ON POLYHARMONIC UNIVALENT MAPPINGS

J. CHEN, A. RASILA and X. WANG

In this paper, we introduce a class of complex-valued polyharmonic mappings, denoted by $HS_p(\lambda)$, and its subclass $HS_p^0(\lambda)$, where $\lambda \in [0,1]$ is a constant. These classes are natural generalizations of a class of mappings studied by Goodman in the 1950s. We generalize the main results of Avci and Złotkiewicz from the 1990s to the classes $HS_p(\lambda)$ and $HS_p^0(\lambda)$, showing that the mappings in $HS_p(\lambda)$ are univalent and sense preserving. We also prove that the mappings in $HS_p^0(\lambda)$ are starlike with respect to the origin, and characterize the extremal points of the above classes.

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1. INTRODUCTION

A complex-valued mapping $F = u + iv$, defined in a domain $D \subset \mathbb{C}$, is called *polyharmonic* (or *p-harmonic*) if F is $2p$ ($p \geq 1$) times continuously differentiable, and it satisfies the polyharmonic equation $\Delta^p F = \Delta(\Delta^{p-1}F) = 0$, where $\Delta^1 := \Delta$ is the standard complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \overline{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

It is well known ([7, 20]) that for a simply connected domain D, a mapping F is polyharmonic if and only if F has the following representation:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_k(z),$$

where G_k are complex-valued harmonic mappings in D for $k \in \{1, \cdots, p\}$. Furthermore, the mappings G_k can be expressed as the form

$$G_k = h_k + \overline{g_k}$$

for $k \in \{1, \cdots, p\}$, where all h_k and g_k are analytic in D ([11, 13]).

Obviously, for $p = 1$ (resp. $p = 2$), F is a harmonic (resp. biharmonic) mapping. The biharmonic model arises from numerous problems in science and

engineering ([16, 18, 19]). However, the investigation of biharmonic mappings in the context of the geometric function theory has started only recently ([1–3, 5, 6, 8–10]). The reader is referred to [7, 20] for discussion on polyharmonic mappings, and [11, 13] for the properties of harmonic mappings.

In [4], Avci and Złotkiewicz introduced the class HS of univalent harmonic mappings F with the series expansion:

$$(1.1) \qquad F(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}$$

such that

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1 - |b_1| \ \ (0 \leq |b_1| < 1),$$

and the subclass HC of HS, where

$$\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \leq 1 - |b_1| \ \ (0 \leq |b_1| < 1).$$

The corresponding subclasses of HS and HC with $b_1 = 0$ are denoted by HS^0 and HC^0, respectively. These two classes constitute a harmonic counterpart of classes introduced by Goodman [15]. They are useful in studying questions of the so-called δ-neighborhoods (Ruscheweyh [22], see also [20]) and in constructing explicit k-quasiconformal extensions (Fait *et al.* [14]). In this paper, we define polyharmonic analogues $HS_p(\lambda)$ and $HS_p^0(\lambda)$, where $\lambda \in [0, 1]$, to the above classes of mappings. Our aim is to generalize the main results of [4] to the mappings of the classes $HS_p(\lambda)$ and $HS_p^0(\lambda)$.

This paper is organized as follows. In Section 3, we discuss the starlikeness and convexity of polyharmonic mappings in $HS_p^0(\lambda)$. Our main result, Theorem 1, is a generalization of ([4], Theorem 4). In Section 4, we find the extremal points of the class $HS_p^0(\lambda)$. The main result of this section is Theorem 2, which is a generalization of ([4], Theorem 6). Finally, we consider convolutions and existence of neighborhoods. The main results in this section are Theorems 3 and 4 which are generalizations of ([4], Theorems 7 (i) and 8) respectively. Note that the bounds for convexity and starlikeness for harmonic mappings given in [4] have been recently improved by Kalaj *et al.* [17].

2. PRELIMINARIES

For $r > 0$, write $\mathbb{D}_r = \{z : |z| < r\}$, and let $\mathbb{D} := \mathbb{D}_1$, *i.e.*, the unit disk. We use H_p to denote the set of all polyharmonic mappings F in $\mathbb{D}$ with a series

expansion of the following form:

$$(2.1)\quad F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \left(h_k(z) + \overline{g_k(z)} \right) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{n=1}^{\infty} (a_{n,k} z^n + \overline{b_{n,k} z^n})$$

with $a_{1,1} = 1$ and $|b_{1,1}| < 1$. Let H_p^0 denote the subclass of H_p for $b_{1,1} = 0$ and $a_{1,k} = b_{1,k} = 0$ for $k \in \{2, \cdots, p\}$.

  In [20], J. Qiao and X. Wang introduced the class HS_p of polyharmonic mappings F with the form (2.1) satisfying the conditions

$$(2.2)$$
$$\begin{cases} \displaystyle\sum_{k=1}^{p} \sum_{n=2}^{\infty} (2(k-1)+n)(|a_{n,k}| + |b_{n,k}|) \leq 1 - |b_{1,1}| - \sum_{k=2}^{p} (2k-1)(|a_{1,k}| + |b_{1,k}|), \\ \displaystyle 0 \leq |b_{1,1}| + \sum_{k=2}^{p} (|a_{1,k}| + |b_{1,k}|) < 1, \end{cases}$$

and the subclass HC_p of HS_p, where

$$(2.3)$$
$$\begin{cases} \displaystyle\sum_{k=1}^{p} \sum_{n=2}^{\infty} (2(k-1)+n^2)(|a_{n,k}| + |b_{n,k}|) \leq 1 - |b_{1,1}| - \sum_{k=2}^{p} (2k-1)(|a_{1,k}| + |b_{1,k}|), \\ \displaystyle 0 \leq |b_{1,1}| + \sum_{k=2}^{p} (|a_{1,k}| + |b_{1,k}|) < 1. \end{cases}$$

  The classess of all mappings F in H_p^0 which are of the form (2.1), and subject to the conditions (2.2), (2.3), are denoted by HS_p^0, HC_p^0, respectively.

  Now, we introduce a new class of polyharmonic mappings, denoted by $HS_p(\lambda)$, as follows: A mapping $F \in H_p$ with the form (2.1) is said to be in $HS_p(\lambda)$ if

$$(2.4)$$
$$\begin{cases} \displaystyle\sum_{k=1}^{p} \sum_{n=2}^{\infty} (2(k-1)+n(\lambda n+1-\lambda))(|a_{n,k}| + |b_{n,k}|) \leq 2 - \sum_{k=1}^{p} (2k-1)(|a_{1,k}| + |b_{1,k}|), \\ \displaystyle 1 \leq \sum_{k=1}^{p} (2k-1)(|a_{1,k}| + |b_{1,k}|) < 2, \end{cases}$$

where $\lambda \in [0,1]$. We denote by $HS_p^0(\lambda)$ the class consisting of all mappings F in H_p^0, with the form (2.1), and subject to the condition (2.4). Obviously, if $\lambda = 0$ or $\lambda = 1$, then the class $HS_p(\lambda)$ reduces to HS_p or HC_p, respectively. Similarly, if $p-1 = \lambda = 0$ or $p = \lambda = 1$, then $HS_p(\lambda)$ reduces to HS or HC.

  If

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{n=1}^{\infty} \left(a_{n,k} z^n + \overline{b_{n,k} z^n} \right)$$

and

$$G(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{n=1}^{\infty} \left(A_{n,k} z^n + \overline{B_{n,k} z^n} \right),$$

then the *convolution* $F * G$ of F and G is defined to be the mapping

$$(F * G)(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{n=1}^{\infty} \left(a_{n,k} A_{n,k} z^n + \overline{b_{n,k} B_{n,k} z^n} \right),$$

while the *integral convolution* is defined by

$$(F \diamond G)(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{n=1}^{\infty} \left(\frac{a_{n,k} A_{n,k}}{n} z^n + \frac{\overline{b_{n,k} B_{n,k}}}{n} \overline{z^n} \right).$$

See [12] for similar operators defined on the class of analytic functions.

Following the notation of J. Qiao and X. Wang [20], we denote the δ-neighborhood of F the set by

$$N_\delta(F(z)) = \left\{ G(z) : \sum_{k=1}^{p} \sum_{n=2}^{\infty} (2(k-1) + n)(|a_{n,k} - A_{n,k}| + |b_{n,k} - B_{n,k}|) \right.$$
$$\left. + \sum_{k=2}^{p} (2k-1)(|a_{1,k} - A_{1,k}| + |b_{1,k} - B_{1,k}|) + |b_{1,1} - B_{1,1}| \le \delta \right\},$$

where $G(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{n=1}^{\infty} \left(A_{n,k} z^n + \overline{B_{n,k} z^n} \right)$ and $A_{1,1} = 1$ (see also Ruscheweyh [22]).

3. STARLIKENESS AND CONVEXITY

We say that a univalent polyharmonic mapping F with $F(0) = 0$ is *starlike* with respect to the origin if the curve $F(re^{i\theta})$ is starlike with respect to the origin for each $0 < r < 1$.

PROPOSITION 1 ([21]). *If F is univalent, $F(0) = 0$ and $\frac{d}{d\theta}\left(\arg F(re^{i\theta}) \right) > 0$ for $z = re^{i\theta} \ne 0$, then F is starlike with respect to the origin.*

A univalent polyharmonic mapping F with $F(0) = 0$ and $\frac{d}{d\theta} F(re^{i\theta}) \ne 0$ whenever $0 < r < 1$, is said to be *convex* if the curve $F(re^{i\theta})$ is convex for each $0 < r < 1$.

PROPOSITION 2 ([21]). *If F is univalent, $F(0) = 0$ and $\frac{\partial}{\partial\theta}\left[\arg\left(\frac{\partial}{\partial\theta} F(re^{i\theta}) \right) \right] > 0$ for $z = re^{i\theta} \ne 0$, then F is convex.*

Let X be a topological vector space over the field of complex numbers, and let D be a set of X. A point $x \in D$ is called an *extremal point* of D if it

has no representation of the form $x = ty + (1 - t)z$ $(0 < t < 1)$ as a proper convex combination of two distinct points y and z in D.

Now, we are ready to prove results concerning the geometric properties of mappings in $HS_p^0(\lambda)$.

THEOREM A ([20], Theorems 3.1, 3.2 and 3.3). *Suppose that* $F \in HS_p$. *Then* F *is univalent and sense preserving in* $\mathbb{D}$. *In particular, each member of* HS_p^0 *(or* HC_p^0*) maps* $\mathbb{D}$ *onto a domain starlike w.r.t. the origin, and a convex domain, respectively.*

THEOREM 1. *Each mapping in* $HS_p^0(\lambda)$ *maps the disk* $\mathbb{D}_r$, *where* $r \leq \max\{\frac{1}{2}, \lambda\}$, *onto a convex domain.*

Proof. Let $F \in HS_p^0(\lambda)$, and let $r \in (0,1)$ be fixed. Then $r^{-1}F(rz) \in HS_p^0(\lambda)$ by (2.4), and we have

$$\sum_{k=1}^{p} \sum_{n=2}^{\infty} \left(2(k-1) + n^2\right)(|a_{n,k}| + |b_{n,k}|)r^{2k+n-3}$$

$$\leq \sum_{k=1}^{p} \sum_{n=2}^{\infty} \left(2(k-1) + n(\lambda n + 1 - \lambda)\right)(|a_{n,k}| + |b_{n,k}|) \leq 1$$

provided that

$$\left(2(k-1) + n^2\right)r^{2k+n-3} \leq 2(k-1) + n(\lambda n + 1 - \lambda)$$

for $k \in \{1, \cdots, p\}$, $n \geq 2$ and $0 \leq \lambda \leq 1$, which is true if $r \leq \max\{\frac{1}{2}, \lambda\}$. Then the result follows from Theorem A. $\square$

Immediately from Theorem A, we get the following.

COROLLARY 1. *Let* $F \in HS_p(\lambda)$. *Then* F *is a univalent, sense preserving polyharmonic mapping. In particular, if* $F \in HS_p^0(\lambda)$, *then* F *maps* $\mathbb{D}$ *onto a domain starlike w.r.t. the origin.*

Example 1. Let $F_1(z) = z + \frac{1}{10}z^2 + \frac{1}{5}\overline{z^2}$. Then $F_1 \in HS_1^0(\frac{2}{3})$ is a univalent, sense preserving polyharmonic mapping. In particular, F_1 maps $\mathbb{D}$ onto a domain starlike w.r.t. the origin, and it maps the disk $\mathbb{D}_r$, where $r \leq \frac{2}{3}$, onto a convex domain. See Figure 1.

This example shows that the class $HS_p^0(\lambda)$ of polyharmonic mappings is more general than the class HS^0 which is studied in [4] even in the case of harmonic mappings (*i.e.* p=1).

Example 2. Let $F_2(z) = z + \frac{1}{101}z^2 + \frac{49}{101}\overline{z^2}$. Then $F_2 \in HS_1^0(\frac{1}{100})$ is a univalent, sense preserving polyharmonic mapping. In particular, F_2 maps $\mathbb{D}$

onto a domain starlike w.r.t. the origin, and it maps the disk $\mathbb{D}_r$, where $r \leq \frac{1}{2}$, onto a convex domain. See Figure 1.
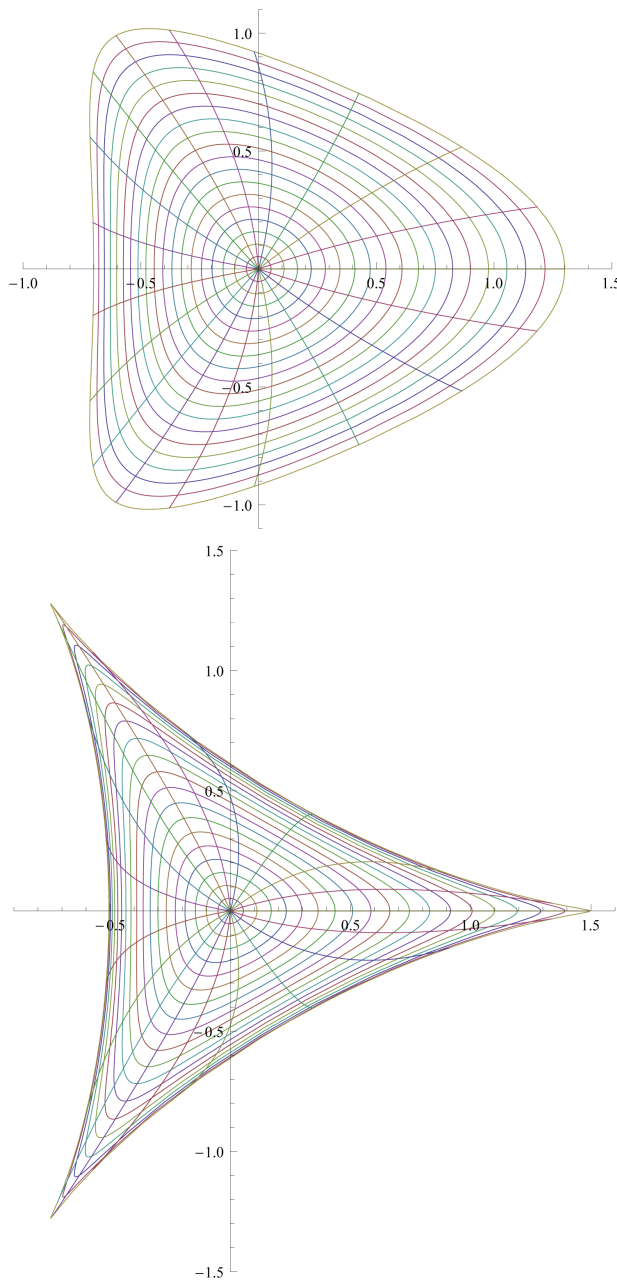

Fig. 1. The images of $\mathbb{D}$ under the mappings $F_1(z) = z + \frac{1}{10}z^2 + \frac{1}{5}\overline{z^2}$ (left) and $F_2(z) = z + \frac{1}{101}z^2 + \frac{49}{101}\overline{z^2}$ (right).

4. **EXTREMAL POINTS**

First, we determine the distortion bounds for mappings in $HS_p(\lambda)$.

LEMMA 1. *Suppose that $F \in HS_p(\lambda)$. Then, the following statements hold:*

(1) For $0 \leq \lambda \leq \frac{1}{2}$,

$$(1 - |b_{1,1}|)|z| - \frac{1 - |b_{1,1}|}{2(1 + \lambda)}|z|^2 \leq |F(z)| \leq (1 + |b_{1,1}|)|z| + \frac{1 - |b_{1,1}|}{2(1 + \lambda)}|z|^2.$$

Equalities are obtained by the mappings

$$F(z) = z + |b_{1,1}|e^{i\mu}\overline{z} + \frac{1 - |b_{1,1}|}{2(1 + \lambda)}e^{i\nu}z^2,$$

for properly chosen real μ and ν;

(2) For $\frac{1}{2} < \lambda \leq 1$,

$$|F(z) \leq (1 + |b_{1,1}|)|z| + \frac{1 - |b_{1,1}| - 3(|a_{1,2}| + |b_{1,2}|)}{2(1 + \lambda)}|z|^2 + (|a_{1,2}| + |b_{1,2}|)|z|^3$$

and

$$|F(z)| \geq (1 - |b_{1,1}|)|z| - \frac{1 - |b_{1,1}| - 3(|a_{1,2}| + |b_{1,2}|)}{2(1 + \lambda)}|z|^2 - (|a_{1,2}| + |b_{1,2}|)|z|^3.$$

Equalities are obtained by the mappings

$$F(z) = z + |b_{1,1}|e^{i\eta}\overline{z} + \frac{1 - |b_{1,1}| - 3(|a_{1,2}| + |b_{1,2}|)}{2(1 + \lambda)}e^{i\varphi}z^2 + (|a_{1,2}| + |b_{1,2}|)e^{i\psi}z|z|^2,$$

for properly chosen real η, φ and ψ.

Proof. Let $F \in HS_p(\lambda)$, where $\lambda \in [0, 1]$. By (2.1), we have

$$|F(z)| \leq (1 + |b_{1,1}|)|z| + \left(\sum_{k=1}^{p}\sum_{n=2}^{\infty}(|a_{n,k}| + |b_{n,k}|) + \sum_{k=2}^{p}(|a_{1,k}| + |b_{1,k}|) \right)|z|^2.$$

For $0 \leq \lambda \leq \frac{1}{2}$, we have

(4.1) $$2(1 + \lambda) \leq 2k - 1,$$

where $k \in \{2, \cdots, p\}$, and

(4.2) $$2(1 + \lambda) \leq 2(k - 1) + n(\lambda n + 1 - \lambda),$$

where $k \in \{1, \cdots, p\}$ and $n \geq 2$. Then (4.1), (4.2) and (2.4) give

$$\sum_{k=1}^{p} \sum_{n=2}^{\infty} (|a_{n,k}| + |b_{n,k}|) + \sum_{k=2}^{p} (|a_{1,k}| + |b_{1,k}|)$$

$$\leq \frac{1}{2(1+\lambda)} \Big(1 - |b_{1,1}| - \sum_{k=1}^{p} \sum_{n=2}^{\infty} \big(2(k-1) + n(\lambda n + 1 - \lambda) $$

$$-2(1+\lambda)\big)(|a_{n,k}| + |b_{n,k}|) - \sum_{k=2}^{p} \big((2k-1) - 2(1+\lambda) \big)(|a_{1,k}| + |b_{1,k}|) \Big),$$

so

$$(1 - |b_{1,1}|)|z| - \frac{1 - |b_{1,1}|}{2(1+\lambda)}|z|^2 \leq |F(z)| \leq (1 + |b_{1,1}|)|z| + \frac{1 - |b_{1,1}|}{2(1+\lambda)}|z|^2.$$

By (2.1), we obtain

$$|F(z)| \leq (1 + |b_{1,1}|)|z| + \left(\sum_{k=1}^{p} \sum_{n=2}^{\infty} (|a_{n,k}| + |b_{n,k}|) + \sum_{k=3}^{p} (|a_{1,k}| + |b_{1,k}|) \right) |z|^2$$

$$+ \quad (|a_{1,2}| + |b_{1,2}|)|z|^3.$$

For $\frac{1}{2} < \lambda \leq 1$, we have

$$(4.3) \qquad\qquad 2(1+\lambda) \leq 2k - 1,$$

where $k \in \{3, \cdots, p\}$, and

$$(4.4) \qquad\qquad 2(1+\lambda) \leq 2(k-1) + n(\lambda n + 1 - \lambda),$$

where $k \in \{1, \cdots, p\}$, $n \geq 2$. Then (4.3), (4.4) and (2.4) imply

$$\sum_{k=1}^{p} \sum_{n=2}^{\infty} (|a_{n,k}| + |b_{n,k}|) + \sum_{k=3}^{p} (|a_{1,k}| + |b_{1,k}|)$$

$$\leq \frac{1}{2(1+\lambda)} \Big(1 - |b_{1,1}| - \sum_{k=1}^{p} \sum_{n=2}^{\infty} \big(2(k-1) + n(\lambda n + 1 - \lambda) $$

$$-2(1+\lambda)\big)(|a_{n,k}| + |b_{n,k}|)$$

$$- \sum_{k=3}^{p} \big(2k - 1 - 2(1+\lambda) \big)(|a_{1,k}| + |b_{1,k}|) - 3(|a_{1,2}| + |b_{1,2}|) \Big).$$

Then

$$|F(z)| \geq (1 - |b_{1,1}|)|z| - \frac{1 - |b_{1,1}| - 3(|a_{1,2}| + |b_{1,2}|)}{2(1+\lambda)}|z|^2 - (|a_{1,2}| + |b_{1,2}|)|z|^3$$

and

$$|F(z)| \leq (1 + |b_{1,1}|)|z| + \frac{1 - |b_{1,1}| - 3(|a_{1,2}| + |b_{1,2}|)}{2(1 + \lambda)}|z|^2 + (|a_{1,2}| + |b_{1,2}|)|z|^3.$$

The proof of this lemma is complete. $\square$

Remark 1. Suppose that $F \in HS_p(\lambda)$ is of the form

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_k(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{n=1}^{\infty} \left(a_{n,k} z^n + \overline{b_{n,k}} \overline{z^n} \right).$$

Then for each $k \in \{1, \cdots, p\}$,

$$|G_k(z)| \leq (|a_{1,k}| + |b_{1,k}|)|z| + \frac{1 - |b_{1,1}|}{2(1 + \lambda)}|z|^2.$$

LEMMA 2. *The family $HS_p(\lambda)$ is closed under convex combinations.*

Proof. Suppose $F_i \in HS_p(\lambda)$ and $t_i \in [0, 1]$ with $\sum_{i=1}^{\infty} t_i = 1$. Let

$$F_i(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{n=1}^{\infty} (a_{n,k}^{(i)} z^n + \overline{b_{n,k}^{(i)}} \overline{z^n}).$$

By Lemma 1, there exists a constant M such that $|F_i(z)| \leq M$, for all $i = 1, \cdots, p$. It follows that $\sum_{i=1}^{\infty} t_i F_i(z)$ is absolutely and uniformly convergent, and by Remark 1, the mapping $\sum_{i=1}^{\infty} t_i F_i(z)$ is polyharmonic. Since $\sum_{i=1}^{\infty} t_i F_i(z)$ is absolutely and uniformly convergent, we have

$$\sum_{i=1}^{\infty} t_i F_i(z) = \sum_{i=1}^{\infty} t_i \sum_{k=1}^{p} |z|^{2(k-1)} \left(\sum_{n=1}^{\infty} a_{n,k}^{(i)} z^n + \sum_{n=1}^{\infty} \overline{b_{n,k}^{(i)}} \overline{z^n} \right)$$

$$= \sum_{k=1}^{p} |z|^{2(k-1)} \left(\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} t_i a_{n,k}^{(i)} z^n + \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} t_i \overline{b_{n,k}^{(i)}} \overline{z^n} \right).$$

By (2.4), we get

$$(4.5) \qquad \sum_{k=1}^{p} \sum_{n=1}^{\infty} \left(2(k-1) + n(\lambda n + 1 - \lambda) \right) \left(\left| \sum_{i=1}^{\infty} t_i a_{n,k}^{(i)} \right| + \left| \sum_{i=1}^{\infty} t_i b_{n,k}^{(i)} \right| \right)$$

$$\leq \sum_{i=1}^{\infty} t_i \left(\sum_{k=1}^{p} \sum_{n=1}^{\infty} \left(2(k-1) + n(\lambda n + 1 - \lambda) \right) (|a_{n,k}^{(i)}| + |b_{n,k}^{(i)}|) \right) \leq 2.$$

It follows from

$$1 \leq \sum_{k=1}^{p} (2k-1) \left(\left| \sum_{i=1}^{\infty} t_i a_{1,k}^{(i)} \right| + \left| \sum_{i=1}^{\infty} t_i b_{1,k}^{(i)} \right| \right) < 2$$

and (4.5) that $\sum_{i=1}^{\infty} t_i F_i \in HS_p(\lambda)$. $\square$

From Lemma 1, we see that the class $HS_p(\lambda)$ is uniformly bounded, and hence, normal. Lemma 2 implies that $HS_p^0(\lambda)$ is also compact and convex. Then there exists a non-empty set of extremal points in $HS_p^0(\lambda)$.

THEOREM 2. *The extremal points of $HS_p^0(\lambda)$ are the mappings of the following form:*

$$F_k(z) = z + |z|^{2(k-1)} a_{n,k} z^n \ or \ F_k^*(z) = z + |z|^{2(k-1)} \overline{b_{m,k}} \overline{z^m},$$

where

$$|a_{n,k}| = \frac{1}{2(k-1) + n(\lambda n + 1 - \lambda)}, \ \ for \ n \geq 2, \ k \in \{1, \cdots, p\},$$

and

$$|b_{m,k}| = \frac{1}{2(k-1) + m(\lambda m + 1 - \lambda)}, \ \ for \ m \geq 2, \ k \in \{1, \cdots, p\}.$$

Proof. Assume that F is an extremal point of $HS_p^0(\lambda)$, of the form (2.1). Suppose that the coefficients of F satisfy the following:

$$\sum_{k=1}^{p} \sum_{n=2}^{\infty} \big(2(k-1) + n(\lambda n + 1 - \lambda)\big)(|a_{n,k}| + |b_{n,k}|) < 1.$$

If all coefficients $a_{n,k} \ (n \geq 2)$ and $b_{n,k} \ (n \geq 2)$ are equal to 0, we let

$$F_1(z) = z + \frac{1}{2(1+\lambda)} z^2 \ \text{and} \ F_2(z) = z - \frac{1}{2(1+\lambda)} z^2.$$

Then F_1 and F_2 are in $HS_p^0(\lambda)$ and $F = \frac{1}{2}(F_1 + F_2)$. This is a contradiction, showing that there is a coefficient, say a_{n_0,k_0} or b_{n_0,k_0}, of F which is nonzero. Without loss of generality, we may further assume that $a_{n_0,k_0} \neq 0$.

For $\gamma > 0$ small enough, choosing $x \in \mathbb{C}$ with $|x| = 1$ properly and replacing a_{n_0,k_0} by $a_{n_0,k_0} - \gamma x$ and $a_{n_0,k_0} + \gamma x$, respectively, we obtain two mappings F_3 and F_4 such that both F_3 and F_4 are in $HS_p^0(\lambda)$. Obviously, $F = \frac{1}{2}(F_3 + F_4)$. Hence, the coefficients of F must satisfy the following equality:

$$\sum_{k=1}^{p} \sum_{n=2}^{\infty} \big(2(k-1) + n(\lambda n + 1 - \lambda)\big)(|a_{n,k}| + |b_{n,k}|) = 1.$$

Suppose that there exists at least two coefficients, say, a_{q_1,k_1} and b_{q_2,k_2} or a_{q_1,k_1} and a_{q_2,k_2} or b_{q_1,k_1} and b_{q_2,k_2}, which are not equal to 0, where $q_1, q_2 \geq 2$. Without loss of generality, we assume the first case. Choosing $\gamma > 0$ small enough and $x \in \mathbb{C}$, $y \in \mathbb{C}$ with $|x| = |y| = 1$ properly, leaving all coefficients of F but a_{q_1,k_1} and b_{q_2,k_2} unchanged and replacing a_{q_1,k_1}, b_{q_2,k_2} by

$$a_{q_1,k_1} + \frac{\gamma x}{2(k_1-1) + q_1(\lambda q_1 + 1 - \lambda)} \ \text{and} \ b_{q_2,k_2} - \frac{\gamma y}{2(k_2-1) + q_2(\lambda q_2 + 1 - \lambda)},$$

or

$$a_{q_1,k_1} - \frac{\gamma x}{2(k_1 - 1) + q_1(\lambda q_1 + 1 - \lambda)} \text{ and } b_{q_2,k_2} + \frac{\gamma y}{2(k_2 - 1) + q_2(\lambda q_2 + 1 - \lambda)},$$

respectively, we obtain two mappings F_5 and F_6 such that F_5 and F_6 are in $HS_p^0(\lambda)$. Obviously, $F = \frac{1}{2}(F_5 + F_6)$. This shows that any extremal point $F \in HS_p^0(\lambda)$ must have the form $F_k(z) = z + |z|^{2(k-1)} a_{n,k} z^n$ or $F_k^*(z) = z + |z|^{2(k-1)} \overline{b_{m,k} z^m}$, where

$$|a_{n,k}| = \frac{1}{2(k - 1) + n(\lambda n + 1 - \lambda)}, \text{ for } n \geq 2, \ k \in \{1, \cdots, p\},$$

and

$$|b_{m,k}| = \frac{1}{2(k - 1) + m(\lambda m + 1 - \lambda)}, \text{ for } m \geq 2, \ k \in \{1, \cdots, p\}.$$

Now, we are ready to prove that for any $F \in HS_p^0(\lambda)$ with the above form must be an extremal point of $HS_p^0(\lambda)$. It suffices to prove the case of F_k, since the proof for the case of F_k^* is similar.

Suppose there exist two mappings F_7 and $F_8 \in HS_p^0(\lambda)$ such that $F_k = tF_7 + (1 - t)F_8 \ (0 < t < 1)$. For $q = 7, 8$, let

$$F_q(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \sum_{n=1}^{\infty} (a_{n,k}^{(q)} z^n + \overline{b_{n,k}^{(q)} z^n}).$$

Then

(4.6) $$|t a_{n,k}^{(7)} + (1 - t) a_{n,k}^{(8)}| = |a_{n,k}| = \frac{1}{2(k - 1) + n(\lambda n + 1 - \lambda)}.$$

Since all coefficients of F_q $(q = 7, 8)$ satisfy, for $n \geq 2$ and $k \in \{1, \cdots, p\}$,

$$|a_{n,k}^{(q)}| \leq \frac{1}{2(k - 1) + n(\lambda n + 1 - \lambda)}, \quad |b_{n,k}^{(q)}| \leq \frac{1}{2(k - 1) + n(\lambda n + 1 - \lambda)},$$

(4.6) implies $a_{n,k}^{(7)} = a_{n,k}^{(8)}$, and all other coefficients of F_7 and F_8 are equal to 0. Thus, $F_k = F_7 = F_8$, which shows that F_k is an extremal point of $HS_p^0(\lambda)$. $\square$

5. CONVOLUTIONS AND NEIGHBORHOODS

Let C_H^0 denote the class of harmonic univalent, convex mappings F of the form (1.1) with $b_1 = 0$. It is known [11] that the below sharp inequalities hold:

$$2|a_n| \leq n + 1, \ 2|b_n| \leq n - 1.$$

It follows from ([11], Theorems 5.14) that if H and G are in C_H^0, then $H * G$ (or $H \diamond G$) is sometime not convex, but it may be univalent or even

convex if one of the mappings H and F satisfies some additional conditions. In this section, we consider convolutions of harmonic mappings $F \in HS_1^0(\lambda)$ and $H \in C_H^0$.

THEOREM 3. *Suppose that* $H(z) = z + \sum_{n=2}^\infty (A_n z^n + \overline{B_n z^n}) \in C_H^0$ *and* $F \in HS_1^0(\lambda)$. *Then for* $\frac{1}{2} \leq \lambda \leq 1$, *the convolution* $F * H$ *is univalent and starlike, and the integral convolution* $F \diamond H$ *is convex.*

Proof. If $F(z) = z + \sum_{n=2}^\infty (a_n z^n + \overline{b_n z^n}) \in HS_1^0(\lambda)$, then for $F * H$, we obtain

$$\sum_{n=2}^\infty n(|a_n A_n| + |b_n B_n|) \leq \sum_{n=2}^\infty n \left(\frac{n+1}{2} |a_n| + \frac{n-1}{2} |b_n| \right)$$

$$\leq \sum_{n=2}^\infty n(\lambda n + 1 - \lambda)(|a_n| + |b_n|) \leq 1.$$

Hence, $(F * H) \in HS^0$. The transformations

$$\int_0^1 \frac{F(z) * H(tz)}{t} dt = (F \diamond H)(z)$$

now show that $(F \diamond H) \in HC^0$. By Theorem A, the result follows. $\square$

Remark 2. The proof of Theorem 3 does not generalize to polyharmonic mappings, when $p \geq 2$. For example, let $p = 2$, and write

$$H(z) = z + \sum_{k=1}^2 |z|^{2(k-1)} \sum_{n=2}^\infty (A_{n,k} z^n + \overline{B_{n,k} z^n})$$

and

$$F(z) = z + \sum_{k=1}^2 |z|^{2(k-1)} \sum_{n=2}^\infty (a_{n,k} z^n + \overline{b_{n,k} z^n}).$$

Suppose that $|A_{n,k}| \leq \frac{n+1}{2}$, $|B_{n,k}| \leq \frac{n-1}{2}$ and $F \in HS_2^0(\lambda)$. Then for $\lambda = 1$, the convolution $F * H$ is univalent and starlike but it is not clear if this is true for $\frac{1}{2} \leq \lambda < 1$. However, the integral convolution $F \diamond H$ is convex for $\frac{1}{2} \leq \lambda \leq 1$.

Example 3. Let $H(z) = \text{Re}\{\frac{z}{1-z}\} + i\text{Im}\{\frac{z}{(1-z)^2}\} \in C_H^0$. Then $H(z)$ maps $\mathbb{D}$ onto the half-plane $\text{Re}\{w\} > \frac{1}{2}$, and let $F(z) = z + \frac{1}{10}z^2 + \frac{1}{5}\overline{z^2} \in HS_1^0(\frac{2}{3})$. Then the convolution $F * H$ is univalent and starlike, and the integral convolution $F \diamond H$ is convex (see Figure 2).
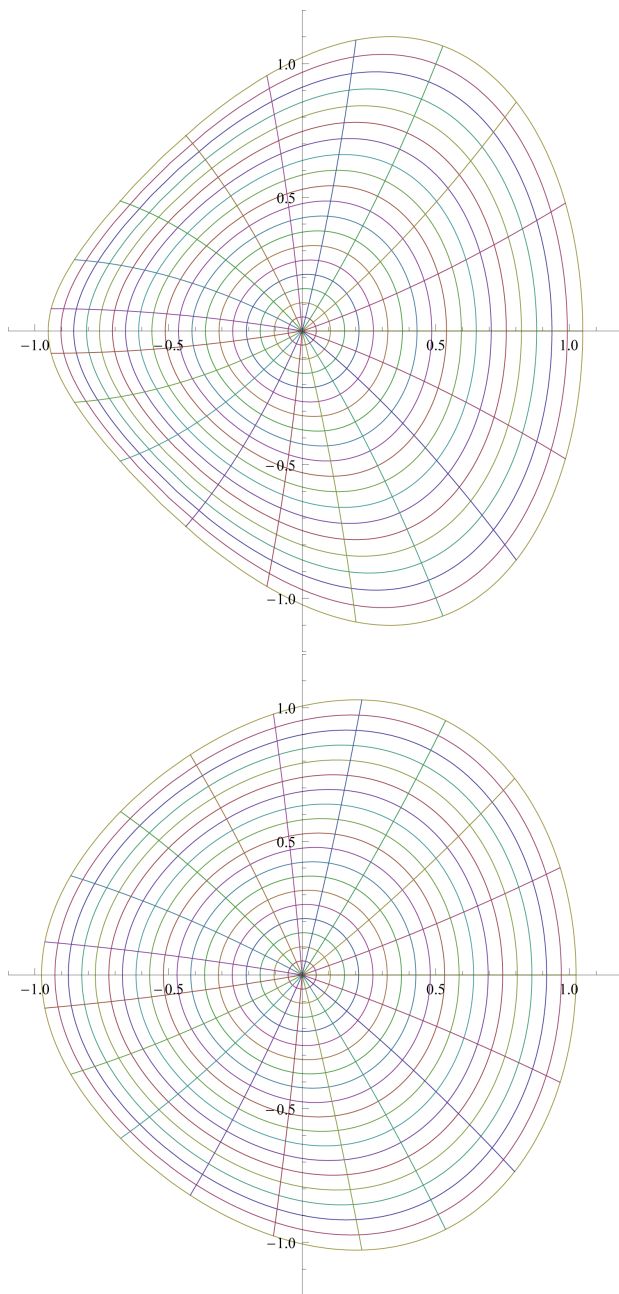
Fig. 2. The images of $\mathbb{D}$ under the mappings $(F * H)(z) = z + \frac{3}{20}z^2 - \frac{1}{10}\overline{z^2}$ (left) and
$(F \diamond H)(z) = z + \frac{3}{40}z^2 - \frac{1}{20}\overline{z^2}$ (right).

Finally, we are going to prove the existence of neighborhoods for mappings in the class $HS_p(\lambda)$.

THEOREM 4. *Assume that $\lambda \in (0,1]$ and $F \in HS_p(\lambda)$. If*

$$\delta \leq \frac{\lambda}{p+\lambda}\left(2 - \sum_{k=1}^{p}(2k-1)(|a_{1,k}| + |b_{1,k}|)\right),$$

then $N_\delta(F) \subset HS_p$.

Proof. Let $H(z) = \sum_{k=1}^{p}|z|^{2(k-1)}\sum_{n=1}^{\infty}(A_{n,k}z^n + \overline{B_{n,k}}\,\overline{z}^n) \in N_\delta(F)$. Then

$$\sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(2(k-1)+n\big)(|A_{n,k}| + |B_{n,k}|) + \sum_{k=2}^{p}(2k-1)(|A_{1,k}| + |B_{1,k}|) + |B_{1,1}|$$

$$\leq \sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(2(k-1)+n\big)(|A_{n,k} - a_{n,k}| + |B_{n,k} - b_{n,k}|)$$

$$+ \sum_{k=2}^{p}(2k-1)(|A_{1,k} - a_{1,k}| + |B_{1,k} - b_{1,k}|) + |B_{1,1} - b_{1,1}|$$

$$+ \sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(2(k-1)+n\big)(|a_{n,k}| + |b_{n,k}|) + \sum_{k=2}^{p}(2k-1)(|a_{1,k}| + |b_{1,k}|) + |b_{1,1}|$$

$$\leq \delta + \sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(2(k-1)+n\big)(|a_{n,k}| + |b_{n,k}|) + \sum_{k=2}^{p}(2k-1)(|a_{1,k}| + |b_{1,k}|) + |b_{1,1}|$$

$$\leq \delta + \frac{p}{p+\lambda}\sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(2(k-1)+n(\lambda n + 1 - \lambda)\big)(|a_{n,k}| + |b_{n,k}|)$$

$$+ \sum_{k=2}^{p}(2k-1)(|a_{1,k}| + |b_{1,k}|) + |b_{1,1}| \leq \delta + \frac{p-\lambda}{p+\lambda} + \frac{\lambda}{p+\lambda}\sum_{k=1}^{p}(2k-1)(|a_{1,k}| + |b_{1,k}|) < 1.$$

Hence, $H \in HS_p$. $\quad\square$

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Hunan Normal University,
Department of Mathematics,
Changsha, Hunan 410081,
People's Republic of China
jiaolongche n@sina.com

Hunan Normal University,
Department of Mathematics,
Changsha, Hunan 410081,
People's Republic of China
and
Aalto University,
Department of Mathematics and Systems Analysis,
P.O. Box 11100, FI-00076 Aalto,
Finland
antti.rasila@iki.fi

Hunan Normal University,
Department of Mathematics,
Changsha, Hunan 410081,
People's Republic of China
xtwang@hunnu.edu.cn