A SURVEY ON EXTREME POINTS, SUPPORT POINTS AND LOEWNER CHAINS IN $\mathbb{C}^n$

IAN GRAHAM, HIDETAKA HAMADA and GABRIELA KOHR

In this paper, we survey some recent results related to extreme points, support points and reachable families of holomorphic mappings generated by the Loewner differential equation on the unit ball $B^n$ in $\mathbb{C}^n$. Certain applications and some conjectures are also considered.

AMS 2010 Subject Classification: Primary 32H02, Secondary 30C45.

Key words: extreme point, Loewner chain, Loewner differential equation, parametric representation, reachable family, support point.

1. INTRODUCTION AND PRELIMINARIES

Loewner chains in $\mathbb{C}^n$ were first studied by Pfaltzgraff [30, 31], who generalized to higher dimensions the Loewner differential equation and obtained existence and uniqueness results for the solutions on $B^n$. Subsequently, the existence and regularity results in the Loewner theory in higher dimensions were refined and many applications were given (see [7], [10–14], [18, 23, 34, 35, 42]). A new geometric approach to Loewner theory in the unit disc and complete hyperbolic complex manifolds may be found in [1, 2, 5, 6]. Recent generalizations of Loewner theory to reflexive complex Banach spaces were obtained in [15] and [16].

It is well known that every function $f \in S$ (the family of normalized univalent functions on $U$) can be embedded as the first element of a Loewner chain. In addition, $f$ has parametric representation, i.e. $f(z) = \lim_{t \to \infty} e^t v(z, t)$ locally uniformly on $U$, where $v = v(z, t)$ is the unique Lipschitz continuous solution on $[0, \infty)$ of the initial value problem

$$\frac{\partial v}{\partial t} = -vp(v, t) \quad a.e. \quad t \geq 0, \quad v(z, 0) = z,$$

for some choice of $p = p(z, t)$ such that $p(\cdot, t) \in \mathcal{P}$ (the Carathéodory family of holomorphic functions $q$ on $U$ such that $q(0) = 1$ and $\Re q(z) > 0$, $z \in U$) for almost all $t \in [0, \infty)$ and $p(z, \cdot)$ is measurable on $[0, \infty)$ for $z \in U$ (see [33]). Becker [4] obtained the general form of solutions to the Loewner differential equation on the unit disc, i.e.

MATH. REPORTS 15(65), 4 (2013), 411–423
\[
\frac{\partial f}{\partial t}(z,t) = zf'(z,t)p(z,t), \quad a.e. \quad t \geq 0, \quad \forall z \in U,
\]

where \(p(\cdot, t) \in \mathcal{P}\) for any fixed \(t \in [0, \infty)\), and \(p(z, \cdot)\) is measurable on \([0, \infty)\) for \(z \in U\). In the case \(n = 1\), there exists a unique normalized univalent solution \(f(z,t) = e^t z + \cdots\) of the above Loewner differential equation, which we call the canonical solution. However, in dimension \(n \geq 2\), the analogous uniqueness result does not hold (see [7] and [10]). Indeed, if \(f(z,t) = e^t z + \cdots\) is a Loewner chain that satisfies the Loewner differential equation

\[
\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad a.e. \quad t \geq 0, \quad \forall z \in B^n,
\]

where \(h(\cdot, t) \in \mathcal{M}\) for \(t \in [0, \infty)\) (notation will be explained in the next section), and \(h(z, \cdot)\) is measurable on \([0, \infty)\) for \(z \in B^n\), and if \(\Phi\) is a normalized biholomorphic mapping on \(\mathbb{C}^n\), then \(g(z,t) = \Phi(f(z,t))\) is another Loewner chain, which satisfies the same Loewner differential equation as \(f(z,t)\).

Recent work on the structure of solutions of the Loewner differential equation in \(\mathbb{C}^n\) appears in [7] (see also [1, 3, 11, 22] and [42]).

In this paper, we survey some recent results in the theory of Loewner chains in \(\mathbb{C}^n\). We present applications to extreme points, support points and reachable families generated by the Loewner differential equation on the unit ball in \(\mathbb{C}^n\). We mention that other extremal problems related to certain compact subsets of \(H(B^n)\) were considered in [20] and [28].

2. EXTREME POINTS AND SUPPORT POINTS FOR COMPACT SUBSETS OF \(H(B^n)\)

Let \(\mathbb{C}^n\) denote the space of \(n\) complex variables \(z = (z_1, \ldots, z_n)\) with the Euclidean inner product \(\langle z, w \rangle = \sum_{j=1}^{n} z_j \bar{w}_j\) and the Euclidean norm \(\|z\| = \langle z, z \rangle^{1/2}\). The open unit ball in \(\mathbb{C}^n\) is denoted by \(B^n\) and the closed unit ball is denoted by \(\overline{B^n}\). In the case \(n = 1\), \(B^1\) is the unit disc \(U\).

Let \(L(\mathbb{C}^n, \mathbb{C}^m)\) denote the space of linear operators from \(\mathbb{C}^n\) into \(\mathbb{C}^m\) with the standard operator norm and let \(I_n\) be the identity in \(L(\mathbb{C}^n)\), where \(L(\mathbb{C}^n) = L(\mathbb{C}^n, \mathbb{C}^n)\). Let \(H(B^n)\) be the family of holomorphic mappings from \(B^n\) into \(\mathbb{C}^n\). If \(f \in H(B^n)\), we say that \(f\) is normalized if \(f(0) = 0\) and \(Df(0) = I_n\). Let \(S(B^n)\) be the family of normalized biholomorphic mappings in \(H(B^n)\). In the case \(n = 1\), the family \(S(B^1)\) is denoted by \(S\). Let \(S^*(B^n)\) be the subfamily of \(S(B^n)\) consisting of starlike mappings on \(B^n\). Also, let \(\mathcal{LS}_n\) be the family of normalized locally biholomorphic mappings on \(B^n\).

If \(A \in L(\mathbb{C}^n)\), we denote by

\[
m(A) = \min\{\Re\langle A(z), z \rangle : \|z\| = 1\}
\]
and by \( k_+(A) \) the upper exponential (Lyapunov) index of \( A \) given by 
\[
k_+(A) = \max\{ \Re \lambda : \lambda \in \sigma(A) \},\]
where \( \sigma(A) \) is the spectrum of \( A \). It is known that 
\[
k_+(A) = \lim_{t \to \infty} \frac{\log \| e^{tA} \|}{t} \text{ (see e.g. \cite{8} and \cite{38, p. 311}).}
\]

For \( f, g \in H(B^n) \), we say that \( f \) is subordinate to \( g \) (\( f \prec g \)) if there exists a Schwarz mapping \( v \) (i.e. \( v \in H(B^n) \) and \( \| v(z) \| \leq \| z \|, z \in B^n \)) such that \( f = g \circ v \).

**Definition 2.1.** A mapping \( f : B^n \times [0, \infty) \to \mathbb{C}^n \) is called a univalent subordination chain if \( f(\cdot,t) \) is biholomorphic on \( B^n \), \( f(0,t) = 0 \) for \( t \geq 0 \), and \( f(\cdot,s) \prec f(\cdot,t) \), \( 0 \leq s \leq t < \infty \). A univalent subordination chain is said to be \( A \)-normalized if \( Df(0,t) = e^{tA} \) for \( t \geq 0 \), where \( A \in L(\mathbb{C}^n) \) with \( m(A) > 0 \). We say that \( f(z,t) \) is a Loewner chain (or a normalized univalent subordination chain) if \( f(z,t) \) is \( I_n \)-normalized.

The above subordination condition is equivalent to the existence of a unique Schwarz mapping \( v = v(z,s,t) \), called the transition mapping associated with \( f(z,t) \), such that \( f(z,s) = f(v(z,s,t),t) \) for \( z \in B^n \) and \( t \geq s \geq 0 \).

The following subset of \( H(B^n) \) plays a central role in the study of Loewner chains and the Loewner differential equation in \( \mathbb{C}^n \) (see \cite{10–12, 18, 19, 30}):
\[
\mathcal{M} = \{ h \in H(B^n) : h(0) = 0, Dh(0) = I_n, \Re \langle h(z), z \rangle > 0, z \in B^n \setminus \{0\} \}.
\]
We remark that \( \mathcal{M} \) is a compact subset of \( H(B^n) \) (see \cite{10}).

**Definition 2.2** (\cite{10, 11, 18, 34}). Let \( f \in H(B^n) \) be a normalized mapping. We say that \( f \) has parametric representation (denoted by \( f \in S^0(B^n) \)) if there exists a mapping \( h : B^n \times [0, \infty) \to \mathbb{C}^n \) such that \( h(\cdot,t) \in \mathcal{M} \) for \( t \in [0, \infty) \), \( h(z,\cdot) \) is measurable on \( [0, \infty) \) for \( z \in B^n \), and \( f(z) = \lim_{t \to \infty} e^{tA}v(z,t) \) locally uniformly on \( B^n \), where \( v = v(z,t) \) is the unique locally absolutely continuous solution on \( [0, \infty) \) of the initial value problem
\[
\frac{\partial v}{\partial t} = -h(v,t) \quad a.e. \quad t \geq 0, \quad v(z,0) = z, \quad \forall z \in B^n.
\]

**Remark 2.3.** The condition in Definition 2.2 is equivalent to the fact that there exists a Loewner chain \( f(z,t) \) such that \( \{e^{-t}f(\cdot,t)\}_{t\geq0} \) is a normal family on \( B^n \) and \( f = f(\cdot,0) \) (see \cite{19}; cf. \cite{10, 18, 35}). In the case of one complex variable, \( S^0(U) = S \) (see \cite{33}). However, \( S^0(B^n) \neq S(B^n) \) for \( n \geq 2 \). In particular, the family \( S^0(B^n) \) is compact, while \( S(B^n) \) is not compact for \( n \geq 2 \) (see \cite{19}). Also, important subclasses of \( S(B^n) \), such as the class \( S^*(B^n) \), are subclasses of \( S^0(B^n) \).

**Definition 2.4.** Let \( X \) be a locally convex linear space over \( \mathbb{C} \) and let \( E \subseteq X \).
(i) A point \( x \in E \) is called an extreme point of \( E \) provided \( x = ty + (1-t)z \), where \( t \in (0,1) \), \( y, z \in E \), implies \( x = y = z \). That is, \( x \in E \) is an extreme point of \( E \) if \( x \) is not a proper convex combination of two points in \( E \).

(ii) A point \( w \in E \) is called a support point of \( E \) if \( RL(w) = \max_{y \in E} RL(y) \) for some continuous linear functional \( L : X \rightarrow \mathbb{C} \) such that \( RL|_E \neq \text{constant} \).

Let \( \text{ex} \ E \) and \( \text{supp} \ E \) be the sets of extreme points of \( E \) and support points of \( E \), respectively. By the Krein-Milman theorem (see e.g. [21], Chapter 4), it is known that if \( E \) is a nonempty compact subset of \( X \) then \( \text{ex} \ E \) is a nonempty subset of \( E \). Also, it is known that if \( E \) is a compact subset of \( X \), which has at least two distinct points, then \( \text{supp} \ E \) is a nonempty subset of \( E \). We shall consider \( X = H(B^n) \).

Remark 2.5. It is well known that no bounded mapping in \( S \) is an extreme point or support point of \( S \). Indeed, if \( f \in \text{ex} \ S \) or \( f \in \text{supp} \ S \), then \( f \) maps the unit disc \( U \) onto the complement of a continuous arc tending to \( \infty \) with increasing modulus (see e.g. [21]).

3. THE LOEWNER VARIATION OF EXTREME POINTS AND SUPPORT POINTS

Pell [29] and Kirwan [25] proved that if \( f \in \text{ex} \ S \) (resp. \( f \in \text{supp} \ S \)) and \( f(z,t) \) is a Loewner chain such that \( f = f(\cdot,0) \), then \( e^{-t}f(\cdot,t) \in \text{ex} \ S \) (resp. \( e^{-t}f(\cdot,t) \in \text{supp} \ S \)), for all \( t \geq 0 \).

Recently, the authors [14] proved the following results related to extreme points and support points for the family \( S^0(B^n) \). These results are generalizations to \( \mathbb{C}^n \) of the above results due to Pell [29] and Kirwan [25]. Analogous results for a more restrictive class of mappings generated using the Roper-Suffridge extension operator were obtained in [20].

Theorem 3.1. Let \( f \in \text{ex} \ S^0(B^n) \) and \( f(z,t) \) be a Loewner chain such that \( f = f(\cdot,0) \) and \( \{e^{-t}f(\cdot,t)\}_{t \geq 0} \) is a normal family on \( B^n \). Then \( e^{-t}f(\cdot,t) \in \text{ex} \ S^0(B^n) \), for all \( t \geq 0 \).

Proof. We sketch some arguments, as given in the proof of ([14], Theorem 2.1). Let \( v_{s,t}(z) = v(z,s,t) \) be the transition mapping associated with \( f(z,t) \). Also, let \( v(z,t) = v_{0,t}(z) \) for \( z \in B^n \) and \( t \geq 0 \). Fix \( t \geq 0 \). Then it is not difficult to deduce that \( e^{-t}f(\cdot,t) \in S^0(B^n) \) for \( t \geq 0 \), and \( e^t g(v(\cdot,t)) \in S^0(B^n) \) for \( t \geq 0 \) and for any mapping \( g \in S^0(B^n) \). On the other hand, if \( e^{-t}f(z,t) = \lambda g(z) + (1-\lambda)h(z) \), \( z \in B^n \), where \( \lambda \in (0,1) \) and \( g, h \in S^0(B^n) \), then \( e^t g(v(\cdot,t)) \equiv e^t h(v(\cdot,t)) \), since \( f(z) = \lambda e^t g(v(z,t)) + (1-\lambda)e^t h(v(z,t)) \) and \( e^t g(v(\cdot,t)) \equiv e^t h(v(\cdot,t)) \in S^0(B^n) \). Finally, the identity theorem for holomorphic mappings yields that \( g \equiv h \). Hence, \( e^{-t}f(\cdot,t) \in \text{ex} \ S^0(B^n) \), as desired. \( \Box \)
Theorem 3.2. Let \( f \in \text{supp} S^0(B^n) \) and let \( f(z,t) \) be a Loewner chain such that \( f = f(\cdot,0) \) and \( \{e^{-t}f(\cdot,t)\}_{t \geq 0} \) is a normal family on \( B^n \). Then there exists \( t_0 > 0 \) such that \( e^{-t}f(\cdot,t) \in \text{supp} S^0(B^n) \) for \( 0 \leq t < t_0 \).

Proof. We shall give some arguments as in the proof of ([14], Theorem 2.5). We keep the same notation as in the proof of Theorem 3.1. Taking into account the fact that \( f \in \text{supp} S^0(B^n) \), there exists a continuous linear functional \( L \) on \( H(B^n) \) such that \( RL \) is nonconstant on \( S^0(B^n) \) and \( RL(f) = \max_{g \in S^0(B^n)} RL(g) \). Now, fix \( t \geq 0 \). Let \( L_t : H(B^n) \to \mathbb{C} \) be given by

\[
L_t(g) = L(e^t g \circ v_t), \quad g \in H(B^n).
\]

Then \( L_t \) is a continuous linear functional on \( H(B^n) \) and \( L_t(e^{-t}f(\cdot,t)) = L(f) \). Also, the functionals \( \{L_t\}_{t \geq 0} \) are weakly continuous in their dependence on \( t \). Since \( e^t g \circ v_t \in S^0(B^n) \) for \( g \in S^0(B^n) \), it follows that

\[
RL_t(e^{-t}f(\cdot,t)) = RL(f) \geq RL(e^t g \circ v_t) = RL_t(g),
\]

for all \( g \in S^0(B^n) \), i.e. \( RL_t(e^{-t}f(\cdot,t)) = \max_{g \in S^0(B^n)} RL_t(g) \). On the other hand, since \( f \in \text{supp} S^0(B^n) \), there is \( h \in S^0(B^n) \) such that \( RL(h) < RL(f) \). Also, since \( L_t(h) \to L(h) \) as \( t \to 0^+ \), we may find a point \( t_0 > 0 \) such that

\[
RL_t(h) < RL(f) = RL_t(e^{-t}f(\cdot,t)), \quad 0 \leq t < t_0.
\]

Thus, \( RL_t|_{S^0(B^n)} \) is nonconstant for \( 0 \leq t < t_0 \), as desired. \( \square \)

As a consequence of the proof of Theorem 3.2, the authors [14] obtained the following generalization to higher dimensions of an extremal principle, called “Basic Lemma” (see [26] and [40]). This result may be useful in proving distortion and coefficient bounds for mappings in \( S^0(B^n) \).

Theorem 3.3. Let \( \lambda : S^0(B^n) \to \mathbb{R} \) be a continuous real-valued functional. If \( f \in S^0(B^n) \) provides the maximum for \( \lambda \) over the family \( S^0(B^n) \) and if \( f(z,t) \) is a Loewner chain such that \( \{e^{-t}f(\cdot,t)\}_{t \geq 0} \) is a normal family on \( B^n \) and \( f = f(\cdot,0) \), then \( e^{-t}f(\cdot,t) \in S^0(B^n) \) provides the maximum for the associated functional \( \lambda_t : S^0(B^n) \to \mathbb{R} \) given by

\[
\lambda_t(g) = \lambda(e^t g \circ v_t), \quad g \in S^0(B^n), \quad t \geq 0,
\]

where \( v_t = v(\cdot,0,t) \) and \( v(z,s,t) \) is the transition mapping associated with \( f(z,t) \). In addition, the two maxima are equal.

A particular case of interest in Theorem 3.3 occurs when the extremal mapping \( f \) is starlike, for example when \( f \) is the Koebe mapping on \( B^n \). In this situation, we have (cf. [14])

Corollary 3.4. Let \( \lambda : S^0(B^n) \to \mathbb{R} \) be a continuous real-valued functional and let \( \lambda_t \) be the functional given by (3.2). If \( f \in S^*(B^n) \) provides the
maximum for \( \lambda \) over the family \( S^0(B^n) \), then \( f \) provides the maximum for \( \lambda_t \) and the two maxima are equal.

In the case \( n = 1 \), interesting applications of the above extremal principle were given by Kirwan and Schober [26], and by Roth [40], to study various properties of functions in the class \( S \), such as upper bounds for two-point distortion results and coefficient bounds for mappings in \( S \) (see also [36] and the references therein).

**Remark 3.5.** (i) Let \( f \in S^0(B^n) \) and let \( f(z,t) \) be a Loewner chain such that \( f = f(\cdot,0) \) and \( \{e^{-t}f(\cdot,t)\}_{t \geq 0} \) is a normal family on \( B^n \). Let \( v_t(z) = v(z,t) = v_{0,t}(z) \) for \( z \in B^n \) and \( t \geq 0 \), where \( v_{s,t}(z) = v(z,s,t) \) is the transition mapping associated with \( f(z,t) \). Then \( e^t v_t \notin \text{ex} S^0(B^n) \) for \( t \geq 0 \) [14, 17]. In particular, \( \text{id}_{B^n} \notin \text{ex} S^0(B^n) \), where \( \text{id}_{B^n} \) is the identity mapping (see [14]).

(ii) Assume that \( f \in \text{supp} S^0(B^n) \). With the same notation as above, if \( e^t v_t \notin \text{supp} S^0(B^n) \), then the conclusion of Theorem 3.2 is that \( e^{-t}f(\cdot,t) \in \text{supp} S^0(B^n) \) for \( t \geq 0 \) (see [14]).

Indeed, if \( t \geq 0 \) is fixed and \( L_t \) is given by (3.1), then

\[
\mathcal{R}L_t(\text{id}_{B^n}) = \mathcal{R}L(e^t v_t) < \mathcal{R}L(f) = \mathcal{R}L_t(e^{-t}f(\cdot,t)),
\]

and thus, \( \mathcal{R}L_t|_{S^0(B^n)} \) is nonconstant, as desired.

In connection with Remark 3.5 (ii), the authors [14] proved the following result related to \( S^0(B^n) \).

**Proposition 3.6.** The identity map \( \text{id}_{B^n} \) is not a support point of \( S^0(B^n) \).

In view of Remark 3.5 (ii), the following conjecture was proposed in [14].

**Conjecture 3.7.** The mapping \( e^t v_t \notin \text{supp} S^0(B^n) \) for \( t \geq 0 \) and \( n \geq 2 \).

We recall that if \( \Omega_1 \subseteq \Omega_2 \subseteq \mathbb{C}^n \) are two domains, then the pair \( (\Omega_1, \Omega_2) \) is called a Runge pair if \( \mathcal{O}(\Omega_2) \) is dense in \( \mathcal{O}(\Omega_1) \), where \( \mathcal{O}(\Omega_j) \) is the family of holomorphic functions on \( \Omega_j \), \( j = 1,2 \). Also, a domain \( \Omega \subseteq \mathbb{C}^n \) is said to be Runge if \( (\Omega, \mathbb{C}^n) \) is a Runge pair (see [37]).

**Remark 3.8.** Very recently Schleissinger ([41], Proposition 2.6) proved that Conjecture 3.7 is true, by using an interesting argument based on the notion of Runge pairs in \( \mathbb{C}^n \). More precisely, let \( L \) be a continuous linear functional on \( H(B^n) \) such that \( \mathcal{R}L|_{S^0(B^n)} \neq \text{constant} \) and \( \mathcal{R}L(e^t v_t) = \max_{F \in S^0(B^n)} \mathcal{R}L(F) \). By using the fact that \( (v_t(B^n), B^n) \) is a Runge pair, in view of ([3], Proposition 5.1), it follows that \( (v_t(B^n), \mathbb{C}^n) \) is also a Runge pair, i.e. \( v_t(B^n) \) is a Runge domain in \( \mathbb{C}^n \). This argument was used to deduce that \( \mathcal{R}L(p) = 0 \), for all \( p \in H(B^n) \) such that \( p(0) = 0 \) and \( Dp(0) = 0_n \) (see [41]). Finally, this step easily implies that \( e^t v(\cdot,t) \notin \text{supp} S^0(B^n) \) for \( t \geq 0 \) (see [41]).
Combining Remarks 3.5 (ii) and 3.8, we deduce that Theorem 3.2 has now the following improvement (see [41], Theorem 1.1), which is a complete generalization to \( \mathbb{C}^n \) of ([29], Theorem).

**Theorem 3.9.** Let \( f \in \text{supp} S^0(B^n) \) and let \( f(z,t) \) be a Loewner chain such that \( f = f(\cdot,0) \) and \( \{e^{-t}f(\cdot,t)\}_{t \geq 0} \) is a normal family on \( B^n \). Then \( e^{-t}f(\cdot,t) \in \text{supp} S^0(B^n) \), for all \( t \geq 0 \).

**Remark 3.10.** Some of the results in this section remain valid under a more general normalization than that used in the definition of the family \( \mathcal{M} \).

Let \( A \in L(\mathbb{C}^n) \) be such that \( k_+(A) < 2m(A) \). We recall that a mapping \( f \in S(B^n) \) has \( A \)-parametric representation \( (f \in S^0_A(B^n)) \) if there exists an \( A \)-normalized univalent subordination chain \( f(z,t) \) such that \( \{e^{-tA}f(\cdot,t)\}_{t \geq 0} \) is a normal family on \( B^n \) and \( f = f(\cdot,0) \) (see [11]; see also [7] and [17]). Note that the set \( S^0_A(B^n) \) is compact for \( k_+(A) < 2m(A) \), by ([11], Theorem 2.15).

We close this section with the following generalization of Theorems 3.1 and 3.9 to the case of mappings which have \( A \)-parametric representation, where \( A \in L(\mathbb{C}^n) \) is such that \( k_+(A) < 2m(A) \) (see [17]). Other extremal results related to the family \( S^0_A(B^n) \) may be found in [17].

**Theorem 3.11.** Let \( A \in L(\mathbb{C}^n) \) be such that \( k_+(A) < 2m(A) \). Also, let \( f \in \text{ex} S^0_A(B^n) \) (respectively, \( f \in \text{supp} S^0_A(B^n) \)) and let \( f(z,t) \) be an \( A \)-normalized univalent subordination chain such that \( f = f(\cdot,0) \) and \( \{e^{-tA}f(\cdot,t)\}_{t \geq 0} \) is a normal family on \( B^n \). Then \( e^{-tA}f(\cdot,t) \in \text{ex} S^0(B^n) \) (respectively, \( e^{-tA}f(\cdot,t) \in \text{supp} S^0(B^n) \)), for all \( t \geq 0 \).

### 4. Extremal Problems

**For Bounded Biholomorphic Mappings on \( B^n \)**

In this section, we indicate some recent progress in some extremal problems related to bounded biholomorphic mappings on \( B^n \) which have parametric representation. To this end, we make use of certain ideas and notions from control theory, to obtain properties of the time-\( T \)-reachable family. One of the main results in this section is a density theorem, which was obtained recently in [17], by using arguments based on the well known Krein-Milman theorem.

The following notions were introduced in [14] and [17] (cf. [24] and [40]):

**Definition 4.1.** Let \( E \subseteq [0,\infty) \) be an interval and let \( \Omega \subseteq H(B^n) \) be a normal family. A mapping \( h = h(z,t) : B^n \times E \to \mathbb{C}^n \) is called a Carathéodory mapping on \( E \) with values in \( \Omega \) if the following conditions hold:

(i) \( h(\cdot,t) \in \Omega \) for \( t \in E \).

(ii) \( h(z,\cdot) \) is a measurable mapping on \( E \) for \( z \in B^n \).
Let $C(E, \Omega)$ be the family of all Carathéodory mappings on $E$ with values in $\Omega$. In terms of control theory, the mapping $h = h(z, t)$ may be called a control function and the family $C(E, \Omega)$ may be called a control system in $H(B^n)$. Also, the family $\Omega$ may be called the input family (cf. [24, 39] and [40]).

**Definition 4.2.** Let $\Omega \subseteq M$, let $h \in C([0, T], \Omega)$ and let $v = v(z, t; h)$ be the unique Lipschitz continuous solution on $[0, T]$ of the initial value problem

$$\frac{\partial v}{\partial t}(z, t) = -h(v(z, t), t) \ a.e. \ t \in [0, T], \quad v(z, 0) = z,$$

for $z \in B^n$, such that $v(\cdot, t; h)$ is a univalent Schwarz mapping and $Dv(0, t; h) = e^{-t}I_n$ for $t \in [0, T]$. Also, let

$$\mathcal{R}_T(id_{B^n}, \Omega) = \left\{ v(\cdot, T; h) : h \in C([0, T], \Omega) \right\}$$

denote the family of all such solutions at $t = T$ generated by all Carathéodory mappings on $[0, T]$ with values in $\Omega$. The family $\mathcal{R}_T(id_{B^n}, \Omega)$ is called the time-$T$-reachable family of (4.1). The set $\Omega$ is called the input set or input family (cf. [40]). Let

$$\tilde{\mathcal{R}}_T(id_{B^n}, \Omega) = e^T \mathcal{R}_T(id_{B^n}, \Omega) \quad \text{for} \quad T \in [0, \infty)$$

and

$$\tilde{\mathcal{R}}_\infty(id_{B^n}, \Omega) = \left\{ \lim_{t \to \infty} e^t v(\cdot, t; h) : h \in C([0, \infty), \Omega) \right\}.$$

The family $\tilde{\mathcal{R}}_T(id_{B^n}, \Omega)$ will be called the normalized time-$T$-reachable family of (4.1) (cf. [24] and [39]).

For $M \in [1, \infty)$, let

$$S^0(M, B^n) = \{ f \in S^0(B^n) : \|f(z)\| < M, \ z \in B^n \}.$$

It is clear that $S^0(M, U) = S(M)$, where $S(M)$ is the subset of $S$ consisting of bounded mappings in $U$.

**Remark 4.3.** It is known that $\tilde{\mathcal{R}}_\infty(id_U, \mathcal{M}) = S$ (see [32] and [33]). Also, if $M \in (1, \infty)$, then $\tilde{\mathcal{R}}_{\log M}(id_U, \mathcal{M}) = S(M)$ (see [9] and [40], Theorem 1.48). On the other hand, $\tilde{\mathcal{R}}_\infty(id_{B^n}, \mathcal{M}) = S^0(B^n)$ (see [10]).

The following result provides concrete examples of mappings in the family $\tilde{\mathcal{R}}_{\log M}(id_{B^n}, \mathcal{M})$, by using starlike mappings on $B^n$ (see [14]).

**Example 4.4.** Let $M > 1$ and $F \in S^*(B^n)$. Also let $F^M : B^n \to \mathbb{C}^n$ be given by $F^M(z) = MF^{-1}(F(z)/M), \ z \in B^n$. Then $F^M \in \tilde{\mathcal{R}}_{\log M}(id_{B^n}, \mathcal{M}).$

**Proof.** Since $F \in S^*(B^n)$, the mapping $F^M$ is well defined. Also, since $F \in S^*(B^n)$, it follows that $F(z, t) = e^t F(z)$ is a Loewner chain and
\( F(z) = F(v(z,t),t) \) for \( z \in B^n \) and \( t \geq 0 \), where \( v(z,t) = F^{-1}(e^{-t}F(z)) \). Obviously, \( F^M(z) = Mv(z,\log M) \) for \( z \in B^n \). Hence, \( F^M \in \tilde{R}_{\log M}(\text{id}_{B^n}, \mathcal{M}) \), as desired. \( \square \)

The following results related to the family \( \tilde{R}_{\log M}(\text{id}_{B^n}, \mathcal{M}) \) were recently proved in [14] (see also [17]).

**Theorem 4.5.** Let \( M > 1 \) and let \( f \in H(B^n) \) be a normalized map. Then \( f \in \tilde{R}_{\log M}(\text{id}_{B^n}, \mathcal{M}) \) if and only if there exists a Loewner chain \( f(z,t) \) such that \( f(\cdot,0) = f \), \( f(\cdot,\log M) = M\text{id}_{B^n} \) and \( \{e^{-t}f(\cdot,t)\}_{t \geq 0} \) is a normal family on \( B^n \). Hence, \( \tilde{R}_{\log M}(\text{id}_{B^n}, \mathcal{M}) \subseteq \mathcal{S}^0(M,B^n) \).

**Remark 4.6.** It was conjectured in [14] that the equality \( \tilde{R}_{\log M}(\text{id}_{B^n}, \mathcal{M}) = \mathcal{S}^0(M,B^n) \) holds for each \( M \in (1,\infty) \) and \( n \geq 2 \). However, we have not been able to prove or disprove this conjecture up to now.

In view of Remark 4.3 and Theorem 4.5, it is easy to see that if \( f_j \in S(M) \), \( j = 1, \ldots, n \), then \( F \in \tilde{R}_{\log M}(\text{id}_{B^n}, \mathcal{M}) \), where \( F(z) = (f_1(z_1), \ldots, f_n(z_n)) \) for \( z = (z_1, \ldots, z_n) \in B^n \). This construction was useful to prove sharpness of the growth result (4.2) for the reachable family \( \tilde{R}_{\log M}(\text{id}_{B^n}, \mathcal{M}) \) (see [14]).

Recall that the Pick function is given by \( p^M_\alpha(z) = M k^{-1}_\alpha(\frac{1}{M} k_\alpha(z)), \zeta \in U \), where \( k_\alpha(z) = \frac{\zeta - z}{(1 - e^{i\alpha z})^2} \) is the rotation of the Koebe function, and \( \alpha \in \mathbb{R} \).

**Theorem 4.7.** Let \( M > 1 \) and \( f \in \tilde{R}_{\log M}(\text{id}_{B^n}, \mathcal{M}) \). Then

\[
(4.2) \quad p^M_m(\|z\|) \leq \|f(z)\| \leq p^M_0(\|z\|), \quad z \in B^n.
\]

These estimates are sharp.

**Remark 4.8.** Taking into account Theorem 4.7 and the fact that any sequence of Loewner chains \( \{f_k(z,t)\}_{k \in \mathbb{N}} \) such that \( \{e^{-t}f_k(\cdot,t)\}_{t \geq 0} \) is a normal family on \( B^n \), contains a subsequence \( \{f_{k_p}(z,t)\}_{p \in \mathbb{N}} \) which converges locally uniformly on \( B^n \) to a Loewner chain \( f(z,t) \) such that \( \{e^{-t}f(\cdot,t)\}_{t \geq 0} \) is a normal family on \( B^n \) (see [19]), the authors [14] (see also [17]) proved that the reachable family \( \tilde{R}_{\log M}(\text{id}_{B^n}, \mathcal{M}) \) is compact.

We recall that if \( P_m : \mathbb{C}^n \to \mathbb{C}^n \) is a homogeneous polynomial of degree \( m \), then \( \|P_m\| \leq k_m |V(P_m)| \), where \( |V(P_m)| \) is the numerical radius of \( P_m \) given by \( |V(P_m)| = \max\{|\langle P_m(w),w\rangle| : \|w\| = 1\} \), \( k_m = m^m/(m-1) \) for \( m > 1 \) and \( k_1 = 2 \) (see [38]). Also, \( \|P_m\| = \max\{|P_m(w)| : \|w\| \leq 1\} \) is the norm of \( P_m \).

The following coefficient bound for the family \( \tilde{R}_{\log M}(\text{id}_{B^n}, \mathcal{M}) \) holds [14].

**Theorem 4.9.** Let \( g \in \tilde{R}_{\log M}(\text{id}_{B^n}, \mathcal{M}) \) and let \( P_2(g) = \frac{1}{2}D^2g(0) \), where \( M > 1 \). Then \( |V(P_2)| \leq 2(1 - 1/M) \). This estimate is sharp. Also, \( \|P_2\| \leq 8(1 - 1/M) \).
The main result of this section is related to the family \( \mathcal{R}_{\log M}(id_{B^n}, ex \mathcal{M}) \), which involves the subfamily \( ex \mathcal{M} \) of \( \mathcal{M} \) consisting of extreme points (for \( n = 1 \), see [40]; cf. [27, 32]). This result was recently proved in a more general form in [17], namely for reachable families generated by \( A \)-normalized univalent subordination chains, where \( A \in L(C^n) \) with \( k_+(A) < 2m(A) \). It is known that if \( n = 1 \), then \( \mathcal{R}_{\log M}(id_U, ex \mathcal{M}) \) is dense in \( S(M) \) for \( M \in (1, \infty) \), and \( \mathcal{R}_\infty(id_U, ex \mathcal{M}) \) is dense in \( S \) (see [27] and [40]; see also [32]).

The proof of Theorem 4.12 involves some auxiliary lemmas of independent interest. We mention them without proofs (see [17]; cf. [40]).

**Lemma 4.10.** Let \( M \in (1, \infty) \) and let \( \Omega_1 \subseteq \mathcal{M} \) be a family such that \( \text{co} \Omega_1 = \mathcal{M} \), where \( \text{co} \Omega_1 \) is the convex hull of \( \Omega_1 \). Then for every \( h = h(z, t) \in C([0, \log M], \mathcal{M}) \), there exists a sequence \( \{h_k\} \subseteq C([0, \log M], \text{co} \Omega_1) \) such that \( \int_0^1 h_k(v(z, s; h), s)ds \to \int_0^1 h(v(z, s; h), s)ds \) locally uniformly on \( B^n \times [0, \log M] \) as \( k \to \infty \), where \( v = v(z, t; h) \) is the unique solution on \( [0, \log M] \) of the initial value problem (4.1).

**Lemma 4.11.** Let \( M \in (1, \infty) \) and let \( \Omega_1 \subseteq \mathcal{M} \). Then \( \mathcal{R}_{\log M}(id_{B^n}, \text{co} \Omega_1) = \mathcal{R}_{\log M}(id_{B^n}, \Omega_1) \).

**Theorem 4.12.** If \( M \in (1, \infty) \), then

\[
\mathcal{R}_{\log M}(id_{B^n}, ex \mathcal{M}) = \mathcal{R}_{\log M}(id_{B^n}, \mathcal{M}).
\]

**Proof.** We only sketch the proof (for details, see [17]). Since \( \mathcal{M} \) is a compact family (see [10]), it follows that \( ex \mathcal{M} \neq \emptyset \). Let \( \Omega_1 = ex \mathcal{M} \) and \( \Omega_2 = \mathcal{M} \). Then \( \Omega_1 \) is a normal family in \( H(B^n) \), and since \( \Omega_2 \) is a nonempty, convex and compact family, we deduce in view of the Krein-Milman theorem that \( \text{co} \Omega_1 = \Omega_2 \). Next, taking into account Lemmas 4.10 and 4.11, the following equality may be proved (cf. [40]):

\[
\mathcal{R}_{\log M}(id_{B^n}, \Omega_1) = \mathcal{R}_{\log M}(id_{B^n}, \Omega_2).
\]

Finally, since \( \mathcal{R}_{\log M}(id_{B^n}, \Omega_2) \) is a compact family, the equality (4.3) holds, as desired. \( \square \)

If \( M = \infty \), then the following consequence of Theorem 4.12 holds (see [17]). This result is an \( n \)-dimensional version of a well known result due to Loewner, who described a dense subset of \( S \) consisting of single-slit maps as a reachable family generated by the Loewner differential equation (see [27]; see also [40]).

**Corollary 4.13.** If \( f \in S^0(B^n) \), then for any \( \varepsilon > 0 \) and for any compact set \( K \subset B^n \), there exist \( M \in (1, \infty) \) and \( f_{\varepsilon,K} \in \mathcal{R}_{\log M}(id_{B^n}, ex \mathcal{M}) \) such that \( \|f(z) - f_{\varepsilon,K}(z)\| < \varepsilon \) on \( K \).
In view of Theorem 4.12 and the above result, the authors [17] proposed the following conjecture, which is Loewner’s density result when \( n = 1 \) [27].

**Conjecture 4.14.** \( \overline{R}_\infty (\text{id}_{B^n}, \text{ex } M) = S^0(B^n) \) for \( n \geq 2 \).

It is also natural to consider the following conjecture, which is true in the case of one complex variable (see [39]). This conjecture was proposed in [17].

**Conjecture 4.15.** Let \( M \in (1, \infty] \) and let \( \Omega \subseteq M \) be a compact and convex family. Then \( R_{\log M}(\text{id}_{B^n}, \Omega) \) is compact and
\[
\overline{R}_{\log M}(\text{id}_{B^n}, \text{ex } \Omega) = R_{\log M}(\text{id}_{B^n}, \Omega), \quad n \geq 2.
\]

**Acknowledgments.** Ian Graham was partially supported by the Natural Sciences and Engineering Research Council of Canada under Grant A9221. Hidetaka Hamada was partially supported by JSPS KAKENHI Grant Number 25400151.

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Received 21 March 2013

University of Toronto,
Department of Mathematics,
Toronto, Ontario M5S 2E4, Canada
graham@math.toronto.edu

Kyushu Sangyo University,
Faculty of Engineering,
3-1 Matsukadai 2-Chome, Higashi-ku,
Fukuoka 813-8503, Japan
h.hamada@ip.kyusan-u.ac.jp

Babeş-Bolyai University,
Faculty of Mathematics and Computer Science,
1 M. Kogălniceanu Str.,
400084 Cluj-Napoca, Romania,
gkohr@math.ubbcluj.ro