightarrow (z \rightarrow y);\)
\((c_{18})\) \(x \rightarrow y \leq (x \circ z) \rightarrow (y \circ z) \leq x \rightarrow (z \rightarrow y), x \rightarrow y \leq (z \circ x) \rightarrow (z \circ y) \leq x \rightarrow (z \rightarrow y);\)
\((c_{19})\) \(x \leq y \text{ implies } x \circ z \leq y \circ z, z \circ x \leq z \circ y.\)

Let \(A = (A, \leq, \rightarrow, \leftrightarrow, 0, 1)\) be a bounded pseudo-BCK algebra. We define two negations \(\neg\) and \(\sim\) ([15]): for all \(x \in A, x^{-} = x \rightarrow 0, x^{\sim} = x \rightarrow 0.\)

Proposition 2.15 ([15]). In a bounded pseudo-BCK algebra the following properties hold:

\((c_{20})\) \(1^{-} = 0 = 1^{\sim}, 0^{-} = 1^{\sim};\)
\((c_{21})\) \(x \leq (x^{-})^{\sim}, x \leq (x^{\sim})^{-};\)
\((c_{22})\) \(x \rightarrow y \leq y^{-} \rightarrow x^{-}, x \rightarrow y \leq y^{\sim} \rightarrow x^{\sim};\)
\((c_{23})\) \(x \leq y \text{ implies } y^{-} \leq x^{-} \text{ and } y^{\sim} \leq x^{\sim};\)
\((c_{24})\) \(x \rightarrow y^{-} = y \rightarrow x^{-} \text{ and } x \rightarrow y^{\sim} = y \rightarrow x^{\sim};\)
\((c_{25})\) \(((x^{-})^{\sim})^{-} = x^{-}, ((x^{\sim})^{-})^{\sim} = x^{\sim}.\)

Proposition 2.16. In a bounded pseudo-BCK algebra the following properties hold:

\((c_{26})\) \(x \rightarrow y^{\sim} = y^{-} \rightarrow x^{-} = x^{-} \rightarrow y^{\sim} \quad \text{and} \quad x \rightarrow y^{-} = y^{\sim} \rightarrow x^{\sim} = x^{\sim} \rightarrow y^{\sim};\)
\((c_{27})\) \(x \rightarrow y^{\sim} = y^{-} \rightarrow x^{-} = x^{-} \rightarrow y^{\sim} \quad \text{and} \quad x \rightarrow y^{\sim} = y^{-} \rightarrow x^{\sim} = x^{\sim} \rightarrow y^{\sim};\)
\((c_{28})\) \((x \rightarrow y^{-})^{\sim} = x \rightarrow y^{-} \quad \text{and} \quad (x \rightarrow y^{\sim})^{\sim} = x \rightarrow y^{\sim}.\)

Proof. \((c_{26})\): By \((c_{24})\) we have \(y \rightarrow x^{-} = x \rightarrow y^{-}.\) Replacing \(y\) by \(y^{-}\) we get \(y^{-} \rightarrow x^{-} = x \rightarrow y^{-}.\) Replacing \(x\) by \(x^{\sim}\) in the last equality we get \(y^{-} \rightarrow x^{\sim} = x^{\sim} \rightarrow y^{-}.\) Hence, \((c_{25})\), we have \(y^{-} \rightarrow x^{-} = x^{\sim} \rightarrow y^{-}.\)

Thus, \(x \rightarrow y^{\sim} = y^{-} \rightarrow x^{\sim} = x^{\sim} \rightarrow y^{-}\) and, similarly, \(x \rightarrow y^{-} = y^{\sim} \rightarrow x^{\sim} = x^{\sim} \rightarrow y^{\sim}.\)

\((c_{27})\) The assertions follow replacing in \((c_{26})\) \(y^{-}\) by \(y\) and \(y^{\sim}\) by \(y^{-}\), and using \((c_{25})\).
(c_{28}) By (c_3) and (c_{27}) we have
\[ 1 = (x \to y^{-}) \sim (x \to y^{-}) = x \to ((x \to y^{-}) \sim y^{-}) = x \to ((x \to y^{-})^{-} \sim y^{-}) = (x \to y^{-})^{-} \sim (x \to y^{-}). \]
Hence \((x \to y^{-})^{-} \leq x \to y^{-}\). On the other hand, by (c_{21}) we have \( x \to y^{-} \leq (x \to y^{-})^{-} \), so \((x \to y^{-})^{-} = x \to y^{-}\). Similarly, \( (x \to y^{-})^{-} = x \to y^{-}\). □

Proposition 2.17. In a bounded pseudo-BCK(\(pP\)) algebra the following properties hold:
\begin{itemize}
  \item [(c_{29})] \((x_{n-1} \to x_n) \circ (x_{n-2} \to x_{n-1}) \circ \ldots \circ (x_1 \to x_2) \leq x_1 \to x_n\) and \((x_1 \to x_2) \circ (x_2 \to x_3) \circ \ldots \circ (x_{n-1} \to x_n) \leq x_1 \to x_n\);
  \item [(c_{30})] \(x \circ 0 = 0 \circ x = 0\);
  \item [(c_{31})] \(x \circ 1 = 1 \circ x = x\);
  \item [(c_{32})] \(x^{-} \circ x = 0 \) and \(x \circ x^{-} = 0\);
  \item [(c_{33})] \(x \leq y^{-} \) if and only if \(x \circ y = 0\) and \(x \leq y^{-}\) if and only if \(y \circ x = 0\);
  \item [(c_{34})] \(x \to y^{-} = (x \circ y)^{-} \) and \(x \to y^{-} = (y \circ x)^{-}\);
  \item [(c_{35})] \(x \leq y^{-} \) if and only if \(y \leq x^{-}\).
\end{itemize}

Proof. (c_{29}): It follows from (c_{15}) by induction.
\begin{itemize}
  \item [(c_{30})]: By (c_{10}), for \(y = 0\) and \(x = 0\).
  \item [(c_{31})]: By (c_{12}), for \(y = 1\) we get \(1 \to x \to (1 \circ x)\), so \(x \to (1 \circ x) = 1\). It follows by (A_6) that \(x \leq 1 \circ x\). On the other hand, by (c_{10}), we have \(1 \circ x \leq x\). Thus, \(1 \circ x = x\). Similarly, \(1 \leq x \to (x \circ 1)\), so \(x \to (x \circ 1) = 1\), that is, \(x \leq x \circ 1\). Therefore, \(x \circ 1 = 1\).
  \item [(c_{32})]: By (c_{11}), for \(y = 0\).
  \item [(c_{33})]: Using \(x \leq y^{-}\). Applying (c_{19}) we get \(x \circ y \leq y^{-} \circ y = 0\), so \(x \circ y = 0\). Conversely, if \(x \circ y = 0\), by (c_{12}) we have \(x \leq y \to (x \circ y) = y \to 0 = y^{-}\). Similarly, \(x \leq y^{-} \) if and only if \(y \circ x = 0\).
  \item [(c_{34})]: By (c_{19}), taking \(z = 0\).
  \item [(c_{35})]: It follows from (c_{33}). □
\end{itemize}

Definition 2.18 ([12]). A bounded pseudo-BCK algebra \(\mathcal{A} = (A, \leq, \to, \sim, 0, 1)\) is with \((pDN)\) (pseudo-Double Negation) condition if it satisfies for all \(x \in A\) the condition \((pDN)\) \((x^{-})^{-} = (x^{-})^{-} = x\).

Proposition 2.19 ([12]). Let \(\mathcal{A}\) be a pseudo-BCK algebra with \((pDN)\) condition. Then for all \(x, y \in A\) the following properties hold:
\begin{itemize}
  \item [(c_{36})] \(x \leq y \) if and only if \(y^{-} \leq x^{-}\);
  \item [(c_{37})] \(x \to y = y^{-} \to x^{-}\), \(x \to y = y^{-} \to x^{-}\);
  \item [(c_{38})] \(x^{-} \to y = y^{-} \to x^{-}\);
  \item [(c_{39})] \(x \to y^{-} \) if and only if \(y \to x^{-}\).
\end{itemize}
Theorem 2.20 ([12]). A bounded pseudo-BCK algebra $\mathcal{A} = (A, \leq, \rightarrow, \sim, 0, 1)$ with (pDN) condition is with (pP) condition.

Definition 2.21. A bounded pseudo-BCK algebra $\mathcal{A}$ is called good if $(x^-)^- = (x^-)^\sim$ for all $x \in A$.

Remark 2.22. It is easy to show that any bounded pseudo-BCK algebra can be embedded into a good one. Indeed, consider the bounded pseudo-BCK algebra $\mathcal{A} = (A, \leq, \rightarrow, \sim, 0, 1)$ and an element $0_1 \notin A$. Consider a new pseudo-BCK algebra $\mathcal{A}_1 = (A_1, \leq, \rightarrow_1, \sim_1, 0_1, 1)$, where $A_1 = A \cup \{0_1\}$ and the operations $\rightarrow_1$ and $\sim_1$ are defined as

$$x \rightarrow_1 y = \begin{cases} x \rightarrow y & \text{if } x, y \in A, \\ 1 & \text{if } x = 0_1, y \in A_1, \\ 0_1 & \text{if } x \in A, y = 0_1, \end{cases}$$

$$x \sim_1 y = \begin{cases} x \sim y & \text{if } x, y \in A, \\ 1 & \text{if } x = 0_1, y \in A_1, \\ 0_1 & \text{if } x \in A, y = 0_1. \end{cases}$$

One can easily check that $\mathcal{A}$ as an subalgebra of $\mathcal{A}_1$ and $\mathcal{A}_1$ is a good pseudo-BCK algebra.

Example 2.23. Consider the pseudo-BCK lattice $\mathcal{A}$ from Example 2.7. Since $(a^-)^\sim = 1$ and $(a^-)^\sim = a$, $\mathcal{A}$ is not good. $\mathcal{A}$ is embedded into the good pseudo-BCK lattice $\mathcal{A}_1 = (A_1, \leq, \rightarrow, \sim_1, 0_1, 1)$, where $A_1 = \{0, a, b, c, d, 1\}$ (in the construction given in Remark 2.22 we replaced $c$ by $d$, $b$ by $c$, $a$ by $b$, $0$ by $a$ and $0_1$ by $0$, so $0 < a < b, c < d < 1$ and $b, c$ are incomparable). The operations $\rightarrow$ and $\sim$ are defined as it the tables

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</table>

One can easily check that $\mathcal{A}_1$ is a good pseudo-BCK algebra. Moreover, we can see that $\min\{z \mid c \leq b \rightarrow z\} = \min\{b, c, d, 1\}$ and $\min\{z \mid b \leq c \sim z\} = \min\{b, c, d, 1\}$ do not hold. Thus, $c \ominus b$ does not exist, so $\mathcal{A}_1$ is without (pP) condition. Since $(A_1, \leq)$ is a lattice, $\mathcal{A}_1$ is a good pseudo-BCK lattice without (pP) condition.

Proposition 2.24. Let $\mathcal{A} = (A, \leq, \rightarrow, \sim, 0, 1)$ be a good pseudo-BCK algebra. Define a binary operation $\oplus$ on $A$ by $x \oplus y := y^\sim \rightarrow x^\sim$. Then for all $x, y \in A$ the following properties hold:
(1) $x \oplus y = x^- \bowtie y^-;\\
(2) x, y \leq x \oplus y;\\
(3) x \oplus 0 = 0 \oplus x = x^-;\\
(4) x \oplus 1 = 1 \oplus x = 1;\\
(5) (x \oplus y)^- \bowtie = x \oplus y = x^- \oplus y^-;\\
(6) \oplus \text{ is associative.}

\textbf{Proof.} (1): It follows from the second identity in \((c_{28})\), replacing \(x\) by \(x^-\).
(2): Since \(x \leq x^- \leq y^- \rightarrow x^-\), we have \(x \leq x \oplus y\). Similarly, \(y \leq y^- \leq x^- \bowtie y^-\), so \(y \leq x \oplus y\).
(3): \(x \oplus 0 = 0^\sim \rightarrow x^- = 1 \rightarrow x^- = x^-\). Similarly, \(0 \oplus x = x^- \rightarrow 0^\sim = x^- \rightarrow 0 = 1 = 1\).
(4): \(1 \oplus x = x^- \rightarrow 1^\sim = x^- \rightarrow 1 = 1\). Similarly, \(x \oplus 1 = 1\).
(5): \((x \oplus y)^-\bowtie = (y^- \rightarrow x^-) \bowtie = y^- \rightarrow x^- = x \oplus y\) (we used \((c_{28})\)).

We also have \(x^- \oplus y^- \bowtie = (y^- \rightarrow x^-) \bowtie = (x^- \rightarrow) \bowtie = y^- \rightarrow x^- = x \oplus y\).
(6): By \((c_{28})\) and \((c_3)\) we have
\[
(x \oplus y) \oplus z = (x^- \bowtie y^-) \oplus z = z^- \rightarrow (x^- \bowtie y^-) \bowtie = z^- \rightarrow (x^- \bowtie y^-) = x^- \bowtie (y \oplus z) = x^- \bowtie (y \oplus z).
\]

\textbf{Remark 2.25.} The definition of \(\oplus\) is inspired from the case of pseudo-BCK algebras (see \([10]\)), where \(x \oplus y := (y^- \circ x^-)\). Since in a good residuated lattice we have
\[
(y^- \circ x^-)^- \bowtie = (y^- \circ x^-)\bowtie, \quad (x \circ y)^- \bowtie = x \rightarrow y^- \text{ and } (x \circ y)^- \bowtie = y \rightarrow x^-,
\]
the definition of \(\oplus\) is valid for the residuated structures, too.

Let \(A\) be a pseudo-BCK algebra. For all \(x, y \in A\), define (see \([11, 15]\))
\[
x \lor y = (x \rightarrow y) \bowtie y, \quad x \lor y = (x \rightarrow y) \rightarrow y.
\]

\textbf{Proposition 2.26} \([11, 15]\). In any bounded pseudo-BCK algebra, for all \(x \in A\),
\[
0 \lor x = x = 0 \cup x \quad \text{and} \quad x \lor 0 = (x^-)^-, \quad x \lor 0 = (x^-)^-.
\]

\textbf{Proposition 2.27.} In any bounded pseudo-BCK algebra \(A\) the following properties hold for all \(x, y \in A\):
(1) \(1 \lor x = x \lor 1 = 1 \cup x = x \cup 1;\)
(2) \(x \leq y \text{ implies } x \lor y = y \text{ and } x \lor y = y;\)
(3) \(x \lor x = x \cup x = x.\)

\textbf{Proof.} (1): We have \(1 \lor x = (1 \rightarrow x) \bowtie x = 1 \lor 1 = (x \rightarrow 1) \bowtie 1 = 1\), so \(1 \lor x = x \lor 1 = 1\). Similarly, \(1 \lor x = x \cup 1 = 1\).
(2): \(x \lor y = (x \rightarrow y) \bowtie y = 1 \bowtie y = y\). Similarly, \(x \lor y = y.\)
(3): By definition of $\vee$ and $\cup$. \hfill $\Box$

**Proposition 2.28.** In any pseudo-BCK algebra, for all $x, y \in A$,

$$x \vee y \rightarrow y = x \rightarrow y \quad \text{and} \quad x \cup y \rightarrow y = x \rightarrow y.$$  

**Proof.** It is a consequence of property $(c_9)$. \hfill $\Box$

**Lemma 2.29.** In any pseudo-BCK algebra $A$:

1. $x \vee (y \vee x)$ is a upper bound of $\{x, y\}$;
2. $x \cup (y \cup x)$ is a upper bound of $\{x, y\}$.

**Proof.** (1): By $(A_2)$ we have $x \leq (x \rightarrow y) \Rightarrow y$. Since, by $(c_6)$, $y \leq (x \rightarrow y) \Rightarrow y$, we conclude that $x, y \leq x \vee y$. Similarly we get $x, y \leq y \vee x$.

(2): Similarly to (1). \hfill $\Box$

**Definition 2.30 ([11], [15]).** Let $A$ be a pseudo-BCK algebra.

(i) If $x \vee y = y \vee x$ for all $x, y \in A$, then $A$ is called $\vee$-commutative.

(i') If $x \cup y = y \cup x$ for all $x, y \in A$, then $A$ is called $\cup$-commutative.

**Lemma 2.31 ([11], [15]).** If $A$ is a pseudo-BCK algebra, then

(i) $A$ is $\vee$-commutative iff it is a semilattice with respect to $\vee$ (under $\leq$);

(i') $A$ is $\cup$-commutative iff it is a semilattice with respect to $\cup$ (under $\leq$).

**Definition 2.32 ([11], [15]).** A pseudo-BCK algebra is called sup-commutative if it is both $\vee$-commutative and $\cup$-commutative.

**Theorem 2.33 ([11], [15]).** A pseudo-BCK algebra is sup-commutative iff it is a semilattice with respect to both $\vee$ and $\cup$.

**Theorem 2.34 ([12]).** Let $A = (A, \leq, \rightarrow, \leadsto, 1)$ be a sup-commutative pseudo-BCK$(pP)$ algebra. Then

1. $(A, \leq)$ is a lattice, where

$$x \vee y = (y \rightarrow x) \leadsto x = (y \leadsto x) \rightarrow x,$$

$$x \wedge y = ([x \rightarrow (x \odot y)] \lor [y \rightarrow (x \odot y)]) \leadsto (x \odot y)$$

$$= ([x \leadsto (x \odot y)] \lor [y \leadsto (x \odot y)]) \rightarrow (x \odot y)$$

for any $x, y \in A$.

2. $(x \rightarrow y) \lor (y \rightarrow x) = 1 = (x \leadsto y) \lor (y \leadsto x)$ for any $x, y \in A$.

**Corollary 2.35 ([12]).** A bounded sup-commutative pseudo-BCK algebra $A$ is with $(pDN)$ condition (and hence it is with $(pP)$ condition).

In a bounded sup-commutative pseudo-BCK algebra $A$ for all $x, y \in A$ define (see [11], [15]) $x \wedge y = (x^{-} \lor y^{-})^{\sim}$, $x \cap y = (x^{-} \cup y^{-})^{\sim}$. 
A bounded sup-commutative pseudo-BCK algebra is a lattice with respect to both \( \lor, \land \) and \( \cup, \cap \) (under \( \leq \)) and for all \( x, y \) we have
\[
x \lor y = x \cup y, \quad x \land y = x \cap y.
\]

Theorem 2.36 ([12]). Let \( \mathcal{A} = (A, \leq, \rightarrow, \sim, 0, 1) \) be a bounded sup-commutative pseudo-BCK algebra. Define \( \Phi_0(\mathcal{A}) = (A, \odot, \oplus, -, \sim, 0, 1) \) by
\[
x^- = x \rightarrow 0, \quad x^\sim = x \sim 0,
\]
\[
x \odot y = (x \rightarrow y^-)^\sim = (y \sim x^-^-),
\]
\[
y \oplus x = (x^- \odot y^-)^\sim = (x^- \sim y^-^-).
\]
Then \( \Phi_0(\mathcal{A}) \) is a pseudo-MV algebra.

(2) Conversely, let \( \mathcal{A} = (A, \odot, \oplus, -, \sim, 0, 1) \) be a pseudo-MV algebra. Define \( \Psi_0(\mathcal{A}) = (A, \leq, \rightarrow, \sim, 0, 1) \) by
\[
x \leq y \text{ iff } x \oplus y^- = 1 \text{ iff } y^- \oplus x = 1,
\]
\[
x \rightarrow y = y \odot x^- = (x \odot y^-^-),
\]
\[
x \sim y = x^- \oplus y = (y^- \odot x)^\sim.
\]
Then \( \Psi_0(\mathcal{A}) \) is a bounded sup-commutative pseudo-BCK algebra.

(3) The maps \( \Phi_0 \) and \( \Psi_0 \) are mutually inverse.

Corollary 2.38 ([12]). Bounded sup-commutative pseudo-BCK algebras are categorically isomorphic with pseudo-MV algebras.

Definition 2.39. Let \( \mathcal{A} \) be a pseudo-BCK algebra. A subset \( D \subseteq A \) is a deductive system of \( \mathcal{A} \) if it satisfies the conditions
\[
(DS_1) \ 1 \in D;
\]
\[
(DS_2) \text{ for all } x, y \in A, \text{ if } x, x \rightarrow y \in D, \text{ then } y \in D.
\]
Condition (DS2) is equivalent to the condition (DS₂) for all \( x, y \in A \), if \( x, x \sim y \in D \), then \( y \in D \).

A deductive system \( D \) of a pseudo-BCK algebra \( \mathcal{A} \) is called proper if \( D \neq A \).

Definition 2.40. A deductive system \( D \) of a pseudo-BCK algebra \( \mathcal{A} \) is called normal if it satisfies the condition
\[
(DS_3) \text{ for all } x, y \in A, \ x \rightarrow y \in D \text{ iff } x \sim y \in D.
\]

Definition 2.41. A deductive system is called maximal if it is proper and not strictly contained in any other deductive system.
Examples 2.42. Consider the pseudo-BCK lattice $\mathcal{A}$ from Example 2.7.
(1) The deductive systems of $\mathcal{A}$ are $D_1 = \{a, c, 1\}$, $D_2 = \{b, c, 1\}$, $D_3 = \{c, 1\}$ and $D_4 = \{1\}$.
(2) $D_1$ and $D_2$ are maximal deductive systems.
(3) $D_3$ is a normal deductive system.
(4) $D_1$ and $D_2$ are not normal deductive systems ($c \rightarrow a = a \in D_1$, while $c \rightleftharpoons a = b \notin D_1$ and $a \rightleftharpoons 0 = b \in D_2$, while $a \rightarrow 0 = 0 \notin D_2$).

Example 2.43. In the pseudo-BCK lattice $\mathcal{A}$ from Example 2.23, $D = \{a, b, c, d, 1\}$ is a maximal normal deductive system.

3. STATES ON PSEUDO-BCK ALGEBRAS

Similarly by to [10], we introduce the notion of a Bosbach state on a bounded pseudo-BCK algebra $\mathcal{A}$.

Definition 3.1. A Bosbach state on a bounded pseudo-BCK algebra $\mathcal{A}$ is a function $s : \mathcal{A} \rightarrow [0, 1]$ such that, for any $x, y \in \mathcal{A}$,

(B1) $s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x)$;
(B2) $s(x) + s(x \rightleftharpoons y) = s(y) + s(y \rightleftharpoons x)$;
(B3) $s(0) = 0$ and $s(1) = 1$.

Example 3.2. Consider the bounded pseudo-BCK lattice $\mathcal{A}$ from Example 2.23. The function $s : \mathcal{A} \rightarrow [0, 1]$ defined by $s(0) = 0$, $s(a) = 1$, $s(b) = 1$, $s(c) = 1$, $s(d) = 1$, $s(1) = 1$, is the unique Bosbach state on $\mathcal{A}$.

However, there are bounded pseudo-BCK algebras, that have no Bosbach state on them.

Example 3.3. Consider the bounded pseudo-BCK lattice $\mathcal{A}$ from Example 2.7. Let us prove that $\mathcal{A}$ has no Bosbach states on it. Indeed, assume that $\mathcal{A}$ admits a Bosbach state $s$ such that $s(0) = 0$, $s(a) = \alpha$, $s(b) = \beta$, $s(c) = \gamma$, $s(1) = 1$. From $s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x)$, taking $x = a$, $y = 0$, $x = b$, $y = 0$ and then $x = c$, $y = 0$ we get $\alpha = 1$, $\beta = 0$, $\gamma = 1$. On the other hand, taking $x = b$, $y = 0$ in $s(x) + s(x \rightleftharpoons y) = s(y) + s(y \rightleftharpoons x)$, we get $\beta + 0 = 0 + 1$, so $0 = 1$, which is a contradiction. Hence $\mathcal{A}$ does not admit a Bosbach state.

Proposition 3.4. Let $\mathcal{A}$ be a bounded pseudo-BCK algebra and $s$ a Bosbach state on it. Then for all $x, y \in \mathcal{A}$ the following properties hold:

(1) $y \leq x$ implies $s(y) \leq s(x)$ and $s(x \rightarrow y) = s(x \rightleftharpoons y) = 1 - s(x) + s(y)$;
(2) $s(x \rightarrow y) = 1 - s(x \vee y) + s(y)$ and $s(x \rightleftharpoons y) = 1 - s(x \uparrow y) + s(y)$;
(3) $s(x \vee y) = s(y \vee x)$ and $s(x \rightleftharpoons y) = s(y \rightleftharpoons x)$;
(4) $s(x^-) = s(x^-) = 1 - s(x)$;
(5) $s(x^-) = s(x^-) = s(x^-) = s(x^-) = s(x)$;
Proposition 2.28, from which is proved using Proposition 3.4(1), so (b) implies (c).

Proof. (1): By (B₁) and (A₆) we have \( s(x) + s(x \to y) = s(y) + 1 \), so

\[
s(x \to y) = 1 - s(x) + s(y).
\]

Similarly, by (B₂) and (A₆) we have \( s(x \vartriangleright y) = 1 - s(x) + s(y) \). Thus, \( s(x \to y) = s(x \vartriangleright y) = 1 - s(x) + s(y) \). Since \( s(y) - s(x) = s(x \to y) - 1 \leq 0 \), we have \( s(y) \leq s(x) \).

(2): Since \( y \leq x \lor y \), using (1), we get \( s(x \lor y \to y) = 1 - s(x \lor y) + s(y) \). Hence, by Proposition 2.28, we have \( s(x \to y) = 1 - s(x \lor y) + s(y) \). Similarly, from \( y \leq x \lor y \) and (1), we get \( s(x \lor y \to y) = 1 - s(x \lor y) + s(y) \). By Proposition 2.28, \( s(x \to y) = 1 - s(x \lor y) + s(y) \).

(3): From the identities \( s(x \to y) = 1 - s(x \lor y) + s(y) \) and \( s(y \to x) = 1 - s(y \lor x) + s(x) \) we get \( s(x \lor y) - s(y \lor x) = s(x \to y) - s(x \to y) - s(x) + s(y) = 0 \) (in the later equation we used (B₁)). Thus, \( s(x \lor y) = s(y \lor x) \) and, similarly, \( s(x \lor y) = s(y \lor x) \).

(4): Take \( y = 0 \) in (B₁) and (B₂).

(5): It follows from (4).

(6): It follows by (c₂₆).

(7): It follows by (c₂₇). \( \square \)

Remark 3.5 ([18]). In the case of a pseudo-BCK semilattice, properties (2), (6) and (7) become

(2′) \( s(x \to y) = s(x \vartriangleright y) = 1 - s(x \lor y) + s(y) \);

(6′) \( s(x \to y) = s(y \leftrightarrow x) = s(x \leftrightarrow y) \) and \( s(x \vartriangleright y) = s(y \leftrightarrow x) = s(x \leftrightarrow y) \) = \( s(x \leftrightarrow y) \);

(7′) \( s(x \leftrightarrow y) = s(y \leftrightarrow x) = s(x \leftrightarrow y) = s(y \leftrightarrow x) \).

Proposition 3.6. Let \( A \) be a bounded pseudo-BCK algebra and a function \( s : A \to [0,1] \) such that \( s(0) = 0 \). Then the following properties are equivalent:

(a) \( s \) is a Bosbach state on \( A \);

(b) \( y \leq x \) implies \( s(x \to y) = s(x \vartriangleright y) = 1 - s(x) + s(y) \) for all \( x, y \in A \);

(c) \( s(x \to y) = 1 - s(x \lor y) + s(y) \) and \( s(x \vartriangleright y) = 1 - s(x \lor y) + s(y) \) for all \( x, y \in A \).

Proof. (a) implies (b): This is proved in Proposition 3.4(1).

(b) implies (c): Assertion (c) is in fact assertion (2) of Proposition 3.4 which is proved using Proposition 3.4(1), so (b) implies (c).
(c) implies (a): From (c) and Proposition 3.4(3) we have \( s(x) + s(x \to y) = s(x) + 1 - s(x \vee y) + s(y) = 1 - s(y \vee x) + s(x) + s(y) = s(y) + s(y \to x) \).
Similarly, \( s(x) + s(x \preceq y) = s(x) + 1 - s(x \cup y) + s(y) = 1 - s(y \cup x) + s(x) + s(y) = s(y) + s(y \preceq x) \). Moreover, by (c) and Proposition 2.27(3) we have \( s(1) = s(x \to x) = 1 - s(x) + s(x) = 1 \). Thus, \( s \) is a Bosbach state on \( A \).

Consider the bounded sup-commutative BCK(P) algebra (i.e. the MV algebra) \( A_L = ([0,1], \leq, \to_L, 0, 1) \), where \( \to_L \) is the Lukasiewicz implication \( x \to_L y = \min\{1 - x + y, 1\} \).

**Definition 3.7.** Let \( A \) be a bounded pseudo-BCK algebra. A state-morphism on \( A \) is a function \( m : A \to [0,1] \) such that
\[
\begin{align*}
(SM_1) & \quad m(0) = 0; \\
(SM_2) & \quad m(x \to y) = m(x \preceq y) = m(x) \to_L m(y).
\end{align*}
\]

**Proposition 3.8.** A state-morphism on a bounded pseudo-BCK algebra \( A \) is a Bosbach state on \( A \).

**Proof.** It is obvious that \( m(1) = m(x \to x) = m(x) \to_L m(x) = 1 \). We also have
\[
m(x) + m(x \to y) = m(x) + m(x) \to_L m(y) = m(x) + \min\{1 - m(x) + m(y), 1\} = \min\{1 + m(y), 1 + m(x)\} = m(y) + \min\{1 - m(y) + m(x), 1\} = m(y) + m(y) \to_L m(x) = m(y) + m(y \to x).
\]
Similarly, \( m(x) + m(x \preceq y) = m(y) + m(y \preceq x) \). Thus, \( s \) is a Bosbach state on \( A \).

**Proposition 3.9.** Let \( A \) be a bounded pseudo-BCK algebra. A Bosbach state \( m \) on \( A \) is a state-morphism if and only if
\[
m(x \vee y) = m(x \cup y) = \max\{m(x), m(y)\}
\]
for all \( x, y \in A \).

**Proof.** If \( m \) is a state-morphism on \( A \), then by Proposition 3.8 \( m \) is a Bosbach state. According to Proposition 3.4(2) we have
\[
m(x \vee y) = 1 + m(y) - m(x \to y) = 1 + m(y) - (m(x) \to_L m(y)) = 1 + m(y) - \min\{1 - m(x) + m(y), 1\} = 1 + m(y) + \max\{-1 + m(x) - m(y), -1\} = \max\{m(x), m(y)\}.
\]
Similarly,
\[
m(x \cup y) = 1 + m(y) - m(x \preceq y) = 1 + m(y) - (m(x) \to_L m(y)) = 1 + m(y) - \min\{1 - m(x) + m(y), 1\} = 1 + m(y) + \max\{-1 + m(x) - m(y), -1\} = \max\{m(x), m(y)\}.
\]
For the converse, assume that \( m \) is a Bosbach state on \( A \) such that
\[
m(x \vee y) = m(x \cup y) = \max\{m(x), m(y)\}
\]
for all \( x, y \in A \).

Then, by Proposition 3.4(2), we have
\[
m(x \rightarrow y) = 1 - m(x \vee y) + m(y) = 1 + m(y) - \max\{(m(x), m(y)) = \\
= 1 + m(y) + \min\{-m(x), -m(y)\} = \min\{1 - m(x) + m(y), 1\} = m(x) \rightarrow_L m(y).
\]
Similarly,
\[
m(x \leadsto y) = 1 - m(x \cup y) + m(y) = 1 + m(y) - \max\{(m(x), m(y)) = \\
= 1 + m(y) + \min\{-m(x), -m(y)\} = \min\{1 - m(x) + m(y), 1\} = m(x) \rightarrow_L m(y).
\]
Thus, \( m \) is a state-morphism on \( A \).

\[\square\]

**Example 3.10.** Consider \( A = \{0, a, b, c, 1\} \) with \( 0 < a < b, c < 1 \) and \( b, c \) incomparable. Assume the operations \( \rightarrow, \leadsto \) given by the tables

\[
\begin{array}{c|cccc}
\rightarrow & 0 & a & b & c & 1 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 \\
a & 0 & 1 & 1 & 1 & 1 \\
b & 0 & a & 1 & c & 1 \\
c & 0 & b & b & 1 & 1 \\
1 & 0 & a & b & c & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\leadsto & 0 & a & b & c & 1 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 \\
a & 0 & 1 & 1 & 1 & 1 \\
b & 0 & c & 1 & c & 1 \\
c & 0 & a & b & 1 & 1 \\
1 & 0 & a & b & c & 1 \\
\end{array}
\]

Then \( A = (A, \leq, \rightarrow, \leadsto, 0, 1) \) is a bounded pseudo-BCK lattice. Since \( (A, \leq) \) is a lattice, \( A \) is a pseudo-BCK lattice. Moreover, we can see that \( c \odot b = \min\{z \mid c \leq b \rightarrow z\} = \min\{b, c, 1\} \) does not exist. Hence, \( A \) is a pseudo-BCK lattice without \((pP)\) condition and the function \( m : A \rightarrow [0, 1] \) defined by
\[
m(0) = 0, \quad m(a) = 1, \quad m(b) = 1, \quad m(c) = 1, \quad m(d) = 1, \quad m(1) = 1
\]
is the unique state-morphism on \( A \). Moreover, \( m(x \vee y) = m(x \cup y) = \max\{m(x), m(y)\} \) for all \( x, y \in A \), hence \( m \) also is a Bosbach state on \( A \).

Similarly to [10], for the case of good pseudo-BL algebras, we define the notion below.

**Definition 3.11.** Let \( A \) be a good bounded pseudo-BCK algebra. Two elements \( x, y \in A \) are called **orthogonal**, and we write \( x \perp y \), iff \( x \sim y \subseteq y \sim \). If \( x, y \in A \) are orthogonal, we define a partial operation \( + \) on \( A \) by \( x + y := x \oplus y \).

**Lemma 3.12.** Let \( A \) be a good bounded pseudo-BCK algebra and \( x, y \in A \).

Then
1. \( x \perp y \iff y \sim \leq x \sim \);  
2. \( x \perp x \sim \) and \( x + x \sim = 1 \);  
3. \( x \sim \perp x \) and \( x \sim + x = 1 \);  
4. \( x \perp 0 \) and \( x + 0 = x \sim \).
One can easily check that

\begin{align*}
(5) & \ 0 \perp x \text{ and } 0 + x = x^-; \\
(6) & \text{if } x \leq y, \text{ then } x \perp y, \ y^- \perp x \text{ and } x + y^- = y \rightarrow x^- , \ y^- + x = y \rightsquigarrow x^- .
\end{align*}

Proof. (1): $x^- \leq y^- \iff y^- \leq x^- \rightsquigarrow = x^-.$
(2): Since $x^- \leq x^- \rightsquigarrow = (x^-)^\sim$, we have $x \perp x$ and $x + x^- = x^- \rightarrow x^- = 1.$
(3): Similarly, from $x^- \leq x^- \rightsquigarrow = (x^-)^\sim$ we get $x^- \perp x$ and $x^- + x = x^- \rightarrow x^- \rightsquigarrow = x^- \rightarrow x^- = 1.$
(4): Since $x^- \leq 1 = 0^\sim$, we have $x \perp 0$ and $x + 0 = 0^- \rightarrow x^- \rightsquigarrow = 1 \rightarrow x^- \rightsquigarrow = x^- .$
(5): Since $x^- \leq 1 = 0^-$, we have $0 \perp x$ and $0 + x = x^- \rightarrow 0^- \rightsquigarrow = x^- \rightarrow 0 = x^- .$
(6): Since $x \leq y$, we have $y^- \leq x^-$, that is, $(y^-)^\sim \leq x^-$, so $x \perp y^-$. Moreover, $x + y^- = y^- \rightarrow x^- = y \rightarrow x^- \rightsquigarrow$ (by (c26)). Similarly, we have $y^- \leq x^-$, so $(y^-)^\sim \leq x^-$, that is, $y^- \perp x$ and $y^- + x = x^- \rightsquigarrow = x^- \rightarrow y^- = y \rightsquigarrow x^- \rightsquigarrow$ (by (c26)). □

Definition 3.13. Let $\mathcal{A}$ be a good bounded pseudo-BCK algebra. A Riečan state on $\mathcal{A}$ is a function $s : \mathcal{A} \rightarrow [0, 1]$ such that, for all $x, y \in \mathcal{A}$,
(R1) if $x \perp y$, then $s(x + y) = s(x) + s(y)$;
(R2) $s(1) = 1$.

Example 3.14. Consider again the good bounded pseudo-BCK algebra $\mathcal{A}$ from Example 2.23. We claim that the Bosbach state $s : \mathcal{A} \rightarrow [0, 1]$ defined by
$s(0) = 0, \ s(a) = 1, \ s(b) = 1, \ s(c) = 1, \ s(d) = 1, \ s(1) = 1$
also is a Riečan state on $\mathcal{A}$. Indeed, the pairs $(x, y)$ of orthogonal elements of $\mathcal{A}$ are given in the table

<p>| | | | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$y$</td>
<td>$x^-$</td>
<td>$y^- x + y$</td>
</tr>
<tr>
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</table>

One can easily check that $s$ is a Riečan state.
Proposition 3.15. If \( s \) is a Riečan state on a good bounded pseudo-BCK algebra \( A \), then for all \( x, y \in A \) the following properties hold:

1. \( s(x^\bot) = s(x^\bot) = 1 - s(x) \);
2. \( s(0) = 0 \);
3. \( s(x^\bot) = s(x^\bot) = s(x^\bot) = s(x) \);
4. if \( x \leq y \), then \( s(x) \leq s(y) \) and \( s(y \rightarrow x^\bot) = s(y \rightarrow x^\bot) = 1 + s(x) - s(y) \);
5. \( s((x \vee y) \rightarrow x^\bot) = s((x \vee y) \rightarrow x^\bot) = 1 - s(x \vee y) + s(x) \) and \( s((x \wedge y) \rightarrow x^\bot) = s((x \wedge y) \rightarrow x^\bot) = 1 - s(x \wedge y) + s(x) \);
6. \( s((x \vee y) \rightarrow y^\bot) = s((x \vee y) \rightarrow y^\bot) = 1 - s(x \vee y) + s(y) \) and \( s((x \wedge y) \rightarrow y^\bot) = s((x \wedge y) \rightarrow y^\bot) = 1 - s(x \wedge y) + s(y) \).

Proof. (1): Since \( x \perp x^\bot \) and \( x + x^\bot = 1 \), we have \( s(x) + s(x^\bot) = s(1) = 1 \), so \( s(x^\bot) = 1 - s(x) \). Similarly, \( s(x^\bot) = 1 - s(x) \).

(2): It follows from the fact that \( 0 \perp 0 \) and \( 0 + 0 = 0 \).

(3): Using the fact that \( x \perp 0 \) and \( x + 0 = x^\bot \), we get \( s(x) = s(x^\bot) \) and, similarly, the other equalities.

(4): Since \( x \leq y \), we have \( x \perp y^\bot \) and \( x + y^\bot = y \rightarrow x^\bot \). Hence \( s(x) + s(y^\bot) = s(y \rightarrow x^\bot) \), so \( s(x) - s(y) = s(y \rightarrow x^\bot) = 1 \), that is, \( s(x) \leq s(y) \). We get \( s(y \rightarrow x^\bot) = 1 + s(y) - s(x) \). Similarly, from \( x \leq y \) we have \( y^\bot \perp x \) and \( y^\bot + x = y \rightarrow x^\bot \), and we get \( s(y \rightarrow x^\bot) = 1 + s(x) - s(y) \).

(5): We use Lemma 3.12. It follows from \( x \leq x \vee y \), that \( x \perp (x \vee y)^\bot \) and \( x + (x \vee y)^\bot = (x \vee y) \rightarrow x^\bot \). Hence \( s(x + (x \vee y)^\bot) = s((x \vee y) \rightarrow x^\bot) \). Thus, \( s((x \vee y) \rightarrow x^\bot) = 1 - s(x \vee y) + s(x) \). Similarly, from \( x \leq x \vee y \) we get \( (x \vee y)^\bot \perp x \) and \( (x \vee y)^\bot + x = (x \vee y) \rightarrow x^\bot \). Hence \( s((x \vee y)^\bot + x) = s((x \vee y) \rightarrow x^\bot) \) and we get \( 1 - s(x \vee y) + s(x) = s((x \vee y) \rightarrow x^\bot) \). Thus, \( s((x \vee y) \rightarrow x^\bot) = s((x \vee y) \rightarrow x^\bot) = 1 - s(x \vee y) + s(x) \) and, similarly, \( s((x \vee y) \rightarrow y^\bot) = s((x \vee y) \rightarrow y^\bot) = 1 - s(x \vee y) + s(y) \).

(6): This follows similarly to (5). \( \square \)

Remark 3.16 ([18]). In the case of a pseudo-BCK semilattice, properties (5) and (6) become

\[ (5') \quad s(x \rightarrow y^\bot) = s(x \rightarrow y^\bot) = 1 - s(x \vee y) + s(y). \]

Theorem 3.17. Let \( A \) be a good bounded pseudo-BCK algebra. Any Bosbach state on \( A \) is a Riečan state.

Proof. Let \( s \) be a Bosbach state on \( A \). Assume \( x \perp y \), i.e. \( x^\bot \leq y^\bot \). By Proposition 3.4(4) and (1) we have \( 1 + s(x^\bot) = s(y^\bot) + s(y^\rightarrow x^\bot) \).

It follows that \( 1 + s(x) = 1 - s(y) + s(y^\rightarrow x^\bot) \), hence \( s(y^\rightarrow x^\bot) = s(x) + s(y) \). Therefore, \( s(x + y) = s(x) + s(y) \). Since by hypothesis \( s(1) = 1 \), \( s \) is a Riečan state on \( A \). \( \square \)
In the next example we show that there exists a Riečan state which is not a Bosbach state.

**Example 3.18.** Consider $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$ and the operations $\to, \Rightarrow$ given by the tables

<table>
<thead>
<tr>
<th>$\to$</th>
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<table>
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<tr>
<th>$\Rightarrow$</th>
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<th>c</th>
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Then $(A, \leq, \to, \Rightarrow)$ is a good pseudo-BCK lattice. (Moreover, since $(A, \leq, \to, \Rightarrow)$ is a chain, it also is a good pseudo-BCK(pP) lattice, that is, a good residuated lattice). The function $s : A \to [0, 1]$ defined by

- $s(0) = 0$,  $s(a) = 1/2$,  $s(b) = 1/2$,  $s(c) = 1$,  $s(1) = 1$

is a Riečan state. Indeed, the orthogonal elements of $A$ are the pairs $(x, y)$ in the table

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$x \sim y$</th>
<th>$y \sim x$</th>
<th>$x + y$</th>
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</table>

One can easily check that $s$ is a Riečan state, but the function $s$ defined above is not a Bosbach state. Indeed, checking condition (B$_2$) we obtain

- $s(a) + s(a \Rightarrow b) = s(a) + s(1) = 1/2 + 1 = 3/2$,
- $s(b) + s(b \Rightarrow a) = s(b) + s(b) = 1/2 + 1/2 = 1$,

so condition (B$_2$) does not hold. We conclude that $s$ is not a Bosbach state. Moreover, the pseudo-BCK algebra $A$ has no Bosbach state on it. Indeed, assume that $A$ admits a Bosbach state $s$ such that $s(0) = 0$, $s(a) = \alpha$, $s(b) = \beta$, $s(c) = 1$, $s(1) = 1$.
\(s(c) = \gamma, s(1) = 1\). From \(s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x)\), taking \(x = a, y = 0, x = b, y = 0\) and then \(x = c, y = 0\), we get \(\alpha = 1/2, \beta = 1/2, \gamma = 1\). But, we already proved that \(s\) is not a Bosbach state.

Remarks 3.19. (1). In the case of good pseudo-BL algebras, Georgescu [10] asked to find an example of Riečan state which is not a Bosbach state. The above example gives a positive answer to this problem in the case of good pseudo-BCK algebras.

(2) In ([9], Theorem 3.8) it is proved that in the case of good bounded non-commutative \(R\ell\)-monoids, the Bosbach and Riečan states coincide. The proof is based on the fact that \((x \rightarrow y)^{\sim} = x^{\sim} \rightarrow y^{\sim}\) and \((x \rightsquigarrow y)^{\sim} = x^{\sim} \rightsquigarrow y^{\sim}\) in any bounded non-commutative \(R\ell\)-monoid (see [9], Lemma 2.1). But these properties are not satisfied in any good pseudo-BCK algebra. Indeed, consider again the good pseudo-BCK\((pP)\) lattice from Example 3.18. We have \((b \rightarrow a)^{\sim} = 0\), but \(b^{\sim} \rightarrow a^{\sim} = 1\). Therefore, the above mentioned result for good bounded non-commutative \(R\ell\)-monoids is not valid in the case of good pseudo-BCK algebras, as we can see in the above example.

(3) Dvurečenskij [6] proved that every linearly ordered pseudo-BL algebra admits a Bosbach state. Since the pseudo-BCK algebra \(A\) from the above example is linearly ordered, we conclude that in contrast to pseudo-BL algebras, there exist linearly ordered pseudo-BCK algebras having no Bosbach state.

(4) In the case of pseudo-BL algebras Georgescu [10] proved that the existence of a state-morphism is equivalent to the existence of a maximal filter which is normal. In the case of pseudo-BCK algebras, this is not true. Indeed, in the pseudo-BCK algebra \(A\) from Example 3.18, \(D = \{c, 1\}\) is a maximal normal deductive system, but we proved that there is no Bosbach state on \(A\). Thus, there is no state-morphism on \(A\).

Theorem 3.20. Every Riečan state on a good pseudo-BCK\((pDN)\) algebra is a Bosbach state.

Proof. Let \(s\) be a Riečan state on a good pseudo-BCK\((pDN)\) algebra \(A\). By Proposition 3.15(2) we have \(s(0) = 0\). Since \(y^{\sim} = y\) for all \(y \in A\), by Proposition 3.15(6) we get \(s(x \lor y \rightarrow y) = 1 - s(x \lor y) + s(y)\) and \(s(x \lor y \rightarrow y) = 1 - s(x \lor y) + s(y)\). By Proposition 2.28, we have \(s(x \rightarrow y) = 1 - s(x \lor y) + s(y)\) and \(s(x \rightsquigarrow y) = 1 - s(x \lor y) + s(y)\). Finally, by Proposition 3.6, \(s\) is a Bosbach state on \(A\).

Theorem 3.21. If \(A\) is a good pseudo-BCK algebra satisfying the identities

\[(x \rightarrow y)^{\sim} = x \lor y \rightarrow y^{\sim}\quad \text{and} \quad (x \rightsquigarrow y)^{\sim} = x \lor y \rightsquigarrow y^{\sim},\]

then the Bosbach and Riečan states coincide.
Proof. Let \( s \) be a Riečan state on a good pseudo-BCK algebra \( A \). By Proposition 3.15(2) we have \( s(0) = 0 \). By Proposition 3.15(3), (5), (6) we get
\[
s(x \rightarrow y) = s((x \rightarrow y)^\sim) = s(x \vee y \rightarrow y^\sim) = 1 - s(x \vee y) + s(y)
\]
and
\[
s(x \rightsquigarrow y) = s((x \rightsquigarrow y)^\sim) = s(x \vee y \rightsquigarrow y^\sim) = 1 - s(x \vee y) + s(y).
\]
Thus, by Proposition 3.6, \( s \) is a Bosbach state on \( A \). \( \square \)

Example 3.22. Consider again the good bounded pseudo-BCK lattice \( A \) from Example 2.23. Since \( x^\sim = 1 \) and \( x \rightarrow y^\sim = 1 \) for all \( x, y \in A \), \( A \) satisfies the identities from Theorem 3.21. Hence the Bosbach and Riečan states coincide, which is in accordance with Example 3.14.

Remark 3.23. In the case of a good pseudo-BCK semilattice it was proved in [18] that \( x \vee y \rightarrow y^\sim = x \rightarrow y^\sim \) and \( x \vee y \rightsquigarrow y^\sim = x \rightsquigarrow y^\sim \) for all \( x, y \in A \). Hence the identities from Theorem 3.21 become
\[
(x \rightarrow y)^\sim = x \rightarrow y^\sim, \quad (x \rightsquigarrow y)^\sim = x \rightsquigarrow y^\sim.
\]
Thus, Theorem 3.21 becomes Theorem 6.11 from [18].

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