BRIOT-BOUQUET STRONG DIFFERENTIAL SUPERORDINATIONS AND SANDWICH THEOREMS

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The notion of differential superordination was introduced by Miller and Mocanu [3] as a dual concept of differential subordination [2] and was developed in [4]. The notion of strong differential subordination was introduced by Antonino and Romaguera [1]. The notion was developed in [8], [9], [10]. In [5] the author introduced the dual concept of strong differential superordinations. In this paper, a Briot-Bouquet strong differential superordination is studied.

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1. INTRODUCTION AND PRELIMINARIES

Let the unit disc of the complex plane

$$U = \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad \overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

Let $\mathcal{H}(U \times \overline{U})$ denote the space of holomorphic functions in $U \times \overline{U}$. For n a positive integer and $a \in \mathbb{C}$, in [7] the authors introduced the classes

$$\mathcal{H}^*[a, n, \xi] = \{f \in \mathcal{H}(U \times \overline{U}) \mid$$
$$f(z, \xi) = a + a_n(\xi)z^n + a_{n+1}(\xi)z^{n+1} + \dots,\ z \in U,\ \xi \in \overline{U}\},$$

with $a_k(\xi)$ holomorphic functions in $\overline{U}$, $k \geq n$, and

$$\mathcal{H}_u(U) = \{f \in \mathcal{H}^*[a, n, \xi] : f(\cdot, \xi) \text{ univalent in } U \text{ for all } \xi \in \overline{U}\},$$

$$K = \left\{f \in \mathcal{H}^*[a, n, \xi] : \operatorname{Re} \frac{zf''(z, \xi)}{f'(z, \xi)} + 1 > 0,\ z \in U \text{ for all } \xi \in \overline{U}\right\}$$

the class of convex functions.

Definition 1 ([6]). We denote by Q the set of function $f(\cdot, \xi)$ that are analytic and injective on the set $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z, \xi) = \infty,\ z \in U,\ \xi \in \overline{U}\right\}$$

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and are such that $f'(z,\xi) \neq 0$ for $\zeta \in \partial U \setminus E(f)$, $\xi \in \overline{U}$.

The subclass of Q for which $f(0,\xi) \equiv a$ is denoted by $Q(a)$.

Definition 2 ([7]). Let $f(z,\xi)$ and $H(z,\xi)$ be analytic in $U \times \overline{U}$. The function $f(z,\xi)$ is said to be strongly **subordinate** to $H(z,\xi)$, or $H(z,\xi)$ is said to be strongly **superordinate** to $f(z,\xi)$, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z,\xi) = H(w(z),\xi)$ for all $\xi \in \overline{U}$. In such a case we write

$$f(z,\xi) \prec\prec H(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

If $H(\cdot\,,\xi)$ is univalent in U, for all $\xi \in \overline{U}$, then $f(z,\xi) \prec\prec F(z,\xi)$ if and only if $f(0,\xi) = F(0,\xi)$ and $f(U \times \overline{U}) \subset F(U \times \overline{U})$.

Remark 1. If $H(z,\xi) \equiv H(z)$, and $f(z,\xi) \equiv f(z)$, then the strong superordination becomes the usual notion of superordination.

Let β and γ be complex numbers, let Ω_ξ and Δ_ξ be sets in the complex plane, and $p(\cdot\,,\xi)$ analytic in $U \times \overline{U}$.

In [6] the authors have determined conditions such that

$$(1) \qquad \left\{p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi)+\gamma}\right\} \subset \Omega_\xi \Rightarrow p(U \times \overline{U}) \subset \Delta_\xi, \quad z \in U,\ \xi \in \overline{U}.$$

In this article we consider the dual problem of determining conditions such that

$$(2) \qquad \Omega_\xi \subset \left\{p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi)+\gamma}\right\} \Rightarrow \Delta_\xi \subset p(U \times \overline{U}), \quad z \in U,\ \xi \in \overline{U}.$$

In particular, we are interested in determining the largest set Δ_ξ in $\mathbb{C}$ for which (2) holds.

If the sets Ω_ξ and Δ_ξ in (1) and (2) are simply connected domains not equal to $\mathbb{C}$, then it is possible to rephrase these expressions very neatly in terms of strong subordination and to obtain

$$(1') \qquad p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi)+\gamma} \prec\prec h_2(z,\xi) \Rightarrow p(z,\xi) \prec\prec q_2(z,\xi),$$

$$(2') \quad h_1(z,\xi) \prec\prec p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi)+\gamma} \Rightarrow g_1(z,\xi) \prec\prec p(z,\xi), \ z \in U,\ \xi \in \overline{U}.$$

The left side of $(1')$ is called a Briot-Bouquet strong differential subordination, and the function q_2 is called a **dominant** of the differential subordination. The **best dominant**, which provides a sharp result, is the dominant that is subordinate to all other dominant.

The left side of $(2')$ is called a Briot-Bouquet strong differential superordination, and the function $q_1(\cdot\,,\xi)$ is called a **subordinant** of the strong differential subordination. The **best subordinant**, which provides a sharp result is the subordinant which is superordinate to all other subordinants.

Definition 3 ([5]). Let Ω_ξ be a set in $\mathbb{C}$ and $q(\cdot\,,\xi) \in \mathcal{H}^*[a,n,\xi]$ with $q'(z,\xi) \neq 0$, $z \in U$, $\xi \in \overline{U}$. The class of admissible functions $\phi_n[\Omega_\xi, q(\cdot\,,\xi)]$ consists of those functions $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition:

$$(A) \qquad \varphi(r,s,t;\zeta,\xi) \in \Omega_\xi$$

whenever

$$r = q(z,\xi), \quad s = \frac{zq'(z,\xi)}{m}, \quad \operatorname{Re}\frac{t}{s} + 1 \le \frac{1}{m}\operatorname{Re}\left[\frac{zq''(z,\xi)}{q'(z,\xi)} + 1\right],$$

where $\zeta \in \partial U$, $z \in U$, $\xi \in \overline{U}$ and $m \ge n \ge 1$. When $n = 1$ we write $\phi_1[\Omega_\xi, q(\cdot\,,\xi)]$ as $\phi[\Omega_\xi, q(\cdot\,,\xi)]$.

In the special case when $h(\cdot\,,\xi)$ is an analytic mapping of $U \times \overline{U}$ onto $\Omega_\xi \neq \mathbb{C}$ we denote this class $\phi_n[h(U \times \overline{U}), q(\cdot\,,\xi)]$ by $\phi_n[h(\cdot\,,\xi), q(\cdot\,,\xi)]$.

If $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ and $q(\cdot\,,\xi) \in \mathcal{H}^*[a,n,\xi]$, then the admissibility condition (A) reduces to

$$(A') \qquad \varphi\left(q(z,\xi), \frac{zq'(z,\xi)}{m}; \zeta, \xi\right) \in \Omega_\xi$$

whenever $r = q(z,\xi)$, $s = \dfrac{zq'(z,\xi)}{m}$, where $z \in U$, $\xi \in \overline{U}$, $\zeta \in \partial U$ and $m \ge n \ge 1$.

Lemma A ([6]). *Let $p(\cdot\,,\xi) \in Q(a)$, and let*

$$q(z,\xi) = a + a_n(\xi)z^n + a_{n+1}(\xi)z^{n+1} + \dots$$

be analytic in $U \times \overline{U}$ with $q(z,\xi) \not\equiv a$ and $a \ge 1$. If $q(\cdot\,,\xi)$ is not subordinate to $p(\cdot\,,\xi)$, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(p)$, and an $m \ge n \ge 1$ for which $q(U_{r_0} \times \overline{U}_{r_0}) \subset p(U \times \overline{U})$,

(i) $q(z_0,\xi) = p(\zeta_0,\xi)$,

(ii) $z_0 q'(z_0,\xi) = m\zeta_0 p'(\zeta_0,\xi)$ *and*

(iii) $\operatorname{Re}\frac{z_0 q''(z_0,\xi)}{q'(z_0,\xi)} + 1 \ge m \operatorname{Re}\left[\frac{\zeta_0 p''(\zeta_0,\xi)}{p'(\zeta_0,\xi)} + 1\right]$.

2. MAIN RESULTS

Theorem 1. *Let $\Omega_\xi \subset \mathbb{C}$, $q(\cdot\,,\xi) \in \mathcal{H}^*[a,n,\xi]$, $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, and suppose that*

$$(3) \qquad \varphi(q(z,\xi), tzq'(z,\xi); \zeta, \xi) \in \Omega_\xi$$

for $z \in U$, $\zeta \in \partial U$, $\xi \in \overline{U}$ *and* $0 < t \le \frac{1}{n} \le 1$. *If* $p(\cdot,\xi) \in Q(a)$ *and* $\varphi(p(z,\xi), zp'(z,\xi); z,\xi)$ *is univalent in* U*, then*

$$\Omega_\xi \subset \{\varphi(p(z,\xi), zp'(z,\xi); z,\xi)\} \tag{4}$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

Proof. Assume $q(z,\xi) \not\prec p(z,\xi)$. By Lemma A, there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \setminus E(p)$, and an $m \ge n \ge 1$ that satisfy conditions (i)–(iii) of Lemma A. Using these conditions with $r = p(\zeta_0,\xi)$, $s = \zeta_0 p'(\zeta_0,\xi)$ and $\zeta = \zeta_0$ in Definition 3 we obtain

$$\varphi(p(\zeta_0,\xi), \zeta_0 p'(\zeta_0,\xi); \zeta_0,\xi) \in \Omega_\xi.$$

Since this contradicts (4) we must have $q(z,\xi) \prec\prec p(z,\xi)$, $z \in U$, $\xi \in \overline{U}$. $\square$

We next consider the special situation when $h(z,\xi)$ is analytic on $U \times \overline{U}$ and $h(U \times \overline{U}) = \Omega_\xi \ne \mathbb{C}$. Then Theorem 1 becomes

Theorem 2. *Let* $h(\cdot\,,\xi)$ *be analytic in* $U \times \overline{U}$, $q(\cdot\,,\xi) \in \mathcal{H}^*[a,n,\xi]$, $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$*, and suppose that*

$$\varphi(q(z,\xi), tzq'(z,\xi); z,\xi) \in H(U \times \overline{U}), \tag{5}$$

for $z \in U$, $\zeta \in \partial U$ *and* $0 < t \le \frac{1}{n} \le 1$. *If* $p(\cdot\,,\xi) \in Q(a)$ *and* $\varphi(p(z,\xi), zp'(z,\xi); z,\xi)$ *is univalent in* U *for all* $\xi \in \overline{U}$*, then*

$$h(z,\xi) \prec\prec \varphi(p(z,\xi), zp'(z,\xi); z,\xi), \quad z \in U,\ \xi \in \overline{U} \tag{6}$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

Furthermore, if

$$\varphi(q(z,\xi), zq'(z,\xi); z,\xi) = h(z,\xi), \quad z \in U,\ \xi \in \overline{U} \tag{7}$$

has a univalent solution $q(\cdot\,,\xi) \in Q(a)$*, then* $q(\cdot\,,\xi)$ *is the best subordinant.*

Theorem 3. *Let* $h(\cdot\,,\xi)$ *be convex in* U*, for all* $\xi \in \overline{U}$ *with* $h(0,\xi) = a$*, and let* θ *and* ψ *be analytic in a domain* $D \subset \mathbb{C}$. *Let* $p(\cdot\,,\xi) \in \mathcal{H}^*[a,1,\xi] \cap Q$ *and suppose that* $\theta[p(z,\xi)] + zp'(z,\xi)\psi[p(z,\xi)]$ *is univalent in* U *for all* $\xi \in \overline{U}$.

If the differential equation

$$\theta[q(z,\xi)] + zq'(z,\xi)\psi[q(z,\xi)] = h(z,\xi), \quad z \in U,\ \xi \in \overline{U} \tag{8}$$

has a univalent solution $q(\cdot\,,\xi)$ *that satisfies* $q(0,\zeta) = a$, $q(U \times \overline{U}) \subset D$*, and*

$$\theta[q(z,\xi)] \prec\prec h(z,\xi), \quad z \in U,\ \xi \in \overline{U}, \tag{9}$$

then

$$h(z,\xi) \prec\prec \theta[p(z,\xi)] + zp'(z,\xi)\psi[p(z,\xi)] \tag{10}$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

Function q is the best subordinant.

Proof. We can assume that $h(\cdot\,,\xi), p(\cdot\,,\xi)$ and $q(\cdot\,,\xi)$ satisfy the conditions of this theorem on the closed $\overline{U} \times \overline{U}$, and that $q'(\zeta,\xi) \neq 0$ for $|\zeta| = 1$. If not, then we can replace $h(\cdot\,,\xi), p(\cdot\,,\xi)$ and $q(\cdot\,,\xi)$ with $h(\rho z,\xi), p(\rho z,\xi)$ and $q(\rho z,\xi)$, where $0 < \rho < 1$. These new functions have the desired properties on $\overline{U} \times \overline{U}$, and we can use them in the proof of the theorem. Theorem 3 would then follow by letting $\rho \to 1$. We will use Lemma A to prove this result. If we let $\varphi(r,s) = \theta[r] + s\psi[r]$, $r = q(z,\xi)$, $s = zq'(z,\xi)$, then (8) becomes

$$\varphi(q(z,\xi), zq'(z,\xi)) = h(z,\xi),$$

and we have

$$\begin{aligned}\varphi(q(z,\xi), tzq'(z,\xi)) &= \theta[p(\zeta,\xi)] + t\zeta p'(\zeta,\xi)\psi[p(\zeta,\xi)]\\ &= (1-t)\theta[p(\zeta,\xi)] + th(\zeta,\xi), \quad 0 < t \leq 1.\end{aligned}$$

From (9) and the convexity of $h(U \times \overline{U})$ we conclude that

$$\varphi(q(z,\xi), tzp'(z,\xi)) \in h(U \times \overline{U}) \quad \text{for } 0 < t \leq 1.$$

Hence condition (5) of Theorem 2 is satisfied and the conclusions of this theorem follow. □

In the special case when $\theta[q(z,\xi)] = q(z,\xi)$ and

$$\theta[q(z,\xi)] = \frac{1}{\beta q(z,\xi) + \gamma}, \quad \beta, \gamma \in \mathbb{C},$$

we obtain the following result for the Briot-Bouquet strong differential superordination.

Corollary 1. *Let $\beta, \gamma \in \mathbb{C}$, and let $h(\cdot\,,\xi)$ be convex in U for all $\xi \in \overline{U}$, with $h(0,\xi) = a$. Suppose that the differential equation*

$$q(z,\xi) + \frac{zq'(z,\xi)}{\beta q(z,\xi) + \gamma} = h(z,\xi), \quad z \in U,\ \xi \in \overline{U} \tag{11}$$

has a univalent solution $q(\cdot\,,\xi)$ that satisfies $q(0,\xi) = a$ and $q(z,\xi) \prec\prec h(z,\xi)$, $z \in U$, $\xi \in \overline{U}$.

If $p(\cdot,\xi) \in \mathcal{H}[a,1] \cap Q$ and $p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi)+\gamma}$ is univalent in U for all $\xi \in \overline{U}$, then

$$h(z,\xi) \prec\prec p(z,\xi) + \frac{zp'(z,\xi)}{\beta p(z,\xi) + \gamma} \tag{12}$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

The function $q(\cdot\,,\xi)$ is the best subordinant.

We can combine that result with Theorem 3 and we obtain the following sandwich theorem.

THEOREM 4. *Let $h_1(\cdot\,,\xi)$ and $h_2(\cdot\,,\xi)$ be convex in $U\times\overline{U}$, for all $\xi\in\overline{U}$ with $h_1(0,\xi)=h_2(0,\xi)=a$, and let θ and ψ be analytic in a domain $D\subset\mathbb{C}$. Let $p(\cdot\,,\xi)\in\mathcal{H}[a,1,\xi]\cap Q$ and suppose that $\theta[p(z,\xi)]+zp'(z,\xi)\psi[p(z,\xi)]$ is univalent in U, for all $\xi\in\overline{U}$. If the differential equations*

$$\theta[q_i(z,\xi)]+zq_i'(z,\xi)\psi[q_i(z,\xi)]=h_i(z,\xi),$$

have univalent solutions q_i that satisfy $q_i(0,\xi)=a$, $q_i(U\times\overline{U})\subset D$, and

$$\theta[q_i(z,\xi)]\prec\prec h_i(z,\xi),$$

for $i=1,2$, then

$$h_1(z,\xi)\prec\prec\theta[p(z,\xi)]+zp'(z,\xi)\psi[p(z,\xi)]\prec\prec h_2(z,\xi)$$

implies

$$q_1(z,\xi)\prec\prec p(z,\xi)\prec\prec q_2(z,\xi),\quad z\in U,\ \xi\in\overline{U}.$$

In the special case when $\theta[p(z,\xi)]=p(z,\xi)$ and

$$\psi[p(z,\xi)]=\frac{1}{\beta p(z,\xi)+\gamma},$$

we obtain the following Briot-Bouquet sandwich result.

COROLLARY 2. *Let $\beta,\gamma\in\mathbb{C}$ and let $h_i(\cdot\,,\xi)$ be convex in U, for all $\xi\in\overline{U}$, with $h_i(0,\xi)=a$, for $i=1,2$. Suppose that the differential equations*

$$q_i(z,\xi)+\frac{zq_i'(z,\xi)}{\beta q_i(z,\xi)+\gamma}=h_i(z,\xi) \tag{13}$$

have univalent solutions $q_i(\cdot\,,\xi)$ that satisfy $q_i(0,\xi)=a$ and $q_i(z,\xi)\prec\prec h_i(z,\xi)$, for $i=1,2$, $z\in U$, $\xi\in\overline{U}$. If

$$p(\cdot\,,\xi)\in\mathcal{H}[a,1,\xi]\cap Q$$

and

$$p(z,\xi)+\frac{zp'(z,\xi)}{\beta p(z,\xi)+\gamma}\in\mathcal{H}_u(U)\quad \text{for all } \xi\in\overline{U},$$

then

$$h_1(z,\xi)\prec\prec p(z,\xi)+\frac{zp'(z,\xi)}{\beta p(z,\xi)+\gamma}\prec\prec h_2(z,\xi)$$

implies

$$q_1(z,\xi)\prec\prec p(z,\xi)\prec\prec q_2(z,\xi),\quad z\in U,\ \xi\in\overline{U}.$$

The functions $q_1(\cdot\,,\xi)$ and $q_2(\cdot\,,\xi)$ are the best subordinant and best dominant respectively.

If $\beta = 0$ and $\gamma \neq 0$ with $\mathrm{Re}\,\gamma \geq 0$, then (13) has univalent (convex) solutions given by

$$q_i(z,\xi) = \frac{\gamma}{z^\gamma}\int_0^z h_i(t,\xi)t^{\gamma-1}\mathrm{d}t, \tag{14}$$

for $i = 1, 2$. In this case we obtain the following sandwich corollary.

COROLLARY 3. *Let $h_1(\cdot\,,\xi)$ and $h_2(\cdot\,,\xi)$ be convex in U, for all $\xi \in \overline{U}$, with $h_1(0,\xi) = h_2(0,\xi) = a$. Let $\gamma \neq 0$ with* $\mathrm{Re}\,\gamma \geq 0$, *and let the functions $q_i(\cdot\,,\xi)$ be defined by* (14) *for $i = 1, 2$. If $p(\cdot\,,\xi) \in \mathcal{H}^*[a,1,\xi] \cap Q$ and $p(z,\xi) + \frac{zp'(z,\xi)}{\gamma}$ is univalent in U for all $\xi \in \overline{U}$, then*

$$h_1(z,\xi) \prec\prec p(z,\xi) + \frac{zp'(z,\xi)}{\gamma} \prec\prec h_2(z,\xi), \quad z \in U,\ \xi \in \overline{U}$$

implies

$$q_1(z,\xi) \prec\prec p(z,\xi) \prec\prec q_2(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

The functions $q_1(z,\xi)$ and $q_2(z,\xi)$ are the best subordinant and best dominant respectively.

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