ON SUB-IMPLICATIVE (α, β)-FUZZY IDEALS OF BCH-ALGEBRAS

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In this paper, we introduce the concept of sub-implicative (α, β)-fuzzy ideal of BCH-algebra where α, β are any of $\{\in, \text{q}, \in \vee \text{q}, \in \wedge \text{q}\}$ with $\alpha \neq \in \wedge \text{q}$, by using belongs to and quasi-coincidence between fuzzy points and fuzzy sets.

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1. INTRODUCTION

The theory of fuzzy sets, which was initiated by Zadeh in his seminal paper [33] in 1965, was applied to generalize some of the basic concepts of algebra. The fuzzy algebraic structures play a vital role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, logic, set theory, real analysis, measure theory etc. Chang applied it to the topological spaces in [5]. Das and Rosenfeld applied it to the fundamental theory of fuzzy groups in [9, 27]. In [15], Hong *et al.* applied the concept to BCH-algebras and studied fuzzy dot subalgebras of BCH-algebras. Jun, give characterizations of BCI/BCH-algebras in [17]. In 2001, Jun *et al.* discussed on imaginable T-fuzzy subalgebras and imaginable T-fuzzy closed ideals in BCH-algebras [18]. Kim [22] studied intuitionistic (T, S)-normed fuzzy closed ideals of BCH-algebras. In [20], Jun *et al.* discussed N-structures applied to closed ideals in BCH-algebras. Jun and Park investigated filters of BCH-algebras based on bipolar-valued fuzzy sets in [19]. In [10], Dudek and Rousseau, give the idea of set-theoretic relations and BCH-algebras with trivial structure. In [21], Kazanci *et al.* studied soft set and soft BCH-algebras. Yin initiated the concepts of fuzzy dot ideals and fuzzy dot H-ideals of BCH-algebras in [32]. In [31], Saeid *et al.* discussed fuzzy n-fold ideals in BCH-algebras.

The concept of a BCH-algebra was initiated by Hu and Li in [13] and gave examples of proper BCH-algebras [14]. Some classifications of BCH-algebras were studied by Dudek [11] and Ahmad [1]. They also have studied several

properties of these algebras. Since then several researchers have applied this notion to various mathematical disciplines. In [6], Chaudhry and Din applied it to BCH-algebras, and they considered the ideals and filters in BCH-algebras. The classes of BCH-algebras were studied in [7, 8, 25]. In [29], Saeid and Namdar applied it to BCH-algebras and they measured on n-fold ideals in BCH-algebras and computation algorithms. The study of Smarandache BCH-algebras is made in [30]. In [28], Roh initiated the notion of radical in BCH-algebras.

In [24], Murali defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. Pu and Liu [26] introduced the concept of quasi-coincidence of a fuzzy point with a fuzzy set, plays a fundamental role to make some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, was published in [4]. In particular,$(\in, \in \vee \text{q})$-fuzzy subgroup is an important and useful generalization of the Rosenfeld's fuzzy subgroups [27]. The $(\in \vee \text{q})$-level subsets was discussed in [2]. In [3], Bhakat studied $(\in, \in \vee \text{q})$-fuzzy normal, quasi-normal and maximal subgroups.

In this paper, we show that any sub-implicative fuzzy ideal of a BCH-algebra must be a fuzzy ideal, but the converse does not hold. We also prove that if μ is a fuzzy set of a BCH-algebra X, then μ_t is a sub-implicative ideal of X for all t $\in$ (0.5, 1] if and only if it satisfies

(A) $\forall x \in X, \mu(0) \vee 0.5 \geq \mu(x)$

(B) $\forall x, y, z \in X, \mu(y^2 * x) \vee 0.5 \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z)$.

We further show that every sub-implicative (α, β)-fuzzy ideal of BCH-algebra X is an (α, β)-fuzzy ideal of X. We prove that a fuzzy set μ of a BCH-algebra X is a sub-implicative $(\in, \in \vee \text{q})$-fuzzy ideal of X if and only if it satisfies conditions

(I) $\mu(0) \geq \mu(x) \wedge 0.5, \forall x \in X$

(J) $\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \wedge 0.5, \forall x, y, z \in X$.

We show that in any implicative BCH-algebra X, every $(\in, \in \vee \text{q})$-fuzzy ideal of X is a sub-implicative $(\in, \in \vee \text{q})$-fuzzy ideal of X. We prove that if I is a sub-implicative ideal of X and μ be a fuzzy set of BCH-algebra X such that

(L) μ(x) = 0 for all x $\in X \setminus I$,

(M) μ(x) $\geq$ 0.5 for all x $\in$ I.

Then μ is a sub-implicative (q, $\in \vee$ q)-fuzzy ideal of X.

In Section 2, we recall some ideal and define sub-implicative ideal of BCH-algebra; in Section 3, we review some fuzzy logic concepts and define sub-implicative fuzzy ideal and discuss some of their level ideal; in Section 4, we define (α, β)-fuzzy subalgebra, (α, β)-fuzzy ideal and sub-implicative (α, β)-fuzzy ideal and investigate some of their related properties.

The definitions and terminologies that we used in this paper are standard.

For other notations, terminologies and applications, the readers are referred to [6–8, 10–11, 13–20].

2. SECTION 2 (CRISP SETS – LEVEL 0)

Throughout this paper X, always means a BCH-algebra without any specification. We also include some basic results that are necessary for this paper.

Definition 2.1 ([20]). By a BCH-algebra, we mean an algebra (X, $*$, 0) of type (2, 0) satisfying the axioms:

(BCH-I) $\mathrm{x} * \mathrm{x} = 0$

(BCH-II) $\mathrm{x} * \mathrm{y} = 0$ and $\mathrm{y} * \mathrm{x} = 0$ imply $\mathrm{x} = \mathrm{y}$

(BCH-III) $(\mathrm{x} * \mathrm{y}) * \mathrm{z} = (\mathrm{x} * \mathrm{z}) * \mathrm{y}$

$\forall x, y, z \in X$.

A BCH-algebra X is said to be a BCI-algebra if it satisfies the identity:

(BCI-I) $((\mathrm{x} * \mathrm{y}) * (\mathrm{x} * \mathrm{z})) * (\mathrm{z} * \mathrm{y}) = 0 \qquad \forall x, y, z \in X.$

BCC-algebras (introduced by Komori [23]) are generalizations of BCK-algebras, weak BCC-algebras are generalizations of BCI-algebras. By many mathematicians, especially from China and Korea, weak BCC-algebras are called BZ-algebras ([12, 34–35]). A weak BCC-algebra satisfying the identity

$$0 * \mathrm{x} = 0$$

is called a BCC-algebra. A weak BCC-algebra satisfying the identity

$$(\mathrm{x} * \mathrm{y}) * \mathrm{z} = (\mathrm{x} * \mathrm{z}) * \mathrm{y}$$

is called a BCI-algebra. A weak BCC-algebra which is neither a BCI-algebra or a BCC-algebra is called proper. A BCC-algebra with the condition

$$(\mathrm{x} * (\mathrm{x} * \mathrm{y})) * \mathrm{y} = 0$$

is called a BCK-algebra. BCK-algebras and BCI-algebras are two important classes of logical algebras introduced by Imai and Iseki [16] in 1966. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Since then, a great deal of literature has been produced on the theory of BCK/BCI-algebras. In [13], Hu and Li introduced a wide class of abstract algebras called BCH-algebras based upon BCK/BCI-algebras, and subsequently gave examples of proper BCH-algebras [14]. For the general development of the BCK/BCI/BCH-algebras the subalgebras play a central role. It is known that every BCI-algebra is a BCH-algebra but not conversely. A BCH-algebra X is

called proper if it is not a BCI-algebra. It is known that proper BCH-algebras exist. In any BCH / BCI-algebra X we can define a partial order $\leq$ by putting $x \leq y$ if and only if $x * y = 0$.

Proposition 2.2 ([3, 14, 29]). *In any BCH-algebra X, the following are true:*

(1) $x * (x * y) = y$
(2) $0 * (x * y) = (0 * x) * (0 * y)$
(3) $x * 0 = x$
(4) $x = 0$ implies $x = 0$

$\forall x, y \in X$.

Definition 2.3 ([15]). A nonempty subset S of a BCH-algebra X is called a subalgebra of X if it satisfies

$$x * y \in S, \qquad \forall\ x, y \in S.$$

Definition 2.4 ([29]). A nonempty subset I of a BCH-algebra X is called an ideal of X if it satisfies (I1) and (I2), where

(I1) $0 \in I$,
(I2) $x * y \in I$ and $y \in I$ imply $x \in I$,

$\forall\ x, y \in X$.

For any elements x and y of a BCH-algebra, $x^n * y$ denotes

$$x * (\ldots\ (x * (x * y))\ \ldots),$$

in which x occurs n times.

Definition 2.5. A nonempty subset I of a BCH-algebra X is called a sub-implicative ideal of X if it satisfies (I1) and (I3), where

(I1) $0 \in I$,
(I3) $(((x^2 * y) * (y * x)) * z) \in I$ and $z \in I$ imply $y^2 * x \in I$,

$\forall\ x, y, z \in X$.

Proposition 2.6. *Any sub-implicative ideal of a BCH-algebra is an ideal, but the converse does not hold.*

Proof. Suppose I is a sub-implicative ideal of X and for all $x, y, z \in X$, we have

$(((x^2 * y) * (y * x)) * z) \in I$ and $z \in I$ imply $y^2 * x \in I$.
Put $y = x$ in above we get
$(((x^2 * x) * (x * x)) * z) \in I$ and $z \in I$ imply $x^2 * x \in I$
$(x * 0) * z \in I$ and $z \in I$ imply $x \in I$ (BCH-I)
$x * z \in I$ and $z \in I$ imply $x \in I$ (by Proposition 2.2(3)).

This means that I satisfies (I2). Combining with (I1) implies that I is an ideal. The last part is shown by the example.

Example 2.7. Let $X = \{0, 1, 2, 3\}$ be a BCH-algebra with Cayley table as follows:

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

$I = \{0\}$ is an ideal of X, but not a sub-implicative ideal of X since

$(((x^2 * y) * (y * x)) * z) \in I$ and $z \in I$ imply $y^2 * x \in I$

Putting $x = 2$, $y = 1$, $z = 0$ in above, we get

$(((2^2 * 1) * (1 * 2)) * 0) \in I$ and $0 \in I$ imply $1^2 * 2 \in I$
$(0 * 0) * 0 \in I$ and $0 \in I$ imply $1 \notin I$
$0 * 0 \in I$ and $0 \in I$ imply $1 \notin I$
$0 \in I$ and $0 \in I$ imply $1 \notin I$
$0 \in \{0\}$ and $0 \in \{0\}$ imply $1 \notin \{0\}$. □

THEOREM 2.8. *Let X be a BCH-algebra. Then an ideal I of X is a sub-implicative ideal of X if and only if the condition*

$$(\forall x, y \in X)((x^2 * y) * (y * x) \in I \Rightarrow y^2 * x \in I)$$

is satisfied.

Proof. Straightforward. □

3. SECTION 3 (LEVEL 1 OF FUZZIFICATION)

We now review some fuzzy logic concepts. A fuzzy set μ of a universe X is a function from X into the unit closed interval [0, 1], that is $\mu : X \to [0, 1]$.

Definition 3.1 ([9]). For a fuzzy set of a BCH-algebra X and $t \in (0, 1]$, the crisp set

$$\mu_t = \{x \in X \mid \mu(x) \geq t\}$$

is called the level subset of μ.

Definition 3.2 ([22]). Let X be a BCH-algebra. A fuzzy set μ of X is said to be a fuzzy subalgebra of X if it satisfies

$$\mu(x * y) \geq \mu(x) \wedge \mu(y), \tag{1}$$

$\forall x, y \in X$.

Definition 3.3. Let μ be a fuzzy set of a BCH-algebra X. Then μ is a fuzzy subalgebra of X if and only if $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is a subalgebra of X for all t $\in$ (0, 1], when $\mu_t \neq \phi$.

Proof. Straightforward. □

Definition 3.4 ([31]). A fuzzy set μ of a BCH-algebra X is called a fuzzy ideal of X if it satisfies (F1) and (F2), where

(F1) $\mu(0) \geq \mu(x)$,

(F2) $\mu(x) \geq \mu(x * y) \wedge \mu(y)$,

$\forall$ x, y $\in$ X.

THEOREM 3.5. *A fuzzy set μ of a BCH-algebra X is a fuzzy ideal of X if and only if $\mu_t \neq \phi$ is an ideal of X.*

Proof. The proof of the following theorem is obvious. □

Definition 3.6. A fuzzy set μ of a BCH-algebra X is called a sub-implicative fuzzy ideal of X if it satisfies (F1) and (F3), where

(F1) $\mu(0) \geq \mu(x)$,

(F3) $\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z)$,

$\forall$ x, y, z $\in$ X.

THEOREM 3.7. *A fuzzy set μ of a BCH-algebra X is a sub-implicative fuzzy ideal of X if and only if $\mu_t \neq \phi$ is a sub-implicative ideal of X.*

Proof. Let μ be a sub-implicative fuzzy ideal of X and $\mu_t \neq \phi$ for t $\in$ (0, 1]. Since $\mu(0) \geq \mu(x) \geq t$ for x $\in \mu_t$, we get $0 \in \mu_t$. If

$$((x^2 * y) * (y * x)) * z \in \mu_t \quad \text{and} \quad z \in \mu_t,$$

then

$$\mu(((x^2 * y) * (y * x)) * z) \geq t \quad \text{and} \quad \mu(z) \geq t.$$

It follows from (F3) that

$$\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \geq t \wedge t \geq t.$$

Hence,

$$y^2 * x \in \mu_t.$$

This shows that μ_t is a sub-implicative ideal of X by (I3). Conversely, suppose that for each t $\in$ (0, 1], μ_t is either empty or a sub-implicative ideal of X. For any x $\in$ X, setting $0 \neq \mu(x) = t$, then x $\in \mu_t$. Since $\mu_t (\neq \phi)$ is a sub-implicative ideal of X, we have $0 \in \mu_t$ and hence,

$$\mu(0) \geq t = \mu(x).$$

If $\mu(\mathrm{x}) = 0$ then obviously

$$\mu(0) \geq 0 = \mu(x).$$

Thus, $\mu(0) \geq \mu(x)$ for all x $\in$ X. Now, we prove that μ satisfies (F3). If not, then there exist $x_1, y_1, z_1 \in X$ such that

$$\mu(y_1^2 * x_1) \leq \mu(((x_1^2 * y_1) * (y_1 * x_1)) * z_1) \wedge \mu(z_1)$$

Select t $\in$ (0, 1] such that

$$\mu(y_1^2 * x_1) < t \leq \mu(((x_1^2 * y_1) * (y_1 * x_1)) * z_1) \wedge \mu(z_1).$$

Hence,

$$((x_1^2 * y_1) * (y_1 * x_1)) * z_1 \in \mu_t \text{ and } z_1 \in \mu_t, \text{ but } y_1^2 * x_1 \notin \mu_t,,$$

which is a contradiction. Therefore,

$$\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z)$$

Consequently μ is a sub-implicative fuzzy ideal of X. □

Next, we investigate the relations between sub-implicative fuzzy ideals and other fuzzy ideals of X.

THEOREM 3.8. *Any sub-implicative fuzzy ideal of a BCH-algebra is a fuzzy ideal, but the converse does not hold.*

Proof. Suppose μ is a sub-implicative fuzzy ideal of X and let y = x in (F3). We obtain

$\mu(\mathrm{x}) = \mu(x^2 * x)$
$\geq \mu(((x^2 * x) * (x * x)) * z) \wedge \mu(z)$
$\geq \mu((x * 0) * z) \wedge \mu(z)$ (BCH-I)
$\geq \mu(x * z) \wedge \mu(z)$ (by Proposition 2.2(3))

for all x, z $\in$ X. This means that μ is a fuzzy ideal of X. □

The last part is shown by the following example.

Example 3.9. Let X = $\{0, 1, 2, 3\}$ be a BCH-algebra with Cayley table as follows:

*	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

We define a map $\mu : X \to [0, 1]$ by $\mu(0) = 1$, and $\mu(1) = \mu(2) = \mu(3) = 1/2$. Then μ is a fuzzy ideal of X. But μ is not a sub-implicative fuzzy ideal of X, because

Put x = 2, y = 1, z = 0 in (F3) we get
$\mu(1^2 * 2) \geq \mu(((2^2 * 1) * (1 * 2)) * 0) \wedge \mu(0)$
$\mu(1) \geq \mu((0 * 0) * 0) \wedge \mu(0)$
$\mu(1) \geq \mu(0 * 0) \wedge \mu(0)$
$\mu(1) \geq \mu(0) \wedge \mu(0)$
$1/2 \geq 1 \wedge 1$
$1/2 \geq 1$
$1/2 \not\geq 1$.

THEOREM 3.10. *Let μ be a fuzzy ideal of a BCH-algebra X. Then*

$$x * y \leq z \text{ implies } \mu(x) \geq \mu(y) \wedge \mu(z)$$

for all x, y, z $\in$ X.

Proof. Straightforward. □

THEOREM 3.11. *A fuzzy ideal μ of a BCH-algebra X is a sub-implicative fuzzy ideal of X if and only if it satisfies the condition*

$$\mu(y^2 * x) \geq \mu((x^2 * y) * (y * x))$$

for all x, y $\in$ X.

Proof. Suppose μ is a sub-implicative fuzzy ideal of X. By (F3), we have

$$\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z).$$

Put z = 0 in above, we get
$\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * 0) \wedge \mu(0)$
$\mu(y^2 * x) \geq \mu((x^2 * y) * (y * x)) \wedge \mu(0)$ (by Proposition 2.2(3))
$\mu(y^2 * x) \geq \mu((x^2 * y) * (y * x))$ (by using condition (F1))
Conversely assume that for all x, y $\in$ X, we have

$$\mu(y^2 * x) \geq \mu((x^2 * y) * (y * x)).$$

Since

$$((x^2 * y) * (y * x)) * (((x^2 * y) * (y * x)) * z) \leq z,$$

by Theorem 3.10 we obtain

$$\mu((x^2 * y) * (y * x)) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z).$$

By given condition we have

$$\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z).$$

Hence, μ is a sub-implicative fuzzy ideal of X. □

4. SECTION 4 (LEVEL 2.1 OF FUZZIFICATION)

In this section, we define (α, β)-fuzzy subalgebra, (α, β)-fuzzy ideal and sub-implicative (α, β)-fuzzy ideal and investigate some of their related properties.

A fuzzy set μ of a BCH-algebra X having the form [31]

$$\mu(\mathrm{y}) = \begin{cases} t \in (0,1] & \text{if } \mathrm{y} = \mathrm{x}, \\ 0 & \text{if } \mathrm{y} \neq \mathrm{x}, \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and a fuzzy set μ of a set X, Pu and Liu [26] gave meaning to the symbol $x_t \alpha \mu$, where $\alpha \in \{\in, \mathrm{q}, \in \vee \mathrm{q}, \in \wedge \mathrm{q}\}$.

A fuzzy point x_t is said to belong to (resp., quasi-coincident with) a fuzzy set μ, written as $x_t \in \mu$ (resp., $x_t q \mu$) if $\mu(\mathrm{x}) \geq \mathrm{t}$ (resp., $\mu(\mathrm{x}) + \mathrm{t} > 1$).

To say that $x_t \in \vee q\mu$ ($x_t \in \wedge q\mu$) means that $x_t \in \mu$ or $x_t q \mu$ ($x_t \in \mu$ and $x_t q \mu$). To say that $x_t \overline{\alpha} \mu$ means that $x_t \alpha \mu$ does not hold.

In what follows let α and β denote any one of $\in$, q, $\in \vee$ q, $\in \wedge$ q unless otherwise specified.

PROPOSITION 4.1. *For any fuzzy set μ of X, the condition (1) is equivalent to the following condition*

$$x_{t_1}, y_{t_2} \in \mu \Rightarrow (x * y)_{t_1 \wedge t_2} \in \mu, \tag{2}$$

for all x, $y \in X$ and $t_1, t_2 \in$ (0, 1].

Proof. Straightforward. □

A fuzzy set μ of a BCH-algebra X is said to be an (α, β)-fuzzy subalgebra of X, where $\alpha \neq \in \wedge$ q , if it satisfies the following condition

$$x_{t_1} \alpha \mu, y_{t_2} \alpha \mu \Rightarrow (x * y)_{t_1 \wedge t_2} \beta \mu \tag{3}$$

for all $t_1, t_2 \in$ (0, 1].

THEOREM 4.2. *Let μ be a fuzzy set of a BCH-algebra X. Then μ_t is a sub-implicative ideal of X for all $t \in$ (0.5, 1] if and only if it satisfies*

(A) $\mu(0) \vee 0.5 \geq \mu(x)$,

(B) $\mu(y^2 * x) \vee 0.5 \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z)$,

$\forall x, y, z \in X$.

Proof. Suppose that μ_t is a sub-implicative ideal of X for all t $\in$ (0.5, 1]. If there is $a \in X$ such that the condition (A) is not valid, that is, there exists $a \in X$ such that

$$\mu(0) \vee 0.5 \leq \mu(a)$$

then

$$\mu(a) \in (0.5, 1] \quad \text{and} \quad a \in \mu_{\mu(a)}.$$

But

$$\mu(0) < \mu(a) \quad \text{implies} \quad 0 \notin \mu_{\mu(a)},$$

a contradiction. Hence, (A) is valid. Suppose that

$$\mu(b^2 * a) \vee 0.5 < \mu(((a^2 * b) * (b * a)) * c) \wedge \mu(c) = \mathrm{u}$$

for some $a, b, c \in X$. Then

$$\mathrm{u} \in (0.5, 1] \quad \text{and} \quad ((a^2 * b) * (b * a)) * c \in \mu_{\mathrm{u}}, c \in \mu_{\mathrm{u}}.$$

But

$$b^2 * a \notin \mu_u \quad \text{since} \quad \mu(b^2 * a) < u.$$

This is a contradiction, and therefore (B) is valid.

Conversely, suppose that μ satisfies conditions (A) and (B). Let t $\in$ (0.5, 1]. For any x $\in \mu_t$, we have

$$\mu(0) \vee 0.5 \geq \mu(x) \geq t > 0.5$$

and so,

$$\mu(0) \geq t.$$

Thus, $0 \in \mu_t$. Let x, y, z $\in$ X be such that

$$((x^2 * y) * (y * x)) * z \in \mu_t, \mathrm{z} \in \mu_t.$$

Then

$$\begin{aligned} \mu(y^2 * x) \vee 0.5 &\geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \\ &\geq \mathrm{t} \wedge \mathrm{t} \\ &\geq \mathrm{t} \\ &> 0.5. \end{aligned}$$

Thus,

$$\mu(y^2 * x) \geq \mathrm{t},$$

that is,

$$y^2 * x \in \mu_t.$$

Hence, μ_t is a sub-implicative ideal of X. □

Definition 4.3. A fuzzy set μ of a BCH-algebra X is called an (α, β)-fuzzy ideal of X, where $\alpha \neq \in \wedge$ q, if it satisfies

(C) $x_t \alpha \mu \Rightarrow 0_t \beta \mu$,

(D) $(x * y)_{t_1} \alpha \mu$, $y_{t_2} \alpha \mu \Rightarrow x_{t_1 \wedge t_2} \beta \mu$,

$\forall t, t_1, t_2 \in (0, 1]$.

Example 4.4. Let X = $\{0, a, b, c, d\}$ be a BCH-algebra with Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	b	d	0

(1) Let μ be a fuzzy set of X defined by $\mu(0) = 0.7$, $\mu(a) = \mu(\mathrm{c}) = 0.3$ and $\mu(\mathrm{b}) = \mu(\mathrm{d}) = 0.2$. Simple calculations show that μ is an $(\in, \in \vee \mathrm{q})$-fuzzy ideal as well as a fuzzy ideal of X.

(2) Let ν be a fuzzy set of X defined by $\nu(0) = 0.6$, $\nu(a) = \nu(\mathrm{c}) = 0.7$ and $\nu(\mathrm{b}) = \nu(\mathrm{d}) = 0.2$. Simple calculations show that ν is an $(\in, \in \vee \mathrm{q})$-fuzzy ideal which is not a fuzzy ideal of X.

THEOREM 4.5. *Every fuzzy ideal of BCH-algebra X is an (α, β)-fuzzy ideal of X.*

Proof. Obvious. □

Definition 4.6. A fuzzy set μ of a BCH-algebra X is called a sub-implicative (α, β)-fuzzy ideal of X, where $\alpha \neq \in \wedge \mathrm{q}$, if it satisfies

(E) $x_t \alpha \mu \Rightarrow 0_t \beta \mu$,

(F) $(((x^2 * y) * (y * x)) * z)_{t_1} \alpha \mu,\ z_{t_2} \alpha \mu \Rightarrow (y^2 * x)_{t_1 \wedge t_2} \beta \mu$,

$\forall t, t_1, t_2 \in (0, 1]$.

Example 4.7. Let $\mathrm{X} = \{0, 1, 2\}$ be a BCH-algebra with Cayley table as follows:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

Let μ be a fuzzy set of X defined by $\mu(0) = \mu(1) = 0.9$ and $\mu(2) = 0.4$. Simple calculations show that μ is a sub-implicative $(\in, \in \vee \mathrm{q})$-fuzzy ideal of X.

Example 4.8. Let $\mathrm{X} = \{0, 1, 2, 3\}$ be a BCH-algebra with Cayley table as follows:

$*$	0	1	2	3
0	0	0	0	3
1	1	0	0	3
2	2	2	0	3
3	3	3	3	0

We define a map $\mu : X \to [0, 1]$ by $\mu(0) = 1$, and $\mu(1) = \mu(2) = \mu(3) = 0.4$. Then μ is an $(\in, \in \vee q)$-fuzzy ideal of X, but μ is not a sub-implicative $(\in, \in \vee q)$-fuzzy ideal of X.

PROPOSITION 4.9. *Every sub-implicative (α, β)-fuzzy ideal of BCH-algebra X is an (α, β)-fuzzy ideal of X.*

Proof. Let μ be an sub-implicative (α, β)-fuzzy ideal of X. Then for all $t_1, t_2 \in (0, 1]$ and x, y, z $\in$ X, we have

$$(*) \qquad (((x^2 * y) * (y * x)) * z)_{t_1} \alpha\mu, z_{t_2} \alpha\mu \Rightarrow (y^2 * x)_{t_1 \wedge t_2} \beta\mu$$

Let y = x in $(*)$, we get

$$(((x^2 * x) * (x * x)) * z)_{t_1} \alpha\mu, z_{t_2} \alpha\mu \Rightarrow (x^2 * x)_{t_1 \wedge t_2} \beta\mu$$
$$((x * 0) * z)_{t_1} \alpha\mu, z_{t_2} \alpha\mu \Rightarrow x_{t_1 \wedge t_2} \beta\mu \quad \text{(BCH-I)}$$
$$(x * z)_{t_1} \alpha\mu, z_{t_2} \alpha\mu \Rightarrow x_{t_1 \wedge t_2} \beta\mu \quad \text{(by Proposition 2.2(3))}$$

This means that μ satisfies (D). Combining with (C) implies that μ is an (α, β)-fuzzy ideal of X. □

THEOREM 4.10. *For any fuzzy set μ of BCH-algebra X, the condition (F1) and (F3) are equivalent to the conditions*

(G) $x_t \in \mu \Rightarrow 0_t \in \mu$,

(H) $(((x^2 * y) * (y * x)) * z)_{t_1} \in \mu,\ z_{t_2} \in \mu \Rightarrow (y^2 * x)_{t_1 \wedge t_2} \in \mu$,

$\forall t, t_1, t_2 \in (0, 1]$ *respectively.*

Proof. Suppose that (F1) is holds and let x $\in$ X and t $\in$ (0, 1] be such that $x_t \in \mu$. Then $\mu(0) \geq \mu(x) \geq t$, and so $0 \in \mu_t$. Assume that (G) is true. Since

$$x_{\mu(x)} \in \mu, \forall x \in X,$$

it follows from (G) that $0_{\mu(x)} \in \mu$ so that

$$\mu(0) \geq \mu(x), \forall x \in X.$$

Suppose that the condition (F3) holds. Let x, y, z $\in$ X and $t_1, t_2 \in (0, 1]$ be such that

$$(((x^2 * y) * (y * x)) * z)_{t_1} \in \mu, z_{t_2} \in \mu.$$

Then

$$\mu(((x^2 * y) * (y * x)) * z) \geq t_1 \quad \text{and} \quad \mu(z) \geq t_2.$$

It follows from (F3) that

$$\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z)$$
$$\geq t_1 \wedge t_2.$$

So,

$$(y^2 * x)_{t_1 \wedge t_2} \in \mu.$$

Finally, suppose that (H) is holds. Note that for every x, y, z ∈ X,

$$(((x^2 * y) * (y * x)) * z)_{\mu(((x^2*y)*(y*x))*z} \in \mu \quad \text{and} \quad z_{\mu(z)} \in \mu.$$

Hence,

$$(y^2 * x)_{\mu(((x^2*y)*(y*x))*z)\wedge\mu(z)} \in \mu \quad \text{by (H)},$$

and thus,

$$\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z). \quad \square$$

THEOREM 4.11. *Every sub-implicative* $(\in \vee q, \in \vee q)$*-fuzzy ideal of a BCH-algebra X is a sub-implicative* $(\in, \in \vee q)$*-fuzzy ideal of X.*

Proof. Let μ be a sub-implicative (∈ ∨ q, ∈ ∨ q)-fuzzy ideal of X. Let x ∈ X and t ∈ (0, 1] be such that $x_t \in \mu$. Then

$$x_t \in \vee q\mu$$

and so

$$0_t \in \vee q\mu.$$

Let x, y, z ∈ X and $t_1, t_2 \in (0, 1]$ be such that

$$(((x^2 * y) * (y * x)) * z)_{t_1} \in \mu \quad \text{and} \quad z_{t_2} \in \mu.$$

Then

$$(((x^2 * y) * (y * x)) * z)_{t_1} \in \vee q\mu \quad \text{and} \quad z_{t_2} \in \vee q\mu.$$

This implies that

$$(y^2 * x)_{t_1 \wedge t_2} \in \vee q\mu.$$

Hence, μ is a sub-implicative $(\in, \in \vee q)$-fuzzy ideal of X. $\square$

THEOREM 4.12. *A fuzzy set* μ *of a BCH-algebra X is a sub-implicative* $(\in, \in \vee q)$*-fuzzy ideal of X if and only if it satisfies conditions*

(I) $\mu(0) \geq \mu(x) \wedge 0.5$,

(J) $\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \wedge 0.5$,

$\forall x, y, z \in X$.

Proof. Assume that μ is a sub-implicative $(\in, \in \vee q)$-fuzzy ideal of X. Let x ∈ X and suppose that $\mu(x) < 0.5$. If $\mu(0) < \mu(x)$, then $\mu(0) < t \leq \mu(x)$ for some t ∈ (0, 0.5) and $x_t \in \mu$ and $0_t \bar{\in} \mu$. Since $\mu(0) + t < 1$, we have $0_t \bar{q} \mu$. It follows that $0_t \overline{\in \vee q}\mu$, a contradiction. Hence,

$$\mu(0) \geq \mu(x).$$

Now, if $\mu(0) \geq 0.5$, then $x_{0.5} \in \mu$ and thus, $x_{0.5} \in \vee q\mu$. So we have

$$\mu(0) \geq 0.5.$$

Otherwise

$$\mu(0) + 0.5 < 0.5 + 0.5 = 1,$$

a contradiction. Consequently,

$$\mu(0) \geq \mu(x) \wedge 0.5, \forall x \in X.$$

Let x, y, z ∈ X and suppose that

$$\mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) < 0.5.$$

Then

$$\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z).$$

If not, then

$$\mu(y^2 * x) < t \leq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z)$$

for some t ∈ (0, 0.5). It follows that

$$(((x^2 * y) * (y * x)) * z)_t \in \mu, z_t \in \mu \quad \text{but} \quad (y^2 * x)_{t \wedge t} = (y^2 * x)_t \overline{\in \vee q} \mu.$$

This is a contradiction. Hence,

$$\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z).$$

Whenever

$$\mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) < 0.5.$$

If

$$\mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \geq 0.5,$$

then

$$(((x^2 * y) * (y * x)) * z)_{0.5} \in \mu \quad \text{and} \quad z_{0.5} \in \mu.$$

This implies that

$$(y^2 * x)_{0.5} = (y^2 * x)_{0.5 \wedge 0.5} \in \vee q \mu.$$

Therefore $\mu(y^2 * x) \geq 0.5$ because if $\mu(y^2 * x) < 0.5$, then

$$\mu(y^2 * x) + 0.5 < 0.5 + 0.5$$
$$= 1,$$

a contradiction. Hence,

$$\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \wedge 0.5.$$

Conversely, assume that μ satisfies conditions (I) and (J). Let x ∈ X and t ∈ (0, 1] be such that $x_t \in \mu$. Then $\mu(x) \geq t$. Suppose that $\mu(0) < t$.

If $\mu(x) < 0.5$, then

$$\begin{aligned}\mu(0) &\geq \mu(x) \wedge 0.5 \\ &= \mu(x) \\ &= t\end{aligned}$$

a contradiction. Hence, we know that $\mu(x) \geq 0.5$ and so

$$\begin{aligned}\mu(0) + t &> 2\mu(0) \\ &\geq 2(\mu(x) \wedge 0.5) \\ &= 1.\end{aligned}$$

Thus,

$$0_t \in \vee q\mu.$$

Let x, y, z $\in$ X and $t_1, t_2 \in (0, 1]$ be such that

$$(((x^2 * y) * (y * x)) * z)_{t_1} \in \mu \quad \text{and} \quad z_{t_2} \in \mu.$$

Then

$$\mu(((x^2 * y) * (y * x)) * z) \geq t_1 \quad \text{and} \quad \mu(z) \geq t_2.$$

Suppose

$$\mu(y^2 * x) < t_1 \wedge t_2.$$

If

$$\mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) < 0.5$$

then

$$\begin{aligned}\mu(y^2 * x) &\geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \wedge 0.5 \\ &= \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \\ &\geq t_1 \wedge t_2\end{aligned}$$

a contradiction, and so

$$\mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \geq 0.5.$$

It follows that

$$\begin{aligned}\mu(y^2 * x) + t_1 \wedge t_2 &> 2\mu(y^2 * x) \\ &\geq 2(\mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \wedge 0.5) \\ &= 1.\end{aligned}$$

So

$$(y^2 * x)_{t_1 \wedge t_2} \in \vee q\mu.$$

Hence, μ is a sub-implicative $(\in, \in \vee \text{ q})$-fuzzy ideal of X. $\square$

THEOREM 4.13. *A $(\in, \in \vee\ q)$-fuzzy ideal μ of a BCH-algebra X is a sub-implicative $(\in, \in \vee\ q)$-fuzzy ideal of X if and only if it satisfies the condition*

$$\mu(y^2 * x) \geq \mu((x^2 * y) * (y * x)) \wedge 0.5$$

for all x, y ∈ X.

Proof. The proof is similar to the proof of Theorem 3.11. □

THEOREM 4.14. *If μ is a sub-implicative $(\in, \in \vee\ q)$-fuzzy ideal of a BCH-algebra X, then the following inequality holds*

(K) $\mu(y^2 * x) \geq \mu((x^2 * y) * (y * x)) \wedge 0.5$,
$\forall x, y \in X$.

Proof. If μ is a sub-implicative $(\in, \in \vee$ q)-fuzzy ideal of a BCH-algebra X, then by taking z = 0 in (J) of Theorem 4.12 and using (I) of Theorem 4.12, we have

$$\begin{aligned}\mu(y^2 * x) &\geq \mu(((x^2 * y) * (y * x)) * 0) \wedge \mu(0) \wedge 0.5\\ &= \mu((x^2 * y) * (y * x)) \wedge \mu(0) \wedge 0.5 \text{ (by Proposition 2.2(3))}\\ &= \mu((x^2 * y) * (y * x)) \wedge 0.5. \quad \square\end{aligned}$$

THEOREM 4.15. *Every $(\in, \in \vee\ q)$-fuzzy ideal μ of a BCH-algebra X satisfying the condition (K) of the Theorem 4.14 is a sub-implicative $(\in, \in \vee\ q)$-fuzzy ideal of X.*

Proof. Let μ be an $(\in, \in \vee$ q)-fuzzy ideal of X. For any x, y, z in X, by conditions (K) of Theorem 4.14 and (J) of Theorem 4.12, we have

$$\mu(y^2 * x) \geq \mu((x^2 * y) * (y * x)) \wedge 0.5.$$

Since μ be an $(\in, \in \vee$ q)-fuzzy ideal of X, so we have

$$\begin{aligned}\mu(y^2 * x) &\geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \wedge 0.5 \wedge 0.5\\ &\geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \wedge 0.5.\end{aligned}$$

Therefore μ is a sub-implicative $(\in, \in \vee$ q)-fuzzy ideal of X. □

THEOREM 4.16. *In an implicative BCH-algebra X, every $(\in, \in \vee\ q)$-fuzzy ideal of X is a sub-implicative $(\in, \in \vee\ q)$-fuzzy ideal of X.*

Proof. Let X be an implicative BCH-algebra and μ be an $(\in, \in \vee$ q)-fuzzy ideal of X. We have

$$\mu(y^2 * x) = \mu(y * (y * x))$$

Since μ is an $(\in, \in \vee$ q)-fuzzy ideal of X, so we have

$$\begin{aligned}\mu(y^2 * x) &\geq \mu((y * (y * x)) * z) \wedge \mu(z) \wedge 0.5\\ &= \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \wedge 0.5.\end{aligned}$$

Therefore μ is a sub-implicative $(\in, \in \vee \text{q})$-fuzzy ideal of X. $\square$

THEOREM 4.17. *Let μ be a sub-implicative $(\in, \in \vee q)$-fuzzy ideal of a BCH-algebra X such that $\mu(x) < 0.5$ for all $x \in X$. Then μ is an sub-implicative $(\in, \in)$-fuzzy ideal of X.*

Proof. Let x $\in$X and t $\in$ (0, 1] be such that $x_t \in \mu$. Then $\mu(x) \geq t$, and so

$$\begin{aligned} \mu(0) &\geq \mu(x) \wedge 0.5 \\ &= \mu(x) \\ &\geq t. \end{aligned}$$

Hence, $0_t \in \mu$. Now let x, y, z $\in$ X and $t_1, t_2 \in (0, 1]$ be such that

$$(((x^2 * y) * (y * x)) * z)_{t_1} \in \mu \quad \text{and} \quad z_{t_2} \in \mu.$$

Then

$$\mu(((x^2 * y) * (y * x)) * z) \geq t_1 \quad \text{and} \quad \mu(z) \geq t_2.$$

It follows from Theorem 4.12(J) that

$$\begin{aligned} \mu(y^2 * x) &\geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \wedge 0.5 \\ &= \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \\ &\geq t_1 \wedge t_2. \end{aligned}$$

So

$$(y^2 * x)_{t_1 \wedge t_2} \in \mu.$$

Hence, μ is an sub-implicative $(\in, \in)$-fuzzy ideal of X. $\square$

THEOREM 4.18. *A fuzzy set μ of BCH-algebra X is a sub-implicative $(\in$ $, \in \vee q)$-fuzzy ideal of X if and only if the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is a sub-implicative ideal of X for all t $\in$ (0, 0.5].*

Proof. Suppose that μ is a sub-implicative $(\in, \in \vee \text{q})$-fuzzy ideal of X and t $\in$ (0, 0.5]. Using Theorem 4.12(I), we have

$$\mu(0) \geq \mu(x) \wedge 0.5 \quad \text{for any} \quad x \in \mu_t.$$

It follows that

$$\begin{aligned} \mu(0) &\geq t \wedge 0.5 \\ &= t. \end{aligned}$$

So $0 \in \mu_t$. Let x, y, z $\in$ X be such that

$$((x^2 * y) * (y * x)) * z \in \mu_t \quad \text{and} \quad z \in \mu_t.$$

Then

$$\mu(((x^2 * y) * (y * x)) * z) \geq t \quad \text{and} \quad \mu(z) \geq t.$$

Using Theorem 4.12(J), we get

$$\begin{aligned}\mu(y^2 * x) &\geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \wedge 0.5\\ &\geq t \wedge t \wedge 0.5\\ &= t \wedge 0.5\\ &= t\end{aligned}$$

and so

$$y^2 * x \in \mu_t.$$

Hence, μ_t is a sub-implicative ideal of X.

Conversely, assume that μ is a fuzzy set in X such that

$$\mu_t = \{x \in X \mid \mu(x) \geq t\}$$

is a sub-implicative ideal of X for all t $\in$ (0, 0.5]. If there is $a \in X$ such that

$$\mu(0) \leq \mu(a) \wedge 0.5.$$

Then

$$\mu(0) < t \leq \mu(a) \wedge 0.5$$

for some t $\in$ (0, 0.5], and so $0 \notin \mu_t$. This is a contradiction. Hence,

$$\mu(0) \geq \mu(x) \wedge 0.5, \forall x \in X.$$

Assume that there exist $a, b, c \in X$ such that

$$\mu(b^2 * a) \geq \mu(((a^2 * b) * (b * a)) * c) \wedge \mu(c) \wedge 0.5.$$

Taking

$$\mathrm{t} = \frac{1}{2}(\mu(b^2 * a) + \mu(((a^2 * b) * (b * a)) * c) \wedge \mu(c) \wedge 0.5).$$

We get

$$\mathrm{t} \in (0, 0.5] \quad \text{and} \quad \mu(b^2 * a) < t \leq \mu(((a^2 * b) * (b * a)) * c) \wedge \mu(c) \wedge 0.5.$$

Thus,

$$((a^2 * b) * (b * a)) * c \in \mu_t \quad \text{and} \quad c \in \mu_t \quad \text{but} \quad b^2 * a \notin \mu_t,$$

a contradiction. Hence,

$$\mu(y^2 * x) \geq \mu(((x^2 * y) * (y * x)) * z) \wedge \mu(z) \wedge 0.5.$$

It follows from Theorem 4.12 that μ is a sub-implicative $(\in, \in \vee \mathrm{q})$-fuzzy ideal of X. $\square$

THEOREM 4.19. *Let I be a sub-implicative ideal of X and let μ be a fuzzy set of BCH-algebra X such that*

(L) *$\mu(x) = 0$ for all $x \in X \setminus I$,*

(M) *$\mu(x) \geq 0.5$ for all $x \in I$.*

Then μ is a sub-implicative $(q, \in \vee\ q)$-fuzzy ideal of X.

Proof. Let x ∈ X and t ∈ (0, 1] be such that $x_t q \mu$. Then

$$\mu(x) + t > 1 \quad \text{and so x } \in \text{ I.}$$

Thus,

$$\mu(x) \geq 0.5 \quad \text{and t } > 0.5.$$

Since 0 ∈ I, it follows that

$$\begin{aligned} \mu(0) + t &> 0.5 + 0.5 \\ &= 1. \end{aligned}$$

So $0_t \in \vee$ qμ. Let x, y, z ∈ X and $t_1, t_2 \in (0, 1]$ be such that

$$(((x^2 * y) * (y * x)) * z)_{t_1} \in \vee \text{q}\mu \quad \text{and} \quad z_{t_2} \in \vee \text{q}\mu.$$

Then

$$\mu(((x^2 * y) * (y * x)) * z) + t_1 > 1 \quad \text{and} \quad \mu(z) + t_2 > 1.$$

Thus,

$$((x^2 * y) * (y * x)) * z \in I \quad \text{and z } \in \text{I.}$$

For, $((x^2 * y) * (y * x)) * z \notin I$ (resp. z ∉ I), then

$$\mu(((x^2 * y) * (y * x)) * z) = 0 \ \text{(resp. } \mu(z) = 0)$$

and so, $t_1 > 1$ (resp. $t_2 > 1$), a contradiction. Since I is a sub-implicative ideal of X, it follows that

$$y^2 * x \in I \quad \text{so that} \quad \mu(y^2 * x) \geq 0.5.$$

If $t_1 \leq 0.5$ or $t_2 \leq 0.5$, then

$$\begin{aligned} \mu(y^2 * x) &\geq 0.5 \\ &\geq t_1 \wedge t_2. \end{aligned}$$

Hence,

$$(y^2 * x)_{t_1 \wedge t_2} \in \mu.$$

If $t_1 > 0.5$ and $t_2 > 0.5$), then

$$\mu(y^2 * x) + t_1 \wedge t_2 > 0.5 + 0.5 = 1.$$

and so

$$(y^2 * x)_{t_1 \wedge t_2} q \mu.$$

Thus, we have

$$(y^2 * x)_{t_1 \wedge t_2} \in \vee \text{q}\mu.$$

Hence, μ is a sub-implicative ideal (q, ∈ ∨ q)-fuzzy of X. □

THEOREM 4.20. *Let I be a sub-implicative ideal of X and let μ be a fuzzy set of a BCH-algebra X such that*

(P) $\mu(x) = 0$ *for all* $x \in X \setminus I$,

(Q) $\mu(x) \geq 0.5$ *for all* $x \in I$.

Then μ is a sub-implicative $(\in, \in \vee q)$-fuzzy ideal of X.

Proof. The proof is similar to the proof of Theorem 4.19. □

5. CONCLUSION

In the study, of fuzzy algebraic system, we see that the sub-implicative fuzzy ideal with special properties always play a central role.

In this paper, we define sub-implicative (α, β)-fuzzy ideal in BCH-algebra and give several characterizations of sub-implicative fuzzy ideas in BCH-algebrs in terms of these notions. We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of fuzzy BCH-algebras and their applications in other branches of algebra. In the future study of fuzzy BCH-algebras, perhaps the following topics are worth to be considered:

(1) To characterize other classes of BCH-algebras by using this notion;

(2) To apply this notion to some other algebraic structures;

(3) To consider these results to some possible applications in computer sciences and information systems in the future.

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