ON QUASIEXTREMAL DISTANCE DOMAINS IN METRIC SPACES

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The purpose of this note is to give some properties of a quasiextremal distance domain in a $Q$-Ahlfors regular Loewner space. In the first part of this note, we recall certain properties about domains and balls in a geodesic space that have been proved in [2]. Next, we shall introduce the notion of weakly quasiconformal accessibility. Gehring and Martio have introduced the class of quasiextremal distance domains in the euclidean case. They proved that a quasiextremal distance domain is linearly locally connected and hence, locally connected on the boundary. We shall prove that, in a locally path connected and $Q$-Ahlfors regular Loewner space, a bounded quasiextremal distance domain is locally connected and weakly quasiconformal accessible on the boundary. Finally, we deal with the extension theorem to boundary for quasiconformal mappings on domains in $Q$-Ahlfors regular Loewner spaces, when one of the domains is a quasiextremal distance domain.

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This note is a continuation of my papers [1, 2] and consequently, the notations are the same like in [1, 2].

At the begining, we recall some definitions and results which we shall use in this paper. Next we give certains properties of a quasiextremal distance domain in a $Q$-Ahlfors regular Loewner space. Finally, we deal with boundary extension theorems for quasiconformal mappings when one of the domains is a quasiextremal distance domain.

Throughout the paper, we shall consider only metric spaces. Let $X$ be a metric space. The space $X$ is said to be path connected (or arc connected) if for any two points $x$ and $y$ in $X$ there exists a continuous function $\gamma$ from the unit interval $[0,1]$ to $X$ with $\gamma(0) = x$ and $\gamma(1) = y$. This function is called a path from $x$ to $y$. We say that $x$ and $y$ are the endpoints of $\gamma$ and that $\gamma$ joins (or connects) the points $x$ and $y$. An arc domain $D$ in $X$ is an open arc connected set in $X$. By a continuum, we mean a compact connected set which contains at least two points.

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A path is rectifiable if its length is a finite number. A metric space is said to be rectifiably connected if any two points can be joined by a rectifiable path. By a curve in a metric space $X$ we mean either a path $\gamma$ or its image. We usually abuse notation and call $\gamma$ both the path and its image.

Note that every path connected metric spaces is connected and the image of a path is always a path connected compact space.

A metric space $X$ is said to be locally (path) connected if for each $x \in X$ and each neighbourhood $U$ of $x$ there exists a (arc) connected neighbourhood $V$ of $x$ such that $V \subseteq U$. It is known that $X$ is locally connected if and only if the following property holds: for each open subset $S$ of $X$, each component of $S$ is an open set ([3], Theorem 5.16).

Let $(X, d)$ be a metric space. We denote by $B(x, r) = \{ y \in X, d(x, y) < r \}$ the open ball of center $x$ in $X$ and radius $r < \text{diam}X$ and its closure $\overline{B}(x, r)$. The closed ball in $X$ centered at the point $x$ and with radius $r$ is the set $B[x, r] = \{ y \in X, d(x, y) \leq r \}$. It is known that $\overline{B}(x, r) \subseteq B[x, r]$ and $B(x, r) \subseteq \text{Int}B[x, r]$. Also, we know that open balls of a metric space are open sets and closed balls are closed sets. In this context, we recall the next theorem.

**Remark 1** ([11], Theorem 5.1.7). Suppose that $(X, d)$ is a metric space, $x \in X$ and $r > 0$. Then:

(i) $\partial B(x, r) \subset \{ y, d(x, y) = r \}$;

(ii) $\partial B[x, r] \subset \{ y, d(x, y) = r \}$.

Next, we recall some results which we shall use in this note.

**PROPOSITION 1** ([2], Theorem 2.1). If $X$ is rectifiably connected and locally path connected space and $D$ is a domain in $X$, then $D$ is an arc domain.

**LEMMA 1** ([14], 11.26). Let $E$ be a connected subset of a topological space $X$. If $A \subset X$ and neither $E \cap A$ nor $E \cap (X \setminus A)$ is empty, then $E \cap \partial A \neq \emptyset$.

**Definition 1** ([10], Definitions 2.2.1, 2.4.1). A geodesic path (or, simply, a geodesic) in a metric space $X$ is a path $\gamma$ which connects two points in $X$ and the length of $\gamma$ is equal to the distance between the points. A metric space is called **geodesic space** if every pair of distinct points can be connected by a geodesic.

Also, we recall some results about domains and balls in geodesic metric spaces.

**PROPOSITION 2** ([2], Theorem 2.2). If $X$ is a geodesic metric space then $\overline{B}(x, r) = B[x, r]$, where $x \in X$ and $r < \text{diam}X$.

**Remark 2**. A ball of a connected metric space doesn’t need to be connected, but it is known that in a compact metric space $X$ with the property
“the closure of any open ball \( B(x, r) \) in \( X \) is the closed ball \( B[x, r] \)”, then any open or closed ball is connected. Therefore, if \((X, d)\) is a compact geodesic metric space, by Proposition 2, \( X \) has previously mentioned property and hence, any ball is connected.

**Proposition 3** ([2], Proposition 2.1). If \((X, d)\) is a geodesic space then any ball \( B(a, r) \) with center \( a \) in \( X \) and radius \( r < \text{diam}X \), is arc connected. Moreover, any geodesic which connects two points in \( B(a, r) \) has length \(< 2r \) and it lies in \( B(a, 2r) \).

**Proposition 4** ([2], Corollary 2.1). If \((X, d)\) is a geodesic metric space then \( X \) is locally path connected, and consequently, locally connected.

**Remark 3.** Using Proposition 3 and Remark 2.3 [2] it follows that, if \((X, d)\) is a geodesic space then any ball in \( X \) is arc connected.

Let us consider \((X, d, \mu)\) a metric measure space, \( D \) a domain in \( X \) and \( b \) a boundary point of \( D \).

**Definition 2.** \( D \) is **locally (arc) connected** at \( b \) if \( b \) has arbitrarily small neighbourhoods \( U \) such that \( U \cap D \) is (arc) connected. This means that for each neighbourhood \( V \) of \( b \) there exists a neighbourhood \( U \) of \( b \), \( U \subset V \) such that \( U \cap D \) is (arc) connected.

Let \((X, d, \mu)\) be a metric measure space and let \( \Gamma \) be a family of (nonconstant) curves in \( X \). We say that a Borel function \( \rho : X \to [0, \infty] \) is admissible for \( \Gamma \) if

\[
\int_{\gamma} \rho \, ds \geq 1
\]

for all locally rectifiable paths \( \gamma \) in \( \Gamma \). The set of all admissible function for \( \Gamma \) is denoted by \( F(\Gamma) \).

**Definition 3.** The (conformal) \( p \)-**modulus** of \( \Gamma \), \( 0 < p < \infty \), is defined as:

\[
M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_X \rho^p \, d\mu.
\]

Note in particular, that the modulus of the collection of all non-locally rectifiable paths is zero.

**Definition 4.** A metric measure space \((X, d, \mu)\) is said to be **\( Q \)-Ahlfors regular**, \( Q > 0 \), if there exists a positive constant \( C < \infty \) such that

\[
\frac{r^Q}{C} \leq \mu(B_r) \leq Cr^Q
\]

for every ball \( B_r \) in \( X \) with radius \( r < \text{diam}X \).
Note that if the above relation holds then $X$ has Hausdorff dimension equal to $Q$.

If $E, F, D$ are subsets of $X$ with $E \subset \overline{D}, F \subset \overline{D}$, we denote by $\Delta(E, F; D)$ the family of all paths which join $E$ and $F$ in $D$. The notation $\Delta(E, F)$ means the family of all paths which join $E$ and $F$ in $X$.

**Definition 5.** Suppose that $(X, d, \mu)$ is a metric measure space of Hausdorff dimension $Q$, $(Q > 0)$ and rectifiably connected. We call $X$ a **Loewner space** (or a $Q$-Loewner space) if there is a function $\varphi : (0, \infty) \to (0, \infty)$ so that

$$M_Q(\Delta(E, F)) \geq \varphi(t)$$

whenever $E$ and $F$ are two disjoint, nondegenerate continua in $X$ and

$$t \geq \delta(E, F) = \frac{\text{dist}(E, F)}{\min\{\text{diam}E, \text{diam}F\}}.$$

**Remark 4.** By Theorem 3.6 [5], it follows that if $X$ is a $Q$-Ahlfors regular Loewner space, $Q > 1$, then there exists a decreasing homeomorphism $\psi : (0, \infty) \to (0, \infty)$ such that $M_Q(\Delta(E, F)) \geq \psi(t)$. Moreover, one can select $\psi$ so as to satisfy $\psi(t) \approx \log \frac{1}{t}$, for all sufficiently small $t$, and $\psi(t) \approx (\log t)^{1-Q}$ for all sufficiently large $t$.

**Remark 5.** By combining Theorem 3.6 [5] and Theorem 3.13 [5], we can deduce that a $Q$-Ahlfors regular Loewner space $(Q > 1)$ is linearly locally connected and quasi-convex.

**Definition 6.** Suppose that $(X, d, \mu)$ is a metric measure space of Hausdorff dimension $Q$, $(Q > 0)$. A domain $D$ in $X$ is called $k$-$\text{quasiextremal distance domain}$, $k \geq 1$, if the following condition holds:

$$M_Q(\Delta(E, F)) \leq k \cdot M_Q(\Delta(E, F; D))$$

for any disjoint continua $E$ and $F$ in $D$. We abbreviate this situation by saying that $D$ is $k$-QED. A domain is QED if it is $k$-QED for some constants $k \geq 1$.

Gehring and Martio [4] introduced the class of quasiextremal distance domains in the euclidean case. They proved that a quasiextremal distance domain is linearly locally connected ([4], 2.11) and hence, locally connected on the boundary. In this note, we shall prove that, in a locally path connected and $Q$-Ahlfors regular Loewner space, a quasiextremal distance domain is locally connected and weakly quasiconformal accessible on the boundary.

Next, we shall introduce the definition of a weakly quasiconformal accessible domain at a boundary point.
Definition 7. Suppose that $(X, d, \mu)$ is a metric measure space. A domain $D$ in $X$ is **weakly quasiconformal accessible** at a boundary point $b$ if for each neighbourhood $U$ of $b$, there exist a neighbourhood $V$ of $b$, $V \subset U$, a continuum $F$ in $D$ and a constant $\delta > 0$ such that

$$M_Q(\Delta(E, F; D)) \geq \delta$$

for all open connected sets $E \subset D$, with $E \cap \partial U \neq \emptyset$ and $E \cap \partial V \neq \emptyset$.

The definition of a domain weakly quasiconformal accessible at a boundary point differs from strictly quasiconformal accessibility at a boundary point ([1], Definition 10) since instead of connected sets we have considered open connected sets.

Obviously, strictly quasiconformal accessibility implies weakly quasiconformal accessibility.

Next, we give some results about quasiextremal distance domains. We consider only rectifiable connected spaces.

**Theorem 1.** Suppose that $X$ is a $Q$-Ahlfors regular Loewner space, $Q > 1$, and locally path connected. If $D \subset X$ is a bounded QED domain then $D$ is locally arc connected on the boundary.

**Proof.** Suppose that $D$ is a $k$-QED domain with $k \geq 1$. Since $X$ is rectifiably connected and locally path connected, by Proposition 1, it follows that $D$ is arc connected. For the first part of the proof, we use arguments like in Lemma 13.18 [8]. Consider a boundary point $b$ of $D$. Let us assume that $D$ is not locally arc connected at $b$. Denote by $d_0 = \sup_{x \in D} d(x, b)$.

Since $D$ is not locally arc connected at $b$ it follows that there is $r_0 \in (0, d_0)$ such that $\mu(D \cap B(b, r_0)) < \infty$ and for every neighbourhood $V \subset B(b, r_0)$ of the point $b$, at least one of the following conditions holds:

(*) $V \cap D$ has at least two arc connected components $C_1$ and $C_2$ such that $b \in \overline{C_1} \cap \overline{C_2}$;

(**) $V \cap D$ has infinitely many arc connected components $C_1, \ldots, C_m, \ldots$ such that $b = \lim_{m \to \infty} b_m$ for some $b_m \in C_m$, and $b \notin \overline{C_m}$ for all $m = 1, 2, \ldots$. Note that $\overline{C_m} \cap \partial V \neq \emptyset$ for all $m = 1, 2, \ldots$ in view of the arc connectedness of $D$.

In particular, either (*) or (**) holds for the neighbourhood $V = B(b, r_0)$. Let $r_1 \in (0, r_0)$. We obtain:

\begin{equation}
M_Q\left(\Delta\left(C_i^*, C_j^*; D\right)\right) \leq \frac{\mu(D \cap B(b, r_0))}{[2(r_0 - r_1)]^Q} < \infty,
\end{equation}

where $C_i^* = C_i \cap \overline{B(b, r_1)}$, $C_j^* = C_j \cap \overline{B(b, r_1)}$, for all $i \neq j$. 
We shall prove the last assertion. One of the admissible functions for the family $\Gamma_{ij}$ of all rectifiable curves in $\Delta \left( C^*_i, C^*_j; D \right)$ is $\rho (x) = \frac{1}{2(r_0 - r_1)}$ for all $x \in B(b, r_0) \setminus B(b, r_1)$ and $\rho (x) = 0$ for all $x \in X \setminus \left( B(b, r_0) \setminus B(b, r_1) \right)$, since components $C_i$ and $C_j$ cannot be connected by a path in $V \cap D$, where $V = B(b, r_0)$, and every path connecting $C^*_i$ and $C^*_j$ in $D$ at least twice intersect the ring $B(b, r_0) \setminus B(b, r_1)$. Using Proposition 13.13 [8], we obtain (1).

We denote $\delta = \frac{\mu(D \cap B(b, r_0))}{2(r_0 - r_1)^q}$ and use (1) we get:

$$M_Q \left( \Delta \left( C^*_i, C^*_j; D \right) \right) \leq \delta,$$

for all $i \neq j$.

Since $D$ is $k$-QED and $X$ is a $Q$-Ahlfors regular Loewner space, there exists a decreasing homeomorphism $\varphi : (0, \infty) \to (0, \infty)$ such that

$$k \cdot M_Q \left( \Delta \left( C^*_i, C^*_j; D \right) \right) \geq M_Q \left( \Delta \left( C^*_i, C^*_j \right) \right) \geq \varphi \left( \delta \left( C^*_i, C^*_j \right) \right),$$

where $\delta \left( C^*_i, C^*_j \right) = \frac{\text{dist}(C^*_i, C^*_j)}{\min\{\text{diam}C^*_i, \text{diam}C^*_j\}}$.

By Remark 4, we can choose $r \in (0, r_1)$ such that $\varphi \left( \delta \left( C^*_{i_0}, C^*_{j_0} \right) \right) \geq k (\delta + 1)$ for some pair $i_0$ and $j_0$, $i_0 \neq j_0$, since $\text{dist} \left( C^*_{i_0}, C^*_{j_0} \right) \to 0$ when $r \to 0$ and in the corresponding $C^*_{i_0}$ and $C^*_{j_0}$ there exists at least one curve intersecting $\partial B(b, r_1)$ and $\partial B(b, r)$. Hence, we get the relation:

$$M_Q \left( \Delta \left( C^*_{i_0}, C^*_{j_0}; D \right) \right) \geq \delta + 1,$$

which contradicts the relation (2).

Therefore, the assumption on the absence of the arc connectedness of $D$ at the point $b$ was not true. □

**Theorem 2.** Suppose that $X$ is a $Q$-Ahlfors regular Loewner space, $Q > 1$, and locally path connected. If $D \subset X$ is a bounded QED domain then $D$ is weakly quasiconformal accessible at each boundary point.

**Proof.** Since $X$ is $Q$-Ahlfors regular Loewner space, it follows that there exists a decreasing function $\varphi : (0, \infty) \to (0, \infty)$ such that

$$M_Q(\Delta(E, F)) \geq \varphi(t)$$

whenever $E$ and $F$ are disjoint, nondegenerate continua in $X$ satisfying

$$\text{dist}(E, F) \leq t \cdot \min\{\text{diam}E, \text{diam}F\}.$$

Let $x$ be a boundary point of $D$ and let $U$ be a neighbourhood of $x$. Hence, there exists a ball $B(x, r)$ of center $x$ and radius $r < \text{diam}D$ such that $B(x, r) \subset U$. We denote $S(x, r) = \{y \in X, d(x, y) = r\}$.
We can take two points \( a \in S(x, \frac{2r}{3}) \cap D \) and \( b \in S(x, \frac{5r}{6}) \cap D \). By Proposition 1, we have that \( D \) is an arc domain and hence, there exists a curve \( \gamma \) in \( D \) which connects \( a \) and \( b \). We can pick a subcurve \( \gamma' \) of \( \gamma \) which connects \( S(x, \frac{2r}{3}) \) and \( S(x, \frac{5r}{6}) \) in \( B[x, \frac{5r}{6}] \setminus B(x, \frac{2r}{3}) \). Thus, \( l(\gamma') \geq d(a, b) \geq \frac{r}{6} \).

We set \( F = \gamma \), which is a continuum in \( D \), and \( V = B(x, \frac{r}{3}) \subset U \). Every open connected set \( E \) in \( D \) which intersects \( \partial U \) and \( \partial V \) will also intersects \( S(x, \frac{2r}{3}) \) and \( S(x, \frac{5r}{6}) \). By Proposition 1, \( E \) is arc connected and hence, we can choose a curve \( \gamma_1 \) in \( E \) which connects \( S(x, \frac{r}{2}) \) and \( S(x, \frac{r}{3}) \) and lies in \( B[x, \frac{r}{2}] \setminus B(x, \frac{r}{3}) \). Set \( E' = \gamma_1 \). Note that \( E' \) is a continuum and \( E' \cap F = \emptyset \).

On the other hand,

\[
\min\{\text{diam}E', \text{diam}F\} \geq \frac{r}{6}
\]

that implies

\[
\frac{\text{dist}(E', F)}{\min\{\text{diam}E, \text{diam}F\}} \leq \frac{6\text{dist}(E', F)}{r} \leq \frac{6\text{diam}D}{r}.
\]

Using (*) and the fact that \( D \) is QED, we obtain

\[
k \cdot M_Q(\Delta(E, F; D)) \geq k \cdot M_Q(\Delta(E', F; D)) \geq M_Q(\Delta(E', F)) \geq \varphi \left( \frac{6\text{diam}D}{r} \right) > 0.
\]

We denote \( \delta = \frac{1}{k} \varphi \left( \frac{6\text{diam}D}{r} \right) \) and hence,

\[
M_Q(\Delta(E, F; D)) \geq \delta,
\]

whenever \( E \subset D \) is a open connected set with \( E \cap \partial U \neq \emptyset \) and \( E \cap \partial V \neq \emptyset \). Thus, we get the desired conclusion. \( \square \)

**Remark 6.** By the proof of Theorem 2 it follows that, in the hypothesis of Theorem 2, \( D \) is arc strictly quasiconformally accessible on the boundary.

Let \((X, d, \mu)\) and \((Y, d', \mu')\) be two \( Q \)-Ahlfors regular metric measure spaces with \( 1 \leq Q < \infty \), and two domains \( D \subset X, D' \subset Y \).

**Definition 8.** A homeomorphism \( f : D \to D' \) is called \( K \)-**(geometrically) quasiconformal**, \( K \in [1, \infty) \) if

\[
\frac{M_Q(\Gamma)}{K} \leq M_Q(f(\Gamma)) \leq KM_Q(\Gamma)
\]

for every family \( \Gamma \) of paths in \( D \). We also say that a homeomorphism \( f : D \to D' \) is (geometrically) quasiconformal if \( f \) is \( K \)-**(geometrically) quasiconformal** for some \( K \in [1, \infty) \), i.e., if the distortion of moduli of path families under the mapping \( f \) is bounded.
To simplify the notation, we write quasiconformal mapping instead of geometrically quasiconformal mapping.

**Remark 7.** Using the results of [5] and [8], it follows that all three definitions (metric quasiconformality, quasisymmetry, and geometric quasiconformality) are equivalent for homeomorphism between \( Q \)-Ahlfors regular Loewner spaces \((Q > 1)\).

Recall that if \( \Omega \) and \( \Omega' \) are open sets in metric spaces \((X, d)\) and \((Y, d')\), correspondingly, and \( f : \Omega \to \Omega' \) is a homeomorphism, then the cluster set of \( f \) at every point \( b \in \partial \Omega \),

\[
C(b, f) = \left\{ b' \in Y : b' = \lim_{n \to \infty} f(x_n), x_n \to b, x_n \in \Omega \right\},
\]

belongs to the boundary of the set \( \Omega' \). ([8], Proposition 13.14)

In euclidean case, Herron and Koskela proved that if \( f : D \to D' \) is a quasiconformal mapping, where \( D \) is locally connected at a boundary point \( b \) and \( D' \) is QED, then \( f \) has a continuous extension to \( D \cup \{b\} \) ([7], Theorem 3.3). We give an analogously theorem for quasiconformal mappings on domains in \( Q \)-Ahlfors regular metric measures spaces.

Next, we consider \( X \) and \( Y \) two \( Q \)-Ahlfors regular metric space, \( Q > 1 \), \( D \) a domain in \( X \) and \( D' \) a domain in \( Y \).

**Theorem 3.** Assume that \( f : D \to D' \) is a quasiconformal mapping. If \( D \) is locally connected at a boundary point \( b \), \( D' \) is weakly quasiconformal accessible at least one point of \( C(f, b) \) and \( \overline{D'} \) is compact, then there exists the limit of \( f \) at \( b \).

**Proof.** Suppose that \( C(f, b) \) contains two distinct points \( b_1', b_2' \) and that \( D' \) is weakly quasiconformal accessible at \( b_1' \). Let \( U \) be a neighbourhood of \( b_1' \) such that \( b_2' \notin \overline{U} \). By the definition of weakly quasiconformal accessibility of \( D' \) at \( b_1' \), there exist a neighbourhood \( V \) of \( b_1' \), \( V \subset U \), a continuum \( F \subset D' \) and a positive number \( \delta > 0 \) such that

\[
(1) \quad M(\Delta(E, F; D')) \geq \delta
\]

whenever \( E \) is a subdomain of \( D' \) with \( E \cap \partial V \neq \emptyset \), \( E \cap \partial U \neq \emptyset \).

Let us denote by \( d_0 = \sup_{x \in D} d(x, b) \) and take \( r_0 \in (0, d_0) \) and a natural number \( n_0 \) such that \( 0 < \frac{1}{n_0} < r_0 \). Since \( D \) is locally connected at \( b \) there is a neighbourhood \( V_{n_0} \) of \( b \) such that \( V_{n_0} \subset B_{n_0} = B(b, \frac{1}{n_0}) \) and \( V_{n_0} \cap D \) is connected.

Next, we consider a ball \( B_{n_1} = B(b, \frac{1}{n_1}) \subset V_{n_0} \) with \( n_1 > n_0 \), where \( n_1 \) is a natural integer. Applying the same procedure, we can choose a sequence \( V_{n_0}, V_{n_1}, \ldots \) of neighbourhoods of \( b \) such that every \( C_{n_i} = V_{n_i} \cap D \) is connected,
open and \( B_{n_{i+1}} \subset V_{n_i} \subset B_{n_i} \) for all \( i \geq 0 \). Note that \( \text{dist}(b, C_{n_i}) \to 0 \) as \( i \to \infty \). Set \( \Gamma_i = \Delta(C_{n_i}, f^{-1}(F); D) \) and \( \Gamma'_i = \Delta(f(C_{n_i}), F; D') \). Since \( f \) is a homeomorphism it follows that \( f(C_{n_i}) \) is connected, open and \( f^{-1}(F) \) is a continuum.

On the other hand, by the inclusion \( C(f, b) \subset f(V_{n_i} \cap D) = f(C_{n_i}) \) we obtain that \( b_1', b_2' \in f(C_{n_i}) \) and using the relation (1) we get

\[
M(\Gamma_i') \geq \delta
\]

for any \( i \). In view of quasiconformality of \( f \), there exists a constant \( K > 0 \) such that \( M(\Gamma_i') \leq K \cdot M(\Gamma_i) \) and therefore,

\[
M(\Gamma_i) \geq \frac{\delta}{K}
\]

for any \( i \).

Since \( f^{-1}(F) \) is a continuum, \( f^{-1}(F) \subset D, D \) is open, \( b \in \partial D \) it follows that \( b \notin f^{-1}(F) \) and for \( i \) large, \( B_{n_i} \cap f^{-1}(F) = \emptyset \) and hence, \( C_{n_i} \cap f^{-1}(F) = \emptyset \).

Fix such an \( i \) and denoted by \( r_i = \frac{1}{n_i} \). Every path connecting \( f^{-1}(F) \) and \( C_{n_{i+j}} \) intersects \( \partial B_{n_i} \) and \( \partial B_{n_{i+j}} \) for \( j \geq 1 \). But \( r_k \to 0 \) as \( k \to \infty \) and hence, \( 0 < 2r_{i+j} < r_i < \infty \) for \( j \) large.

By Lemma 3.14 [5] there exists a constant \( C_0 \) (depending only on \( Q \) and the constant \( C \) by \( Q \)-regularity of \( X \) ) so that

\[
M(\Gamma_{i+j}) \leq C_0(\log \frac{r_i}{r_{j+i}})^{1-Q}.
\]

Since \( (\log \frac{r_i}{r_{j+i}})^{1-Q} \to 0 \) as \( j \to \infty \), the relation (4) contradicts the relation (3). Consequently, \( C(f, b) \) has, at most, one point for that \( D' \) is weakly quasiconformal accessible and hence, we get the desired conclusion. \( \square \)

**Corollary 1.** Suppose that \( Y \) is a \( Q \)-Loewner space, locally path connected and let \( f : D \to D' \) be a quasiconformal mapping. If \( D \) is locally connected at a boundary point \( b \), \( D' \) is QED and \( \overline{D'} \) is compact, then there exists the limit of \( f \) at \( b \).

**Proof.** Since \( Y \) is a \( Q \)-Loewner space, locally path connected, by Theorem 2, it follows that \( D' \) is weakly quasiconformal accessible on the boundary. Using Theorem 3, we obtain the desired conclusion. \( \square \)

**Corollary 2.** Suppose that \( Y \) is a \( Q \)-Loewner space, locally path connected and let \( f : D \to D' \) be a quasiconformal mapping. If \( D \) is locally connected on the boundary, \( D' \) is QED and \( \overline{D'} \) is compact, then \( f \) can be extended to a continuous mapping \( f^* : \overline{D} \to \overline{D'} \).
**Definition 9** ([2], Definition 2.3). We say that a ball $B(a,r)$ in a geodesic metric space $X$ is a **GC-ball** if any geodesic which connects any two points $x$ and $y$ in $B[a,r]$ lies in $B[a,r]$.

**Definition 10** ([2], Definition 2.4). We say that a metric space $X$ is a **GC-geodesic space** if the following conditions are satisfied:

(i) $X$ is a geodesic metric space;

(ii) any ball $B(a,r)$ with center $a$ in $X$ and radius $r < diamX$ is a GC-ball.

**Corollary 3.** Suppose that $X$ is a GC-geodesic metric space, $Y$ is a $Q$-Loewner space, locally path connected (or geodesic), and let $f : B(a,r) \to D'$ be a quasiconformal mapping, where $B(a,r)$ is a ball with center $a$ in $X$ and radius $r < diamX$. If $D'$ is a QED domain and $\overline{D'}$ is compact then $f$ can be extended to a continuous mapping $f^* : \overline{D} \to \overline{D'}$.

**Proof.** Since $X$ is a GC-geodesic metric space, by Proposition 2.4 [2], it follows that $B(a,r)$ is locally arc connected on the boundary. Using Corollary 2, we get the desired conclusion. □

To prove the next result, we recall the following proposition:

**Remark 8** ([5], Corollary 4.11). Suppose that $X$ and $Y$ are $Q$-Ahlfors regular metric spaces with $Q > 1$, that $X$ is a Loewner space and that $Y$ is linearly locally connected. Then the inverse of a (metric) quasiconformal map $f$ from $X$ onto $Y$ is (metric) quasiconformal.

**Corollary 4.** Suppose that $X$, $Y$ are $Q$-Ahlfors regular Loewner spaces $(Q > 1)$, locally path connected and $f : D \to D'$ is a quasiconformal mapping, where $D$, $D'$ are QED. If $\overline{D}$ and $\overline{D'}$ are compact sets, then $f$ can be extended to a homeomorphism $f^* : \overline{D} \to \overline{D'}$.

**Proof.** The proof follows by Theorems 1, 2, 3, Remarks 5, 7 and 8. □

**Remark 9** ([6], Theorem 3). In the last three results, we can consider proper and geodesic spaces which admit a $(1,Q)$-Poincaré inequality instead of $Q$-Loewner spaces. In order to prove this statement, we use the next result:

"Suppose that $(X,d,\mu)$ is a proper and geodesic $Q$-regular metric space, $Q > 1$. Then $X$ is a Loewner space if and only if it admits a $(1,Q)$-Poincaré inequality."

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