ON THE WEAK SOLVABILITY OF SCHRÖDINGER TYPE EQUATIONS WITH BOUNDARY CONDITIONS

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This paper is concerned with the weak solvability of the boundary value problems for Schrödinger type equations. For such equations with a real (admissible) spectral parameter $\mu$, the nonhomogeneous Dirichlet, respectively, homogeneous Dirichlet-Neumann problem on a weak Stokes domain lying in a Riemann domain is shown to admit a unique weak solution by making use of a Schrödinger product. If the parameter (denoted by $\lambda$) is complex-valued, the weak solvability (in a suitable sense) of a homogeneous Dirichlet-Neumann boundary value problem for a generalized Schrödinger equation with $L^2$-density is proved in the case where $\lambda$ does not belong to $\mathbb{R}_-$.

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1. INTRODUCTION

One of the most important example of a singular differential operator (in mathematical physics) is the Schrödinger operator

$$\mathcal{G}(\psi) := -\Delta \psi + (\mu - V) \psi$$

in $L^2(\mathbb{R}^n)$, where $\mu$ is a real constant (usually called the spectral parameter) and $V$ a real-valued $L^2$-function (the potential). For wider applications, one may wish to take $\mu$ and/or $V$ to be complex-valued. In the following, let $Y$ denote a complex space of pure positive dimension $m$ and $D \subseteq Y$ a relatively compact open subset, and $\tilde{D} \subseteq Y$ an open neighborhood of $D$. Assume throughout this paper that $(\tilde{D}, p)$ is a Riemann subdomain of $Y$, meaning that $\tilde{D}$ (as a subspace of $Y$) admits a holomorphic map $p : \tilde{D} \to \mathbb{C}^m$ with discrete fibers. Denote by $\Delta_p$ the pull-back under $p$ of the Laplace operator of the Euclidean metric on $\mathbb{C}^m$ to (the largest) open dense subset $D^* \subseteq D$ where $p$ is locally biholomorphic. Observe that if $\psi \in C^2(D)$, the Schrödinger operator $
abla$ $\tilde{\mathcal{G}}_{p, \lambda, V}[\psi] := -\tilde{\Delta}_p[\psi] + (\lambda - V)[\psi]$ associated with a constant $\lambda \in \mathbb{C}$ and $V \in L^1_{loc}(D)$, regarded as acting on $C^2_c(D)$, is induced on $D^*$ by the function

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\[ (1.1) \quad \mathcal{S}_{p,\lambda,V}(\psi) := -\Delta_p \psi + (\lambda - V) \psi. \]

If \( D \) is a weak Stokes domain ([6], p. 568) lying in a Riemann domain, for the Schrödinger equation

\[ (1.2) \quad \mathcal{S}_{p,\lambda,V}(\Psi) = -\Delta_p \Psi + (\lambda - V) \Psi = g \]

in \( D^* \) with a real spectral parameter \( \lambda \) and density \( g \in L^2(D) \) (or more generally, \( g \in H^{-1}(D) \)), sufficient conditions can be given for the solvability, in a weak sense, of the equation (1.2) subject to a boundary condition (Theorem 4.1). Similarly, one may consider, for a generalized Schrödinger equation, a mixed homogeneous Dirichlet-Neumann problem relative to a partition

\[ (1.3) \quad \partial D \setminus A = (\partial D)_\Sigma \cup (\partial D)_\Gamma, \]

where \( A \) is a thin analytic subset of \( X \) (see Theorems 4.2 and 4.3). This is done by making use of a Schrödinger product \( S_{\lambda,h} \) defined on a (suitable) Sobolev space \( H^1_{\lambda,h}(D) \).

For the metaharmonic operator \( \mathcal{M}_{p,\lambda} := \mathcal{S}_{p,\lambda,0} \) with a real spectral parameter \( \lambda \), it is possible to construct a Green's operator (with explicitly determined kernel) (at least) on suitable pseudoballs in a complex space. The general question remains as to for what parameters \( \lambda \) the equation (1.2) admits a classical or a weak solution (in a suitable sense). If \( D \) is a domain in the complex plane, Garnir [4] proved by an ingenious method that the equation "\( \mathcal{M}_{p,\lambda}(\Psi) = g \)" admits a distributional solution, provided \( \lambda \) belongs to \( \mathbb{C} \setminus \mathbb{R}^- \). This method can be extended to the case of a generalized Schrödinger operator \( \mathcal{P}_{\lambda,h} w \), defined for complex parameters \( \lambda \) and elements \( w \) of a linear subspace of \( L^1_{\text{loc}}(D) \) (see (3.6)), by making use of a generalized Poincaré's inequality (Lemma 3.1) and introducing a Schrödinger map and its weak Green's operator for a real (admissible) spectral parameter \( \mu \) (compare [4]). The solvability of a homogeneous Dirichlet-Neumann problem (see (4.9)) for an (admissible) \( L^2 \)-density \( g \) can then be ascertained, in the case \( \lambda \in \mathbb{C} \setminus \mathbb{R}^- \), in a weak sense, namely, there exists an element \( w \) (in a Sobolev space) satisfying suitable boundary conditions such that the associated Schrödinger functional satisfies the equation

\[ (1.4) \quad \langle \mathcal{P}_{\lambda,h} w, v \rangle_D = \langle [g], v \rangle_D, \]

where \( v \) ranges over a (suitable) test space (Theorem 5.1). For Neumann problems for related Schrödinger type equations in a (possibly) irregular domain in \( \mathbb{R}^n \), see [1, 2]. The author is indebted to the referee for suggestions which led to improvements of this paper.
2. PRELIMINARIES

In what follows, every complex space is assumed to be reduced and has a countable topology. For the definition and basic properties of differential forms on a complex space, see ([5], §4.1). In particular, the exterior differentiation \( d \), the operators \( \partial \), \( \overline{\partial} \) and \( d^c := (1/4\pi i)(\partial - \overline{\partial}) \) are well-defined ([5], Chap. 4). Denote by \( \|z\| \) the Euclidean norm of \( z = (z_1, \cdots, z_m) \in \mathbb{C}^m \), where \( z_j = x_j + iy_j \). Let the space \( \mathbb{C}^m \) be oriented so that the form \( \upsilon^m := (dd^c\|z\|^2)^m \) is positive. If \( p : Y \to \mathbb{C}^m \) is a holomorphic map, set \( a' := p(a) \), \( p[a] := p - a' \) and \( r_a := \|p[a]\| \), for \( a \in Y \). Clearly the form

\[
\upsilon_p := dd^c r_a^2 = \left(\frac{i}{2\pi}\right) \partial \overline{\partial} r_a^2
\]

is nonnegative and independent of \( a \).

For an open subset \( G \) of \( Y \), denote by \( C^\nu_k(G) \) the set of \( \mathbb{C} \)-valued \( k \)-forms of class \( C^\nu \) on \( G \), \( C^\nu_{k,c}(G) \) the subspace of compactly supported \( k \)-forms (dropping the degree if \( k = 0 \)), with \( \nu = \beta \) to mean locally bounded, \( \nu = m \) measurable, and \( \nu = \lambda \) locally Lipschitzian ([5], §4). Similarly, for \( C^\lambda_k(\mathcal{G}) \).

A measurable function \( \psi \) on \( Y \) is said to be \( \text{locally integrable} \) (\( f \in L^1_{\text{loc}}(Y) \)) if so is every \( 2m \)-form \( \psi \chi \) with \( \chi \in C^0_{2m}(Y) \). Denote by \( d\upsilon \) the Euclidean volume element of \( \mathbb{C}^m \) and define the semivolume form \( d\tilde{\upsilon} := p^*(d\upsilon) \) on \( Y \). If \( f, \psi \in L^1_{\text{loc}}(D) \), set

\[
(f, \psi)_D := \int_D f \psi d\tilde{\upsilon},
\]

provided the integral exists. Each element \( f \in L^1_{\text{loc}}(D) \) gives rise naturally to a top-dimensional current, \( T = [f] : \chi \mapsto \int f \wedge \chi \), with induced functional

\[
\langle [f], \phi \rangle := (-1)^{m(m-1)/2} \frac{\pi^m}{m!} \int T \wedge \phi v_p^m = (f, \phi)_D, \quad \forall \phi \in C^1_c(D).
\]

3. THE SCHröDINGER PRODUCT

In ([5], §3) the \textit{Dirichlet product} of locally Lipschitz functions \( \psi, \eta : D \to \mathbb{C} \) is defined:

\[
[\psi, \eta]_{[D]} := \int_D d\psi \wedge d^c\overline{\eta} \wedge v_p^{m-1},
\]

provided the integral exists. It is shown in ([7], (3.6)) that the Dirichlet product is Hermitian symmetric. The Hilbert scalar product on \( L^2(D) \) is defined by the pairing \( (f, g) \mapsto (f, \overline{g})_D \) (with induced norm \( \| \cdot \|_{L^2(D)} \)). A generalization of both the Hilbert and Dirichlet product is considered below.
If \( \psi = v \in C^1(D) \), denote by \( \partial_\nu \psi, 1 \leq \nu \leq 2m \), the \( \nu \)-th first order derivative \( \frac{\partial \psi}{\partial t_\nu} \) on \( D^* \), namely,

\[
\frac{\partial \psi}{\partial t_\nu} := \frac{\partial v}{\partial x_k}, \quad \text{if } \nu = 2k - 1, \quad \frac{\partial \psi}{\partial t_\nu} := \frac{\partial v}{\partial y_k}, \quad \text{if } \nu = 2k.
\]

Define \( C^{1,1}(\overline{D}) := \{ \phi \in C^1(\overline{D}) \mid \phi_{\bar{x}_j} \text{ and } \phi_{\bar{y}_j} \in C^j(\overline{D}), 1 \leq j \leq m \} \). If \( \psi \in L^1_{\text{loc}}(D) \), denote by (the same) \( \partial_\nu \psi \) the \( \nu \)-th distributional weak derivative of \( \psi \) on \( D \), provided it exists as an element of \( L^1_{\text{loc}}(D) \). Let \( \mathcal{M}_D \) be the subspace of \( L^1_{\text{loc}}(D) \) consisting of all elements \( w \) such that: (i) the first order weak derivatives \( \partial_\nu w = \phi_\nu, \nu = 1, \cdots, 2m \), exist (as an element of \( L^1_{\text{loc}}(D) \)), namely,

\[
\langle \partial_\nu[w], v \rangle = - (w, \frac{\partial v}{\partial t_\nu})_{D^*} = (\phi_\nu, v)_{D^*}, \quad \forall v \in C^1_c(D);
\]

(ii) each weak derivative \( \partial_\nu w \in C^\beta(D) \). In the following, let \( h \) denote an \((2m+1)\)-tuple \((h_0, \cdots, h_{2m})\), where each component \( h_j \) is a locally integrable function on \( D \), with \( h_j \geq 0 \) for \( 1 \leq j \leq 2m \), and set \( V := -h_0 \). Then it is easy to show that if \( w, v \in \mathcal{M}_D \), then each of the forms \( h_j(\partial_j w)(\partial_j \bar{v}) d\bar{v} \) is integrable on \( D \). It follows that for any given complex constant \( \lambda \), the sesquilinear form

\[
S_{\lambda,h}(w,v) := ((\lambda - V)w, \bar{v})_D + \sum_{j=1}^m \int_D \left( h_{2j-1} \frac{\partial w}{\partial x_j} \frac{\partial \bar{v}}{\partial \bar{x}_j} + h_{2j} \frac{\partial w}{\partial y_j} \frac{\partial \bar{v}}{\partial \bar{y}_j} \right) d\bar{v}
\]

is well-defined on \( \mathcal{M}_D \times \mathcal{M}_D \). Since an element of \( C^1(\overline{D}) \) gives rise to a member of \( \mathcal{M}_D \), the Schrödinger form \( S_{\lambda,h} : C^1(\overline{D}) \times C^1(\overline{D}) \to \mathbb{C} \) is well-defined by (3.1). In the following, the reference to ”\( \mu, V \)” will be abbreviated simply as ”\( \mu, V \)” whenever \( h = h^{(1)} := (-V,1,\cdots,1) \). Thus,

\[
S_{\lambda,V}(w,v) = S_{\lambda,h^{(1)}}(w,v) = ((\lambda - V)w, \bar{v})_D + S_{0,(0,h_1,\cdots,h_m)}(w,v).
\]

Green’s first identity ([6], (5.9)) yields a relation between the operator \( \tilde{S}_{p,\lambda,V} \) and the form \( S_{\lambda,h} \):

\[
\langle \tilde{S}_{p,\lambda,V}[w], v \rangle = S_{\lambda,V}(w, \bar{v}), \quad \forall (w,v) \in C^1(\overline{D}) \times C^2_c(D).
\]

In the following, let \( \mu \) denote a real constant. Then \( S_{\mu,h} \) defines a Hermitian symmetric, continuous form on \( C^1(\overline{D}) \) (with induced seminorm denoted accordingly by \( \| \cdot \|_{\mu,h} \)). Let \( \Delta \) be the branch locus of \( p \) and set \( \Delta' = p(\Delta) \). If \( a \in D_0 := D \setminus p^{-1}(\Delta') \), a connected neighborhood \( U \Subset D \) can be chosen so that \( p|U \) is a covering projection onto a ball \( U' \Subset \mathbb{C}^m[1] \) with sheets \( U_\nu \). In terms of such \( U = U^\alpha \), a \( C^\infty \)-partition of unity \( \{(U_\nu^\alpha, p_\nu^\alpha)\} \), each \( U_\nu^\alpha \) being a sheet of \( U^\alpha \), is said to be distinguished for \( D_0 \), if, for any real-valued
\( \phi \in C^1(\overline{D}) \), setting \( \phi_\nu^\alpha := \rho_\nu^\alpha \phi \), one has

\[
\int_{D_0} \sum_{j=1}^{2m} \left( \sum_{\alpha \neq \beta} h_j \frac{\partial \phi_\nu^\alpha}{\partial t_j} \frac{\partial \phi_\nu^\beta}{\partial t_j} \right) d\tilde{\nu} \geq 0. \tag{1}
\]

**Definition 1.** A pair \((\mu, h)\) (as above) is said to be **admissible for** \(D\) if:

1. \( h_D := \min \{ \text{ess inf}_D h_j \mid j = 1, \ldots, 2m \} > 0 \);
2. \( D_0 \) admits a distinguished partition of unity; and
3. denoting by \( P_{cm[1]} \) the Poincaré’s constant for the Euclidean unit ball, \( \mu - \text{ess sup}_D V > -\frac{h_D}{(P_{cm[1]})^2} \).

A pair \((\mu, V)\) is said to be **admissible for** \(D\) if so is \((\mu, h\{1\})\).

If the map \(p\) exhibits \(D\) as a covering space of an open all in \(\mathbb{C}^m\), then the condition (ii) above holds trivially; in which case, every pair \((\mu, 0)\) with \(\mu \geq 0\) is admissible for \(D\).

**Lemma 3.1 (Generalized Poincaré’s inequality).** If \((\mu, h)\) is admissible for \(D\), then there exists a positive constant \(c_{D,\mu,h}\) (depending on a distinguished covering of \(\overline{D}\)) such that

\[
\|\phi\|_{L^2(D)} \leq c_{D,\mu,h} \|\phi\|_{\mu,h}, \quad \forall \phi \in C^1(\overline{D}). \tag{3.4}
\]

**Proof.** It suffices to consider a real-valued \(\phi \in C^1(\overline{D})\). In terms of a distinguished covering for \(D_0\), by the Poincaré’s inequality for the Euclidean unit ball, one has

\[
\int_D \frac{h_D}{(P_{cm[1]})^2} \int_{(U^\alpha)'} \|\phi\|^2 d\tilde{\nu} \leq \int_D \frac{h_D}{(P_{cm[1]})^2} \sum_{\alpha,\nu} \int_{(U^\alpha)'} \|\phi_\nu^\alpha\|^2 d\nu
\]

\[
\leq h_D \sum_{\alpha,\nu} \int_{(U^\alpha)'} \|\nabla (\phi_\nu^\alpha)\|^2 d\nu \leq \int_{D_0} \sum_{\alpha,\nu} \sum_{j=1}^{2m} h_j \left| \frac{\partial (\phi_\nu^\alpha)'}{\partial t_j} \right|^2 d\tilde{\nu}
\]

\[
\leq \int_{D_0} \sum_{j=1}^{2m} h_j \sum_{\alpha,\beta} \left( \frac{\partial (\phi_\nu^\alpha)'}{\partial t_j} \frac{\partial (\phi_\nu^\beta)'}{\partial t_j} \right) d\tilde{\nu} = S_{0,(0,h_1,\ldots,h_m)}(\phi, \phi),
\]

where \((\phi_\nu^\alpha)':[U^\alpha]' \to \mathbb{C}\) is induced by the restriction of \(\phi\) to a sheet of \(U^\alpha\). Therefore, one has

\[
\|\phi\|^2_{\mu,h} \geq (\mu - \text{ess sup}_D V + \frac{h_D}{(P_{cm[1]})^2}) \|\phi\|^2_{L^2(D)} =: \kappa_{D,\mu,h} \|\phi\|^2_{L^2(D)}. \tag{3.5}
\]

\(^1\)This condition is likely to be superfluous by making judicious choice of the partition functions.
In the following, assume (unless otherwise mentioned) that $(\mu, h)$ is admissible for $D$. Then the Hermitian form $S_{\mu, h}$ is elliptic, hence defines a scalar product on $C^1(D)$. With respect to the induced norm $\| \|_{\mu, h}$ the completion of $C^1(D)$ (respectively, $C^\infty_c(D)$) gives rise to the Sobolev space $H^\mu := H^1_{\mu, h}(D)$ (respectively, $H^1_{\mu, h,c}(D)$) equipped with the (naturally extended) scalar product $S_{\mu, h}$ given by the right-hand side expression of (3.1). It follows that, if $\phi \in H^\mu$, then the generalized Poincaré’s inequality (3.4) holds for all $\phi \in H^\mu$. Clearly, this inequality implies that $H^\mu \subseteq L^2(D)$. Taking $(\mu, V) = (1, 0)$, the Sobolev spaces $H^1(D)$ and $H^1_c(D)$ are defined. Moreover, the Schrödinger form (3.2) extends to a scalar product $\langle \cdot , \cdot \rangle_{1,0,D} = S_{1,(0,1,\ldots,1)}$ on $H^1(D)$ with associated energy norm denoted by $\| \|_{1,0,D}$.

In what follows, let $\lambda$ denote a complex (possibly real) constant, and $(\lambda, h)$ a pair with $h = \{-V, h\}$ where each $h_j \in C^\lambda(D)$ (unless otherwise mentioned). It can be shown that, if $w \in M_D$, the integral
\[
\langle \partial_k[h_j \partial_j w], \bar{\phi} \rangle = -(h_j \partial_j w, \partial_k \bar{\phi})_{D^*}
\]
exists for all $\phi \in C^1_c(D)$ and $j, k = 1, \ldots, 2m$ ([8], §3, Remark 1). Therefore, each such element $w$ gives rise to a Schrödinger functional $\mathfrak{P}_{\lambda, h} w : C^1_c(D) \to \mathbb{C}$ defined by
\[
\langle \mathfrak{P}_{\lambda, h} w, v \rangle_D := S_{\lambda, h}(w, v), \quad \forall v \in C^1_c(D).
\]

Note also, that the right-hand side of (3.6) serves to extend (i) the action of the functional $\mathfrak{P}_{\mu, h} w$ to $H^1_{\mu, h}(D)$ and (ii) its definition to $w \in H^1_{\mu, h}(D)$. If $w \in C^{1,1}(D)$, the functional
\[
\langle \mathfrak{S}_{\lambda, h}(w), \phi \rangle_{D^*} := ((\lambda - V)w, \bar{\phi})_D - \sum_j \int_{D^*} \partial_j(h_j \partial_j w) \bar{\phi} \, d\bar{\nu}
\]
is well-defined for all elements $\phi \in C^1_c(D)$. Consequently, the functional $\phi \mapsto \langle \mathfrak{S}_{\mu, h}(w), \phi \rangle_{D^*}$ extends naturally to an operator (denoted by the same) on $H^1_{\mu, h}(D)$. In the following, assume that $D \subseteq Y$ is a weak Stokes domain with nonempty (maximal) boundary manifold $dD$ of $D_{\text{reg}}$ (oriented to the exterior of the manifold $D_{\text{reg}}$ of simple points of $D$) ([5], p. 218).

**Proposition 3.1.** Assume that $w \in C^{1,1}(D)$. Then for all $\phi \in C^{1,1}(D)$,
\[
\langle \mathfrak{S}_{\lambda, h}(w), \phi \rangle_{D^*} + \sum_{j=1}^{2m} \int_{dD} \bar{\phi} n_j(h_j \partial_j w) \, d\sigma = S_{\lambda, h}(w, \phi),
\]
where $d\sigma$ denotes the surface element of $dD$, $n$ the unit outward normal to $dD$. If further $\lambda = \mu \in \mathbb{R}$, the above formula remains valid for all $\phi \in H^1_{\mu, h,c}(D)$. 

Proof. For all \( w, \phi \in C^{1,1}(\bar{D}) \), the formula (3.8) is a consequence of the Green’s formula. Assume now that \( \lambda = \mu \in \mathbb{R} \). An arbitrary element \( \phi \in H^{1}_{\mu,h,c}(D) \) is approximable in the \( L^2 \)-norm on \( L^2(D) \) by a sequence \( \phi_n \in C_c^{\infty}(D) \). Then

\[
S_{\lambda,h}(w,\phi_n) = ((\mu - V)w,\bar{\phi}_n)_{D} + \sum_{j=1}^{2m} \left( \int_{dD} \bar{\phi}_n n_j (h_j \partial_j w) \, d\sigma - \int_{D} \bar{\phi}_n \partial_j (h_j \partial_j w) \, d\tilde{\nu} \right)
\]

\[
= \langle \mathcal{S}_{\mu,h}(w),\phi_n \rangle_{D^*} + \sum_{j=1}^{2m} \int_{dD} \bar{\phi}_n n_j (h_j \partial_j w) \, d\sigma
\]

\[
\rightarrow \langle \mathcal{S}_{\mu,h}(w),\phi \rangle_{D^*} + \sum_{j=1}^{2m} \int_{dD} \bar{\phi} n_j (h_j \partial_j w) \, d\sigma, \text{ as } n \to \infty,
\]

thus, proving formula (3.8). \( \square \)

4. WEAK SOLVABILITY OF SCHRÖDINGER TYPE EQUATIONS

A pair \((\mu, h)\) (with \( h \) as in (3.1)) is said to be strictly admissible for \( D \), if

\[
\|\phi\|_{1,0,D} \leq \text{Const.} \|\phi\|_{\mu,h}, \quad \forall \phi \in C_c^{\infty}(\tilde{D}),
\]

(notting that \( C_c^{\infty}(\tilde{D}) \subset C_c^1(\bar{D}) \)). To each such pair \((\mu, h)\) the associated ”\( S_{\mu,h} \)” is positive definite, hence defines a scalar product on \( H^{1}_{\mu,h}(D) \).

It is shown in ([8], §3) that the \( k \)-th partial derivatives \( \partial_{\tilde{x}_k} T \) and \( \partial_{\tilde{y}_k} T \) of a regular current \( T = [\psi] \) (where \( \psi \in L^1_{\text{loc}}(D) \)) can be defined as \( (2m-1) \)-dimensional quasicurrents on \( D \) (which are currents if \( D \) is nonsingular). As a consequence, given \( g = (g_0, \ldots, g_{2m}) \) with \( g_j \in L^2(D), j = 0,1,\ldots,2m \), by extending each \( g_j \) to \( \tilde{D} \) by setting \( \tilde{g}_j = 0 \) on \( \tilde{D} \setminus D \), there is an associated quasicurrent

\[
[g] := \tilde{g}_0 + \sum_{j=1}^{m} \left( \partial_{\tilde{x}_k}[\tilde{g}_j] + \partial_{\tilde{y}_k}[\tilde{g}_j] \right)
\]

on \( \tilde{D} \), and an induced functional (denoted by the same) acting on \( C_c^{\infty}(\tilde{D}) \):

\[
\langle [g], \phi \rangle_{D} := (g_0, \bar{\phi})_{D^*} - \sum_{j=1}^{m} \left( (g_j, \frac{\partial \bar{\phi}}{\partial \tilde{x}_j})_{D^*} + (g_j, \frac{\partial \bar{\phi}}{\partial \tilde{y}_j})_{D^*} \right).
\]

Let \( H^{-1}(D) \) denote the linear space of all such \( 2m \)-tuples \( g \). It follows from (4.3) and Hölder’s inequality that, if \((\mu, h)\) is strictly admissible for \( D \),
then
\begin{equation}
\langle [g], \phi \rangle_D \leq \text{Const.} \max_{0 \leq j \leq 2m} \|g_j\|_{L^2(D)} \|\phi\|_{\mu, h}, \quad \forall \phi \in C^\infty_c(\bar{D}).
\end{equation}

Thus, the linear operator \([g]\) is continuous on \(C^\infty_c(D)\) with respect to the norm \(\| \|_{\mu, h}\) (or \(\| \|_{\mu, V}\), if \(h = h^{(1)}\)). The subspace \(C^\infty_c(D)\) being dense in \(H^1_{\mu, \{h\}, c}(D)\), the operator \([g]\) admits a unique continuous linear extension (denoted by the same) to the latter (relative to \(\| \|_{\mu, h}\)), hence also to \(H^1_{\mu, h}(D)\), by the Hahn-Banach’s theorem. The operator \([g]\) is said to be \textit{representable} by an element \(\eta \in L^2(D)\), if the action of \([g]\) on \(C^\infty_c(D)\) agrees with the (conjugated) action of \([\eta]\). Note that if each \(g_j\) has weak derivative \(\partial_j g_j\) belonging to \(L^2(D)\) for \(\nu = 1, \ldots, 2m\), then \([g]\) is representable by an element of \(L^2(D)\). The subspace of \textit{“Schrödinger functions”} of type \((\mu, V)\), namely,
\[S_{\mu, V}(D) := \{w \in H^1_{\mu, V}(D) | \mathfrak{P}_{\mu, h^{(1)}}^\perp w = 0\},\]
is given precisely by the orthogonal complement of \(H^1_{\mu, V, c}(D)\) (with respect to \(S_{\mu, V}\)):
\begin{equation}
H^1_{\mu, V}(D) = H^1_{\mu, V, c}(D) \oplus S_{\mu, V}(D).
\end{equation}

This is an easy consequence of the identity (3.3).

\textbf{Definition 2.} Let \(g \in H^{-1}(D)\) and \(f \in C^{1,1}(\bar{D})\). An element \(\Psi = w \in H^1_{\mu, V}(D)\) is called a \textit{weak solution} of the Dirichlet problem
\begin{equation}
\mathcal{G}_{\mu, V}(\Psi) = -\Delta_p \Psi + (\mu - V) \Psi = [g], \quad \Psi = f \text{ on } dD,
\end{equation}
provided that \(w \equiv f \mod (H^1_{\mu, V, c}(D))\) and
\[S_{\mu, V}(w, \rho) = ((\mu - V)w, \bar{\rho})_D + S_{0, 0}(w, \rho) = (\langle [g], \rho \rangle_D, \forall \rho \in H^1_{\mu, V, c}(D)).\]

It is a consequence of Green’s first identity that, if \(g \in L^2(D)\), every weak solution \(\Psi = \psi \in C^{1,1}(\bar{D})\) of the Dirichlet problem
\begin{equation}
\mathcal{G}_{\mu, V}(\Psi) = g \quad \text{in } D^*, \quad \Psi = 0 \text{ on } dD,
\end{equation}
is a classical solution of the Schrödinger equation (1.2) in \(D^*\).

\textbf{Theorem 4.1.} Assume that either (i) \(g \in L^2(D)\) and \((\mu, V)\) is admissible for \(D\) or (ii) \(g \in H^{-1}(D)\) such that \([g]\) is representable by an element \(\eta \in L^2(D)\) and \((\mu, V)\) is strictly admissible for \(D\). Then for each \(f \in C^{1,1}(\bar{D}) \cap C^2(D^*)\) with \(\Delta_p f \in L^2(D)\), the Dirichlet problem (4.6) admits a unique weak solution in \(H^1_{\mu, V}(D)\).

\textit{Proof.} Let \(F := \Delta_p f\). The homogeneous Dirichlet problem (4.7) with \textit{“g”} replaced by \textit{“}\(\hat{\gamma} := \eta + F - (\mu - V) f\)”, where \(\eta = g\) whenever \(g \in L^2(D)\),
admits a weak solution \( \psi = w_0 \in H^1_{\mu,V,c}(D) \), by Riesz’s representation theorem (applied to the functional \([\hat{g}]\)). Then the function \( \Psi := w_0 + f \) gives a weak solution of the Dirichlet problem (4.6). For the equation (3.3) (with \( \mu = 0 \)) and the Hermitian symmetry of the Dirichlet product imply that

\[
S_{0,0}(f, \phi) = - (\triangle_p f, \bar{\phi})_D, \quad \forall \phi \in C^\infty_c(D),
\]
hence, by virtue of definition (3.2), one has

\[
S_{\mu,V}(\Psi, \phi) = S_{\mu,V}(w_0, \phi) + S_{\mu,V}(f, \phi) = \langle [\hat{g}], \phi \rangle_D + ((\mu - V) f, \bar{\phi})_D + S_{0,0}(f, \phi) = \langle [g], \phi \rangle_D,
\]
thus, proving the above claim.

The difference \( \psi = \Psi_1 - \Psi_2 \) of two solutions of (4.6) belongs to \( H^1_{\mu,V,c}(D) \) and the functional "\( S_{\mu,V}(\psi, \cdot) \)" annihilates \( H^1_{\mu,V,c}(D) \). Hence, the decomposition (4.5) implies that \( \psi = 0 \). □

To consider a mixed Dirichlet-Neumann boundary value problem with a partition (1.3) of \( \partial D \setminus A \), let \( \mathcal{N} \) be an open neighborhood of \( (\partial D)_N \) contained in \( \tilde{D} \) and set \( \hat{D} := D \cup \mathcal{N} \). Denote by \( \mathcal{R}_{0,\delta} \) the closure in \( H^1_{\mu,h}(D) \) of the subspace consisting of \( \phi \in C^\infty_c(\hat{D}) \) that vanishes in the intersection with \( D \) of some neighborhood of \( (\partial D)_\mathcal{D} \) (possibly depending on \( \phi \)). Let \( \mathcal{A}_{\mu,h} \) be a maximal linear subspace of \( H^1_{\mu,h}(D) \) admitting a (Schrödinger) continuous linear mapping \( s_{\mu,h} : \mathcal{A}_{\mu,h} \to L^2(D) \) satisfying the equation

\[
\langle \Psi_{\mu,h} w, v \rangle_D = (s_{\mu,h}(w), \bar{v})_D, \quad \forall v \in C^\infty_c(D).
\]

It can be shown that, for each \( w \in C^{1,1}(\overline{D}) \cap \mathcal{A}_{\mu,h} \), the (conjugated) action of \( [s_{\mu,h}(w)] \) on \( \mathcal{R}_{0,\delta} \) agrees with the action of \( [\mathcal{G}_{\mu,h}(w)] \). Define

\[
\mathcal{N}^\mu_{0,\delta} := \{ w \in \mathcal{A}_{\mu,h} \cap \mathcal{R}_{0,\delta} | S_{\mu,h}(w, \phi) = (s_{\mu,h}(w), \bar{\phi})_D, \forall \phi \in \mathcal{R}_{0,\delta} \}.
\]

**Definition 3.** A weak solution to the Dirichlet-Neumann problem

\[
\mathcal{G}_{\mu,h}(w) = [g], \quad w \in \mathcal{N}^\mu_{0,\delta},
\]
where \( g \in H^{-1}(D) \), is an element \( w \in \mathcal{N}^\mu_{0,\delta} \) satisfying the equation

\[
\langle \Psi_{\mu,h} w, \phi \rangle_D = \langle [g], \phi \rangle_D, \quad \forall \phi \in \mathcal{R}_{0,\delta}.
\]

**Theorem 4.2.** Assume that either (i) \( g \in L^2(D) \) and \( (\mu, h) \) is admissible for \( D \) or (ii) \( g \in H^{-1}(D) \) such that \([g]\) is representable by an element of \( L^2(D) \) and \( (\mu, h) \) is strictly admissible for \( D \). Then the Dirichlet-Neumann problem (4.9) admits a unique weak solution.
Proof. Let \( \mathcal{B}_{\mu,b} \) be the set of all elements \( w \in H^1_{\mu,b}(D) \) such that the anti-linear form \( v \mapsto \langle \Psi_{\mu,b} w, v \rangle_D \) is continuous on \( H^1_{\mu,b,c}(D) \) with respect to the \( L^2 \)-norm on \( L^2(D) \). \( \mathcal{B}_{\mu,b} \) is clearly a linear subspace of \( H^1_{\mu,b}(D) \). According to the Hahn-Banach’s theorem, the said anti-linear form is extendable to a unique continuous anti-linear form on \( L^2(D) \), and this extension is given by taking the scalar product of a unique element of \( L^2(D) \), to be denoted by \( s_{\mu,b}(w) \), with \( v \). Thus the equation (4.8) holds for all \( v \in H^1_{\mu,b}(D) \). With \( g \) given as either in (i) or, in (ii), the functional \( \phi \mapsto \langle [g], \phi \rangle_D \) is well-defined and bounded on \( \mathcal{R}_{0,\mathcal{D}} \) (by the inequality (4.4)). Hence, the Riesz’s representation theorem implies that there is a unique element \( w_0 \in \mathcal{R}_{0,\mathcal{D}} \) satisfying the equation

\[
\langle [g], \phi \rangle_D = S_{\mu,b}(w_0, \phi), \quad \forall \phi \in \mathcal{R}_{0,\mathcal{D}}
\]

with

\[
\|w_0\|_{\mu,b} \leq \text{Const.} \max_{0 \leq j \leq 2m} \|g_j\|_{L^2(D)}.
\]

If \( [g] \) is representable (on \( C^\infty_c(D) \)) by an element \( \eta \in L^2(D) \) (or, if \( g \in L^2(D) \)), then by the formula (3.6), the equation (4.8) holds with \( s_{\mu,b}(w_0) = \eta \) (or \( g \)), for all \( v \in C^\infty_c(D) \). One has, for such \( v \), \( |\langle \Psi_{\mu,b} w_0, v \rangle_D| \leq \|\eta\|_{L^2(D)} \|v\|_{L^2(D)} \), thereby showing that \( w_0 \in \mathcal{B}_{\mu,b} \). Moreover, by the equation (4.11), this element \( w_0 \) belongs to \( \mathcal{N}_{0,\mathcal{D}}^\mu \), thus giving a weak solution to the problem (4.9).

The difference \( \psi = w_1 - w_2 \) of two weak solutions of (4.9) satisfies the conditions \( \psi \in \mathcal{N}_{0,\mathcal{D}}^\mu \subset \mathcal{R}_{0,\mathcal{D}} \) and \( \langle \Psi_{\mu,b} w_1, \psi \rangle_D = 0 \) on \( \mathcal{R}_{0,\mathcal{D}} \). Hence, it follows from the Poincaré’s inequality (3.4) (or (4.1)) that \( \|\psi\|_{L^2(D)} = 0 \), and thus \( \psi = 0 \).

**Theorem 4.3**. If \( (\mu, h) \) is admissible for \( D \), then the Schrödinger map \( s_{\mu,b} : A_{\mu,b} \rightarrow L^2(D) \) is invertible with inverse given by a continuous, self-adjoint and strictly positive isomorphism \( \mathcal{G}_{\mu,b} : L^2(D) \rightarrow \mathcal{N}_{0,\mathcal{D}}^\mu \) (called the weak Green’s operator for (4.9)); in particular, for \( g \in L^2(D) \), the element \( w = \mathcal{G}_{\mu,b}g \) gives the unique weak solution to the Dirichlet-Neumann problem (4.9). Moreover, for admissible \( (\mu_j, h) \), \( j = 1, 2 \), the operators \( \mathcal{G}_{\mu_j,b} \) are commutative.

**Proof**. Given \( g \in L^2(D) \), the association \( g \mapsto \mathcal{G}_{\mu,b}g := w \), the unique weak solution to Dirichlet-Neumann problem (4.9) is well-defined by Theorem 4.2. The equations (4.11)–(4.12) imply that

\[
S_{\mu,b}(\mathcal{G}_{\mu,b}g, \phi) = (g, \bar{\phi})_D, \quad \forall \phi \in \mathcal{R}_{0,\mathcal{D}},
\]

with

\[
\|\mathcal{G}_{\mu,b}g\|_{\mu,b} \leq \text{Const.} \|g\|_{L^2(D)}.
\]
Furthermore, the induced mapping $G_{\mu, h} : L^2(D) \to N_{0, D}^{\mu}$ is a bounded linear operator giving the left-inverse of $s_{\mu, h}|N_{0, D}^{\mu}$. Setting $\phi = G_{\mu, h} f$ with $f \in L^2(D)$ in (4.13) yields
\begin{equation}
S_{\mu, h}(G_{\mu, h} g, G_{\mu, h} f) = (g, G_{\mu, h} f)_D.
\end{equation}

Also, by (4.13) one has
\begin{equation}
S_{\mu, h}(G_{\mu, h} f, G_{\mu, h} g) = (f, G_{\mu, h} g)_D.
\end{equation}

Consequently,
\begin{equation}
(g, G_{\mu, h} f)_D = (f, G_{\mu, h} g)_D = (G_{\mu, h} g, \bar{f})_D,
\end{equation}
thus proving that $G_{\mu, h}$ is a self-adjoint endomorphism of $L^2(D)$. Formula (4.15) shows that the integral $(G_{\mu, h} g, \bar{g})_D$ is real-valued, and thus, by virtue of (3.4), one has, by relations (4.15) and (4.16),
\begin{equation}
(G_{\mu, h} g, \bar{g})_D \geq (c_{D, \mu, h})^{-2} \|G_{\mu, h} g\|^2_{L^2(D)}.
\end{equation}

On the other hand, the Hölder’s and the generalized Poincaré inequalities imply that
\begin{equation}
(G_{\mu, h} g, \bar{g})_D \leq \text{Const.} \|G_{\mu, h} g\|_{L^2(D)} \|g\|_{L^2(D)}.
\end{equation}

Therefore, by the inequality (4.14),
\begin{equation}
\|G_{\mu, h} g\|^2_{L^2(D)} \leq \text{Const.} \|g\|^2_{L^2(D)},
\end{equation}
so that $G_{\mu, h}$ is a continuous endomorphism of $L^2(D)$. Furthermore, $G_{\mu, h}$ is strictly positive, since in fact, the condition $G_{\mu, h} g = 0$ gives $g = s_{\{\mu, h\}}(G_{\mu, h} g) = 0$, so that, if $g \in L^2(D) \setminus \{0\}$, then $(G_{\mu, h} g, \bar{g})_D > 0$. Finally, for admissible $(\mu_j, h_j)$, $j = 1, 2$, one has
\begin{align*}
G_{\mu_1, h} g &= G_{\mu_1, h}(s_{\mu_0, h}(G_{\mu_0, h} g)) = G_{\mu_1, h}([s_{\mu_1, h} - (\mu_1 - \mu_0)] G_{\mu_0, h} g) \\
&= G_{\mu_1, h}(s_{\mu_1, h}(G_{\mu_0, h} g)) - (\mu_1 - \mu_0) G_{\mu_1, h}(G_{\mu_0, h} g),
\end{align*}
and thus, the identity
\begin{equation}
G_{\mu_1, h} - G_{\mu_0, h} = -(\mu_1 - \mu_0) G_{\mu_1, h} G_{\mu_0, h}
\end{equation}
holds. In particular, the commutativity of the $G_{\mu_j, h}$ follows. \qed

5. THE GENERALIZED GARNIR’S THEOREM

In the rest of this paper, assume that $h = (-V, h_1, \cdots, h_{2m})$ is given (as in (3.1)) with $h_D > 0$. The purpose of this section is to consider, for a given $g \in L^2(D)$, the solvability of the operator equation (1.4) with $v$ ranging
over $R_{0,2}$. If $\lambda \in \mathbb{C} \setminus \mathbb{R}_-\text{,}$ a positive number $\mu_0$ is called an admissible value for $(V, D, \lambda)$, provided $(\mu_0, V)$ is admissible for $D$ and $\lambda - \mu_0 \in \mathbb{C} \setminus \mathbb{R}_-\text{.}$ The method of Garnir [4] (compare [3], 5.13.8) can be extended to yield the following:

**Theorem 5.1 (Generalized Garnir’s Theorem).** Let $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$ and $\mu_0$ be an admissible value for $(V, D, \lambda)$. Then there exists a continuous endomorphism $G_{\lambda, h}$ of $L^2(D)$ (independent of the choice of $\mu_0$) taking values in $N_{0,2}^{\mu_0}$, such that the element $w := G_{\lambda, h}g$ gives a weak solution, in the sense of (4.17), of the Dirichlet-Neumann Problem (4.9) (relative to a partition (1.3)), for each $g \in L^2(D)$.

*Proof.* The existence of the endomorphism $G_{\lambda, h}$, for each $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$, will be shown in Lemma 6.1 (of the Appendix). Observe that for such $\lambda$ one has

$$\langle \mathfrak{p}_{\lambda, h}G_{\lambda, h}g, v \rangle_D = \lambda \langle G_{\lambda, h}g, \bar{v} \rangle_D + \langle \mathfrak{p}_{0, h}G_{\lambda, h}g, v \rangle_D$$

$$= \lambda \langle G_{\lambda, h}g, \bar{v} \rangle_D + \langle s_{0, h}(G_{\lambda, h}g), \bar{v} \rangle_D.$$

for all $g, v \in L^2(D)$. As will be shown in Lemma 6.2, the terms on the right-hand side of this equation are holomorphic in $\mathbb{C} \setminus \mathbb{R}_-$ (as a function of $\lambda$). The equation (1.4) itself is known to hold for all $v \in R_{0,2}$, in case $\lambda = \mu$ is a real ”admissible” spectral parameter, namely, $\mu \geq \text{ess sup}_D V$. Therefore, the principle of analytic continuation ensures its validity for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$. □

6. APPENDIX: GREEN OPERATORS
WITH COMPLEX SPECTRAL PARAMETERS

**Lemma 6.1.** Let $\lambda \in \mathbb{C} \setminus \mathbb{R}_-\text{.}$ Then for each admissible $\mu_0$ for $(V, D, \lambda)$, the mapping

$$G_{\lambda, h} := G_{\mu_0, h}[I + (\lambda - \mu_0)G_{\mu_0, h}]^{-1}$$

is a continuous endomorphism of $L^2(D)$ (independent of the choice of $\mu_0$), taking values in $N_{0,2}^{\mu_0}$.

*Proof.* Since $G_{\mu_0, h}$ is a continuous, positive, self-adjoint endomorphism of $L^2(D)$, the inverse map $[I + (\lambda - \mu_0)G_{\mu_0, h}]^{-1}$ exists as a continuous endomorphism of $L^2(D)$ (by [3], Lemma, p. 359). To show that the operator $G_{\lambda, h}$ is well-defined by (6.1), choose admissible $\mu_j > 0$, $j = 0, 1$ for $(V, D, \lambda)$. Then, equivalent to the identity (4.17), one has

$$[I + (\lambda - \mu_0)G_{\mu_0, h}]G_{\mu_1, h} = [I + (\lambda - \mu_1)G_{\mu_1, h}]G_{\mu_0, h},$$

One the other hand one has, clearly,

$$G_{\mu_1, h}[I + (\lambda - \mu_0)G_{\mu_0, h}] = [I + (\lambda - \mu_0)G_{\mu_0, h}]G_{\mu_1, h}$$

$$= [I + (\lambda - \mu_1)G_{\mu_1, h}]G_{\mu_0, h}.$$
Multiplying (6.3) on the right by \([I + (\lambda - \mu_0) \mathcal{G}_{\mu_0, h}]^{-1}\) and on the left by \([I + (\lambda - \mu_1) \mathcal{G}_{\mu_1, h}]^{-1}\) yields
\[
[I + (\lambda - \mu_1) \mathcal{G}_{\mu_1, h}]^{-1} \mathcal{G}_{\mu_1, h} = \mathcal{G}_{\mu_0, h}[I + (\lambda - \mu_0) \mathcal{G}_{\mu_0, h}]^{-1}.
\]

It will be shown below that the following identity holds:
\[
[I + (\lambda - \mu_1) \mathcal{G}_{\mu_1, h}]^{-1} \mathcal{G}_{\mu_1, h} = [I + (\lambda - \mu_0) \mathcal{G}_{\mu_0, h}]^{-1} \mathcal{G}_{\mu_0, h}.
\]

It follows from this that
\[
\mathcal{G}_{\mu_0, h}[I + (\lambda - \mu_0) \mathcal{G}_{\mu_0, h}]^{-1} = [I + (\lambda - \mu_0) \mathcal{G}_{\mu_0, h}]^{-1} \mathcal{G}_{\mu_0, h},
\]
and the same identity holds with \(\mu_0\) replaced by \(\mu_1\). Consequently from this and (6.4) one concludes, as desired, that
\[
\mathcal{G}_{\mu_0, h}[I + (\lambda - \mu_0) \mathcal{G}_{\mu_0, h}]^{-1} = \mathcal{G}_{\mu_1, h}[I + (\lambda - \mu_1) \mathcal{G}_{\mu_1, h}]^{-1}.
\]

To prove the identity (6.4), observe that the operators \(I + (\mu - \mu_1) \mathcal{G}_{\mu_1}\) and \(I + (\mu - \mu_0) \mathcal{G}_{\mu_0}\) commute; hence, setting \(w := [I + (\mu - \mu_1) \mathcal{G}_{\mu_1}]^{-1} v\), one has
\[
[I + (\mu - \mu_1) \mathcal{G}_{\mu_1}][I + (\mu - \mu_0) \mathcal{G}_{\mu_0}] w = [I + (\mu - \mu_0) \mathcal{G}_{\mu_0}] v.
\]

Then,
\[
[I + (\mu - \mu_0) \mathcal{G}_{\mu_0}][I + (\mu - \mu_1) \mathcal{G}_{\mu_1}]^{-1} v = [I + (\mu - \mu_1) \mathcal{G}_{\mu_1}]^{-1} [I + (\mu - \mu_0) \mathcal{G}_{\mu_0}] v.
\]

Therefore,
\[
[I + (\mu - \mu_0) \mathcal{G}_{\mu_0}][I + (\mu - \mu_1) \mathcal{G}_{\mu_1}]^{-1} = [I + (\mu - \mu_1) \mathcal{G}_{\mu_1}]^{-1} [I + (\mu - \mu_0) \mathcal{G}_{\mu_0}].
\]

Consequently, the identity (6.4) will result from the relation
\[
\mathcal{G}_{\mu_0, h} = [I + (\lambda - \mu_0) \mathcal{G}_{\mu_0, h}][I + (\lambda - \mu_1) \mathcal{G}_{\mu_1, h}]^{-1} \mathcal{G}_{\mu_1, h}
\]
\[
= [I + (\lambda - \mu_1) \mathcal{G}_{\mu_1, h}]^{-1} [I + (\lambda - \mu_0) \mathcal{G}_{\mu_0, h}] \mathcal{G}_{\mu_1, h},
\]

namely, the above relation (6.2). \(\square\)

**Lemma 6.2.** Given a pair \((g, \rho) \in L^2(D) \times L^2(D)\), the function \(g_h(\lambda) := (s_{0,h}(\mathcal{G}_{\lambda,h} g), \bar{\rho})_D\) is holomorphic in \(\mathbb{C} \setminus \mathbb{R}_-\). The same is true for the function \(\hat{g}_h(\lambda) := (\mathcal{G}_{\lambda,h} g, \bar{\rho})_D\).

**Proof.** Let \(\lambda_0 \in \mathbb{C} \setminus \mathbb{R}_-\) be given and choose \(\mu_0\) for \((V, D, \lambda_0)\) as in Lemma 6.1. Write \(\lambda' = \lambda_0 + \varepsilon\). Taking \(\varepsilon\) with sufficiently small modulus, the same \(\mu_0\) can be used in defining \(\mathcal{G}_{\lambda',h}\). With \(\hat{\mathcal{K}} := [I + (\lambda_0 - \mu_0) \mathcal{G}_{\mu_0, h}]^{-1}\), the definition (6.1) implies that
\[
g_h(\lambda') - g_h(\lambda_0) = (s_{0,h}(\mathcal{G}_{\mu_0, h} (\hat{\mathcal{K}}^{-1} + \varepsilon \mathcal{G}_{\mu_0, h}^{-1} g), \bar{\rho})_D - (s_{0,h}(\mathcal{G}_{\mu_0, h} \hat{\mathcal{K}} g), \bar{\rho})_D
\]
\[
= (s_{0,h}(\mathcal{G}_{\mu_0, h} (I + \varepsilon \hat{\mathcal{K}} \mathcal{G}_{\mu_0, h}^{-1} \hat{\mathcal{K}} g), \bar{\rho})_D - (s_{0,h}(\mathcal{G}_{\mu_0, h} \hat{\mathcal{K}} g), \bar{\rho})_D.
\]
If $|\varepsilon|$ is so small that
\[
|\varepsilon| \|G_{\mu_0,h}\|_{L^2(D)} \leq (\|R\|_{L^2(D)})^{-1},
\]
then the series $\sum_{n \geq 0} (-1)^n \varepsilon^n (\Re G_{\mu_0,h})^n$ converges (relative to the $L^2$ operator norm) and its sum is none other than $(I + \varepsilon \Re G_{\mu_0,h})^{-1}$. Consequently, one has
\[
s_{0,h}(g_b(\lambda')) - s_{0,h}(g_b(\lambda_0)) = \sum_{n \geq 1} (-1)^n \varepsilon^n (s_{0,h} \circ G_{\mu_0,h}) (\Re G_{\mu_0,h})^n \Re g), \rho)\),
\]
the series on the right being necessarily convergent for $\varepsilon$ with sufficiently small modulus. The holomorphy of $g_b(\lambda)$ in a neighborhood of $\lambda_0$ is thereby established. □

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