# SOME UNIQUENESS RESULTS RELATED TO MEROMORPHIC FUNCTION THAT SHARE A SMALL FUNCTION WITH ITS DERIVATIVE

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In this paper we investigate the uniqueness problem of meromorphic function as well as its power which share a small function with its derivative counterpart. The paper improves, supplement and rectify some recent results of Chen,Wang and Zhang [5].

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*Key words:* meromorphic function, derivative, small function, weighted sharing.

## 1. INTRODUCTION, DEFINITIONS AND RESULTS

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane $\mathbb{C}$. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [6]. It will be convenient to let $E$ denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function $h$, we denote by $T(r,h)$ the Nevanlinna characteristic of $h$ and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$, as $r \longrightarrow \infty$ and $r \notin E$.

Let $f$ and $g$ be two non-constant meromorphic functions and let $a$ be a complex number. We say that $f$ and $g$ share $a$ CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share $a$ IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition, we say that $f$ and $g$ share $\infty$ CM, if $1/f$ and $1/g$ share 0 CM, and we say that $f$ and $g$ share $\infty$ IM, if $1/f$ and $1/g$ share 0 IM.

A meromorphic function $a$ is said to be a small function of $f$ provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \longrightarrow \infty$, $r \notin E$.

In 1979 Mues and Steinmetz considered the following uniqueness result of an entire function and its derivative when they share two values IM.

Theorem A ([13]). *Let $f$ be a non-constant entire function. If $f$ and $f'$ share two distinct values $a$, $b$ IM then $f' \equiv f$.*

R. Brück [4] first dealt with the uniqueness problem of an entire function sharing one value with its derivative and obtained the following result.

Theorem B ([4]). *Let $f$ be a non-constant entire function. If $f$ and $f'$ share the value 1 CM and if $N(r,0;f') = S(r,f)$ then $\frac{f'-1}{f-1}$ is a nonzero constant.*

Brück's result inspired a large number of authors such as Yang [14], Zhang [17], Yu [16], Liu-Gu [11], Zhang-Yang [19] to further extended and generalised the same result under weaker hypothesis. Next we define the notion of weighted sharing of values appeared in the uniqueness literature in 2001 which measured how close a shared value is to be shared IM or to be shared CM.

*Definition* 1.1 ([7, 8]). Let $k$ be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that $f, g$ share the value $a$ with weight $k$.

The definition implies that if $f$, $g$ share a value $a$ with weight $k$ then $z_0$ is an $a$-point of $f$ with multiplicity $m$ $(\leq k)$ if and only if it is an $a$-point of $g$ with multiplicity $m$ $(\leq k)$ and $z_0$ is an $a$-point of $f$ with multiplicity $m$ $(> k)$ if and only if it is an $a$-point of $g$ with multiplicity $n$ $(> k)$, where $m$ is not necessarily equal to $n$.

We write $f$, $g$ share $(a,k)$ to mean that $f$, $g$ share the value $a$ with weight $k$. Clearly if $f$, $g$ share $(a,k)$, then $f$, $g$ share $(a,p)$ for any integer $p$, $0 \leq p < k$. Also we note that $f$, $g$ share a value $a$ IM or CM if and only if $f$, $g$ share $(a,0)$ or $(a,\infty)$ respectively.

If $a$ is a small function we define that $f$ and $g$ share $a$ IM or $a$ CM or with weight $l$ according as $f-a$ and $g-a$ share $(0,0)$ or $(0,\infty)$ or $(0,l)$ respectively. Though we use the standard notations and definitions of the value distribution theory available in [6], we explain some definitions and notations which are used in the paper.

*Definition* 1.2 ([10]). Let $p$ be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r,a;f \mid\geq p)$ $(\overline{N}(r,a;f \mid\geq p))$ denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not less than $p$.

(ii) $N(r,a;f \mid\leq p)$ $(\overline{N}(r,a;f \mid\leq p))$ denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not greater than $p$.

*Definition* 1.3 ([15]). For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer $p$ we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2) + \ldots + \overline{N}(r,a;f \mid\geq p)$. Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

*Definition* 1.4. For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer $m$, we denote by $\overline{N}(r, a; f \mid g \neq a)$ ($\overline{N}(r, a; f \mid g = a)$) the reduced counting function of those $a$-points of $f$ which are not the $a$-points of $g$ (common $a$-points of $f$ and $g$).

*Definition* 1.5 ([18]). For a positive integer $p$ and $a \in \mathbb{C} \cup \{\infty\}$ we put

$$\delta_p(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N_p(r, a; f)}{T(r, f)}$$

Clearly $0 \leq \delta(a; f) \leq \delta_p(a; f) \leq \delta_{p-1}(a; f) \ldots \leq \delta_2(a; f) \leq \delta_1(a; f) = \Theta(a; f)$.

*Definition* 1.6 ([1]). Let $f$ and $g$ be two non-constant meromorphic functions such that $f$ and $g$ share the value 1 IM. Let $z_0$ be a 1-point of $f$ with multiplicity $p$, a 1-point of $g$ with multiplicity $q$. We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of $f$ and $g$ where $p > q$, by $N_E^{1)}(r, 1; f)$ the counting function of those 1-points of $f$ and $g$ where $p = q = 1$ and by $\overline{N}_E^{(2}(r, 1; f)$ the counting function of those 1-points of $f$ and $g$ where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r, 1; g)$, $N_E^{1)}(r, 1; g)$, $\overline{N}_E^{(2}(r, 1; g)$.

*Definition* 1.7 ([7, 8]). Let $f$, $g$ share a value $(a, 0)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those $a$-points of $f$ whose multiplicities differ from the multiplicities of the corresponding $a$-points of $g$.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

Nowadays, the idea of weighted sharing is being used immensely for further investigation of the Brück's result (see [3, 10, 18, 20]). Recently, in [5] Chen, Wang and Zhang generalise and improve the results of Yu [16], by considering the problem of uniqueness of $f$ and $(f^n)^{(k)}$, when they share a small function and proved the following theorem.

THEOREM C. *Let $f$ be a non-constant meromorphic function and $k(\geq 1)$, $n(\geq 1)$ be integers and $a \equiv a(z)(\not\equiv 0, \infty)$ be a non-constant meromorphic small function. Suppose that $f - a$ and $(f^n)^{(k)} - a$ share $(0, \infty)$ and*

$$(k + 3)\, \Theta(\infty; f) + \delta_2(0; f) + \delta_{2+k}(0; f) > k + 4 \tag{1.1}$$

*or $f - a$ and $[f^n]^{(k)} - a$ share $(0, 0)$ and*

$$(6 + 2k)\, \Theta(\infty; f) + 3\, \Theta(0; f) + 2\delta_{2+k}(0; f) > 2k + 10 \tag{1.2}$$

*then $f \equiv (f^n)^{(k)}$*

*Remark* 1.1. In [5] while proving Theorem 1.2 the authors have made some mistakes in sub cases 1.1 and 1.2. Lemma 2.3 in [5] implies that

$$\begin{aligned} & 2\overline{N}(r,0;F') + \overline{N}(r,0;F) \leq 2N_2(r,0;f) + \overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) \\ \nleq & \ 3\overline{N}(r,0;f) + 2\overline{N}(r,\infty;f) \end{aligned}$$

and

$$\begin{aligned} \overline{N}(r,0;G') & \leq N_2\left(r,0;(f^n)^{(k)}\right) + \overline{N}(r,\infty;f) + S(r,f) \\ & \leq N_{k+2}\left(r,0;f^n\right) + (k+1)\overline{N}(r,\infty;f) + S(r,f) \\ & \leq (k+2)\overline{N}(r,0;f) + (k+1)\overline{N}(r,\infty;f) + S(r,f). \end{aligned}$$

But instead of considering the counting function $(k+2)\overline{N}(r,0;f)$ in the second inequality the authors mistakenly consider the counting function $N_{k+2}(r,0;f)$ which can be easily identified if one minutely proceed in the proof of the corresponding theorem. Consequently (1.1) and (1.2) in [5] are not correct.

The above discussion is sufficient enough to make oneself inquisitive to investigate the actual form of (1.1) and (1.2) in [5] for which the conclusion of Theorem C is true. In the paper we will not only rectify Theorem C but also improve and supplement the same theorem by relaxing the nature of sharing. We also pay our attention to the uniqueness of more generalised form of a function namely $f^m$ and $(f^n)^{(k)}$ sharing a small function for two arbitrary positive integers $n$ and $m$ as this type of uniqueness results was not considered earlier by any other researchers. Following theorems are the main results of the paper.

THEOREM 1.1. *Let $f$ be a non-constant meromorphic function and $k(\geq 1)$, $n(\geq 1)$, $l(\geq 0)$ be integers and $a(z)(\not\equiv 0,\infty)$ be a non-constant meromorphic small function. Suppose $f-a$ and $(f^n)^{(k)}-a$ share $(0,l)$. If $l=2$ and*

$$3\,\Theta(\infty;f) + (k+2)\,\Theta(0;f) + \delta_2(0;f) > k+6-n \tag{1.3}$$

*or $l=1$ and*

$$4\,\Theta(\infty;f) + (k+3)\ \Theta(0;f) + \delta_2(0;f) > k+8-n \tag{1.4}$$

*or $l=0$ and*

$$(2k+6)\ \Theta(\infty;f) + (3k+5)\ \Theta(0;f) + \delta_2(0;f) > 5k+12-n \tag{1.5}$$

*then $f \equiv (f^n)^{(k)}$.*

THEOREM 1.2. *Let $f$ be a non-constant meromorphic function and $k(\geq 1)$, $n;m(\geq 1)$, $l(\geq 0)$ be integers and $a(z)(\not\equiv 0,\infty)$ be a non-constant meromorphic small function. Suppose $f^m-a$ and $(f^n)^{(k)}-a$ share $(0,l)$. If $l=2$*

*and*

$$3\,\Theta(\infty;f)+(k+4)\,\Theta(0;f)>k+7-n \tag{1.6}$$

*or* $l=1$ *and*

$$4\,\Theta(\infty;f)+(k+5)\;\Theta(0;f)>k+9-n \tag{1.7}$$

*or* $l=0$ *and*

$$(2k+6)\,\Theta(\infty;f)+(3k+7)\,\Theta(0;f)>5k+13-n \tag{1.8}$$

*then* $f^m\equiv(f^n)^{(k)}$.

Theorem 1.3. *Let* $f$ *be a non-constant meromorphic function and* $k(\geq 1)$, $n;m(\geq 1)$, $l(\geq 0)$ *be integers and* $a(z)(\not\equiv 0,\infty)$ *be a non-constant meromorphic small function. Suppose* $f^m-a$ *and* $(f^n)^{(k)}-a$ *share* $(0,l)$. *If* $l=2$ *and*

$$(k+3)\,\Theta(\infty;f)+(k+4)\,\Theta(0;f)>2k+7-m \tag{1.9}$$

*or* $l=1$ *and*

$$\left(k+\frac{7}{2}\right)\,\Theta(\infty;f)+\left(k+\frac{9}{2}\right)\,\Theta(0;f)>2k+8-m \tag{1.10}$$

*or* $l=0$ *and*

$$(2k+6)\,\Theta(\infty;f)+(2k+7)\,\Theta(0;f)>4k+13-m \tag{1.11}$$

*then* $f^m\equiv(f^n)^{(k)}$.

## 2. LEMMAS

In this section, we present some lemmas which will be needed in the sequel. Let $F$, $G$ be two non-constant meromorphic functions. Henceforth, we shall denote by $H$ the following function.

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1 ([20]). *Let* $f$ *is a non-constant meromorphic function and* $p$ *and* $k$ *are two positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right)\leq T\left(r,f^{(k)}\right)-T(r,f)+N_{p+k}(r,0;f)+S(r,f),$$

$$N_p\left(r,0;f^{(k)}\right)\leq N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.2 ([2]). *Let $f$, $g$ share $(1,0)$. Then*

$$\overline{N}_L(r,1;f)+\overline{N}_E^{(2}(r,1;f)\leq\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r).$$

Lemma 2.3 ([12]). *Let $f$ be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum\limits_{k=0}^{n}a_kf^k}{\sum\limits_{j=0}^{m}b_jf^j}$$

*be an irreducible rational function in $f$ with constant coefficients $\{a_k\}$ and $\{b_j\}$where $a_n\neq 0$ and $b_m\neq 0$ Then*

$$T(r,R(f))=dT(r,f)+S(r,f),$$

*where $d=\max\{n,m\}$.*

## 3. PROOFS OF THE THEOREMS

*Proof of Theorem 1.1.* Let $F=\frac{f}{a}$ and $G=\frac{(f^n)^{(k)}}{a}$. Then $F-1=\frac{f-a}{a}$ and $G-1=\frac{(f^n)^{(k)}-a}{a}$. Since $f-a$ and $(f^n)^{(k)}-a$ share $(0,l)$ it follows that $F$, $G$ share $(1,l)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

**Case 1.** Let $H\not\equiv 0$.
**Subcase 1.1.** $l\geq 1$.

From (2.1) it can be easily calculated that the possible poles of $H$ occur at (i) multiple zeros of $F$ and $G$, (ii) those 1 points of $F$ and $G$ whose multiplicities are different, (iii) those poles of $F$ and $G$ whose multiplicities are different, (iv) zeros of $F'$ ($G'$) which are not the zeros of $F(F-1)$ ($G(G-1)$). Since $H$ has only simple poles we get.

$$\begin{aligned}N(r,\infty;H)\ \leq\ &\overline{N}(r,\infty;F)+\overline{N}_*(r,1;F,G)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)\\ &+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+\overline{N}(r,0;a)+\overline{N}(r,\infty;a),\end{aligned}\tag{3.1}$$

where $\overline{N}_0(r,0;F')$ denotes the reduced counting function of the zeros of $F'$ which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is defined analogously. Let $z_1$ be a simple zero of $F-1$ but $a(z_1)\neq 0,\infty$. Then $z_1$ is a simple zero of $G-1$ and a zero of $H$. So

$$\begin{aligned}N(r,1;F\mid=1)&\leq N(r,0;H)+N(r,\infty;a)+N(r,0;a)\\ &\leq N(r,\infty;H)+S(r,f).\end{aligned}\tag{3.2}$$

Hence,

$$\begin{aligned}(3.3)\qquad \overline{N}(r,1;G) &\leq N(r,1;F\mid=1)+\overline{N}(r,1;F\mid\geq 2)\\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F\mid\geq\ 2)+\overline{N}(r,0;G\mid\geq 2)\\ &\quad+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F')\\ &\quad+\overline{N}_0(r,0;G')+S(r,f).\end{aligned}$$

Noting that $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)$ by the Milloux's basic result (p. 57 [6]), from (3.3) we get

$$\begin{aligned}&(3.4)\ \ T(r,f^n)\\ &\leq \overline{N}(r,\infty;f^n)+N(r,0;f^n)+\overline{N}(r,1;\frac{(f^n)^{(k)}}{a})-N_{\oplus}(r,0;(\frac{(f^n)^{(k)}}{a})')+S(r,f)\\ &=\overline{N}(r,\infty;f)+N(r,0;f^n)+\overline{N}(r,1;G)-N_{\oplus}(r,0;G')+S(r,f)\\ &\leq 2\overline{N}(r,\infty;f)+N(r,0;f^n)+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}(r,0;F\mid\geq 2)\\ &\quad+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')-N_{\oplus}(r,0;G')\\ &\quad+S(r,f),\end{aligned}$$

where $N_{\oplus}(r,0;G')$ is the counting function which only counts those points such that $G'=0$ but $G-1\neq 0$.

We observe that

$$\begin{aligned}&N(r,0,f^n)+\overline{N}(r,0;G\mid\geq 2)-N_{\oplus}(r,0;G')+\overline{N}_0(r,0;G')\\ \leq\ &N(r,0,f^n\mid\leq k+1)+N(r,0,f^n\mid\geq k+2)-\{N(r,0;G'\mid G=0)\\ &-\overline{N}(r,0;G'\mid G=0)\}+S(r,f)\\ \leq\ &N_{2+k}(r,0;f^n)+S(r,f).\end{aligned}$$

While $2\leq l<\infty$ we see that

$$\begin{aligned}(3.5)\qquad &\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0\left(r,0;F'\right)\\ \leq\ &\overline{N}(r,1;F\mid\geq l+1)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0\left(r,0;F'\right)\\ \leq\ &N\left(r,0;F'\mid F\neq 0\right)\\ \leq\ &\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

So, using (3.5) in (3.4) and noting that $N_{2+k}(r,0;f^n)\leq(2+k)\overline{N}(r,0;f)$, we have

$$nT(r,f)\leq 3\,\overline{N}(r,\infty;f)+(2+k)\overline{N}(r,0;f)+N_2(r,0;f)+S(r,f),$$

*i.e.*

$$3\Theta(\infty;f)+(2+k)\Theta(0;f)+\delta_2(0;f)\leq k+6-n,$$

which contradicts (1.3).

While $l=1$ (3.5) changes to

$$\begin{aligned} &\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0\left(r,0;F'\right) \\ \leq\; & 2\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F') \\ \leq\; & 2N\left(r,0;F'\mid F\neq 0\right) \\ \leq\; & 2\overline{N}(r,0;f)+2\overline{N}(r,\infty;f)+S(r,f). \end{aligned} \tag{3.6}$$

So from (3.4), using (3.6) and noting that $N_{2+k}(r,0;f^n)\leq(2+k)\overline{N}(r,0;f)$, we get

$$nT(r,f)\leq 4\,\overline{N}(r,\infty;f)+(3+k)\overline{N}(r,0;f)+N_2(r,0;f)+S(r,f),$$

*i.e.*,

$$4\Theta(\infty;f)+(3+k)\Theta(0;f)+\delta_2(0;f)\leq k+8-n,$$

which contradicts (1.4).

**Subcase 1.2.** $l=0$.

In this case $F$ and $G$ share $(1,0)$ except the zeros and poles of $a(z)$. So in view of Definition 1.7, (3.1) changes to

$$\begin{aligned} N(r,\infty;H)\leq\; & \overline{N}(r,\infty;F)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2) \\ & +\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f). \end{aligned} \tag{3.7}$$

Let $z_0$ be a zero of $F-1$ with multiplicity $p$ and a zero of $G-1$ with multiplicity $q$. It is easy to see that

$$N_E^{1)}(r,1;F)=N_E^{1)}(r,1;G)+S(r,f)$$

$$\overline{N}_E^{(2}(r,1;F)=\overline{N}_E^{(2}(r,1;G)+S(r,f)$$

and

$$N_E^{1)}(r,1;F)\leq N(r,\infty;H)+S(r,f). \tag{3.8}$$

By the Milloux Basic result, using (3.7) and (3.8) we get

$$\begin{aligned} T(r,f^n)\leq\; & \overline{N}(r,\infty;f)+N(r,0;f^n)+N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_E^{(2}(r,1;F) \\ & +\overline{N}_L(r,1;G)-N_{\oplus}(r,0;G')+S(r,f) \\ \leq\; & 2\overline{N}(r,\infty;f)+\overline{N}(r,0;F\mid\geq 2)+N(r,0;f^n)+\overline{N}(r,0;G\mid\geq 2) \\ & +\overline{N}_0(r,0;G')-N_{\oplus}(r,0;G')+2\,\overline{N}_L(r,1;F)+2\,\overline{N}_L(r,1;G) \\ & +\overline{N}_E^{(2}(r,1;F)+\overline{N}_0\left(r,0;F'\right) \\ \leq\; & 2\overline{N}(r,\infty;f)+\overline{N}(r,0;F\mid\geq 2)+N_{2+k}(r,0;f^n)+2N(r,0;F'\mid F\neq 0) \\ & +2N(r,0;G'\mid G\neq 0)+S(r,f). \end{aligned}$$

From Lemma 2.1 for $p=1$ and $k=1$ we get

$$N(r,0;G'\mid G\neq 0)\leq\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\leq N_{1+k}(r,0;f^n)+(k+1)\overline{N}(r,\infty;f).$$

So from above we get

$$nT(r,f) \leq (6+2k)\overline{N}r,\infty;f) + (5+3k)\overline{N}(r,0;f) + N_2(r,0;f) + S(r,f),$$

that is,

$$(6+2k)\Theta(\infty;f) + (5+3k)\Theta(0;f) + \delta_2(0,f) \leq 5k+12-n,$$

which contradicts (1.5).

**Case 2.** Let $H \equiv 0$.
On integration we get from (2.1)

$$\frac{1}{F-1} \equiv \frac{C}{G-1} + D, \tag{3.9}$$

where $C$, $D$ are constants and $C \neq 0$. We will prove that $D=0$.

**Subcase 2.1.** Suppose $D \neq 0$. If $z_0$ be a pole of $f$ with multiplicity $p$ such that $a(z_0) \neq 0, \infty$, then it is a pole of $G$ with multiplicity $np+k$ respectively. This contradicts (3.9). It follows that $N(r,\infty;f) \leq N(r,0;a) + N(r,\infty;a) = S(r,f)$ and hence, $\Theta(\infty;f) = 1$. Also it is clear that $\overline{N}(r,\infty;F) = \overline{N}(r,\infty;G) = S(r,f)$. From (1.3)–(1.5) we know respectively

$$(k+2)\Theta(0;f) + \delta_2(0;f) > k+3-n, \tag{3.10}$$

$$(k+3)\Theta(0;f) + \delta_2(0;f) > k+4-n, \tag{3.11}$$

and

$$(3k+5)\Theta(0;f) + \delta_2(0;f) > 3k+6-n. \tag{3.12}$$

**Subcase 2.1.1.** Suppose $n \leq k+1$.
Since $D \neq 0$, from (3.9) we get

$$-\frac{D\left(F-1-\frac{1}{D}\right)}{F-1} \equiv C\frac{1}{G-1}.$$

So

$$\overline{N}\left(r, 1+\frac{1}{D}; F\right) = \overline{N}(r,\infty;G) = S(r,f).$$

If $D \neq -1$, using the second fundamental theorem for $F$ we get

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1+\frac{1}{D};F) \\ &\leq \overline{N}(r,0;F) + S(r,f), \end{aligned}$$

*i.e.*,

$$T(r,f) \leq \overline{N}(r,0;f) + S(r,f) \leq T(r,f) + S(r,f).$$

So we have $T(r,f) = \overline{N}(r,0;f)$ and so $\Theta(0;f) = 0$, which contradicts (3.10)–(3.12).

If $D = -1$, then

$$\frac{F}{F-1} \equiv C\frac{1}{G-1}. \tag{3.13}$$

Clearly, we know from above $\overline{N}(r,0;F) = \overline{N}(r,\infty;G) = S(r,f)$ and hence, $\overline{N}(r,0;f) = S(r,f)$.

If $C \neq -1$ we know from (3.13) that $\overline{N}(r,1+C;G) = \overline{N}(r,\infty;F) = S(r,f)$. So from Lemma 2.1 and the second fundamental theorem we get

$$\begin{aligned} T\left(r,(f^n)^{(k)}\right) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1+C;G) + S(r,f) \\ &\leq \overline{N}\left(r,0;(f^n)^{(k)}\right) + S(r,f) \\ &\leq T\left(r,(f^n)^{(k)}\right) - T(r,f^n) + N_{k+1}(r,0;f^n) + S(r,f), \end{aligned}$$

*i.e.*,

$$nT(r,f) \leq (k+1)\overline{N}(r,0;f) + S(r,f) = S(r,f),$$

which is absurd.

So $C = -1$ and we get from (3.13) that $FG \equiv 1$, which implies $\left[\frac{(f^n)^{(k)}}{f^n}\right] = \frac{a^2}{f^{n+1}}$.

In view of the first fundamental theorem we get from above

$$\begin{aligned} (n+1)T(r,f) &\leq T\left(r,\frac{(f^n)^{(k)}}{f^n}\right) + S(r,f) \\ &= N\left(r,\infty;\frac{(f^n)^{(k)}}{f^n}\right) + S(r,f) \\ &\leq k[\overline{N}(r,\infty;f) + \overline{N}(r,0;f)] + S(r,f) = S(r,f), \end{aligned}$$

which is impossible.

**Subcase 2.1.2.** Suppose $n > k+1$.

Since $D \neq 0$, from (3.9) we get

$$\frac{D\left(G-1+\frac{C}{D}\right)}{G-1} \equiv \frac{1}{F-1}.$$

So,

$$\overline{N}\left(r,1-\frac{C}{D};G\right) = \overline{N}(r,\infty;F) = S(r,f).$$

If $D \neq C$, using the second fundamental theorem for $G$ and using Lemma 2.1 we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1-\frac{C}{D};G) \\ &\leq \overline{N}(r,0;G) + S(r,f) \\ &\leq T\left(r,(f^n)^{(k)}\right) - T(r,f^n) + N_{1+k}(r,0;f^n) + S(r,f), \end{aligned}$$

*i.e.*,

$$nT(r,f) \leq (k+1)T(r,f) + S(r,f),$$

which is impossible since $n > k+1$.

If $D = C$, then

$$C\frac{G}{G-1} \equiv \frac{1}{F-1}. \tag{3.14}$$

From above it is clear that $\overline{N}(r,0;G) = \overline{N}(r,\infty;F) = S(r,f)$ and since $n > k+1$, we have $\overline{N}(r,0;f) = S(r,f)$.

If $C \neq -1$ we know from (3.14) that $\overline{N}(r,1+\frac{1}{C};F) = \overline{N}(r,\infty;G) = S(r,f)$. So from the second fundamental theorem we get

$$T\left(r,F\right) \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,1+\frac{1}{C};F),$$

*i.e.*,

$$T(r,f) \leq \overline{N}(r,0;f) + S(r,f) = S(r,f),$$

which is absurd.

Hence, $C = -1$. Therefore, from (3.14) we get $FG \equiv 1$, which implies $\left[\frac{(f^n)^{(k)}}{f^n}\right] = \frac{a^2}{f^{n+1}}$. So, using the same argument as in Subcase 2.1.1 we can again get a contradiction.

**Subcase 2.2.** $D = 0$ and so from (3.9) we get

$$G - 1 \equiv C\ (F-1).$$

If $C \neq 1$, then

$$F \equiv \frac{1}{C}\ (G-1+C)$$

and

$$\overline{N}(r,0;F) = \overline{N}\left(r,1-C;G\right).$$

By the second fundamental theorem and Lemma 2.1 for $p = 1$ and Lemma 2.3 we have

$$\begin{aligned} &T(r,G) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\left(r,1-C;G\right) + S(r,G) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;G) + \overline{N}(r,0;F) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + T(r,(f^n)^{(k)}) - T(r,f^n) + N_{1+k}(r,0;f^n) + \overline{N}(r,0;f) + S(r,f), \end{aligned}$$

*i.e.*,

$$T(r,f^n)\le \overline{N}(r,\infty;f)+(2+k)\overline{N}(r,0;f)+S(r,f).$$

Hence,

$$\Theta(\infty;f)+(2+k)\Theta(0;f)\le k+3-n.$$

So, it follows that

$$\begin{aligned}&3\Theta(\infty;f)+(k+2)\Theta(0;f)+\delta_2(0;f)\\ \le\ &2\Theta(\infty;f)+\Theta(\infty;f)+(k+2)\Theta(0;f)+\delta_2(0;f)\\ \le\ &k+6-n,\end{aligned}$$

$$4\Theta(\infty;f)+(k+3)\ \Theta(0;f)+\delta_2(0;f)\le k+8-n,$$

and

$$(2k+6)\Theta(\infty;f)+(3k+5)\Theta(0;f)+\delta_2(0;f)\le 5k+12-n.$$

This contradicts (1.3)–(1.5). Hence, $C=1$ and so $F\equiv G$, that is $f\equiv (f^n)^{(k)}$. This completes the proof of the theorem. □

*Proof of Theorem 1.3.* Let $F=\frac{f^m}{a}$ and $G=\frac{(f^n)^{(k)}}{a}$. Then $F-1=\frac{f^m-a}{a}$ and $G-1=\frac{(f^n)^{(k)}-a}{a}$. Since $f^m-a$ and $(f^n)^{(k)}-a$ share $(0,l)$ it follows that $F$, $G$ share $(1,l)$ except the zeros and poles of $a(z)$. Now we consider the following cases.

**Case 1.** Let $H\not\equiv 0$.
**Subcase 1.1.** $l\ge 1$.

Here with the same argument as used in the proof of Theorem 1.1 we can obtain (3.1) and (3.2).

By the second fundamental theorem we see that

$$\begin{aligned}T(r,F)+T(r,G)\le\ &\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F)\\ &+\overline{N}(r,0;G)+\overline{N}(r,1;F)+\overline{N}(r,1;G)\\ &-N_0(r,0;F')-N_0(r,0;G')+S(r,F)+S(r,G).\end{aligned}\tag{3.15}$$

Using (3.1) and (3.2) we get

$$\begin{aligned}\overline{N}(r,1;F)+\overline{N}(r,1;G)\le\ &N(r,1;F\mid=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)\\ &+\overline{N}_E^{(2}(r,1;F)+\overline{N}(r,1;G)+S(r,f)\\ \le\ &\overline{N}(r,0;F\mid\ge 2)+\overline{N}(r,0;G\mid\ge 2)+\overline{N}(r,\infty;F)\\ &+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\ &+\overline{N}(r,1;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')\\ &+S(r,f).\end{aligned}\tag{3.16}$$

While $l \geq 2$ we see that

$$2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}(r,1;G)\leq N(r,1;G)+S(r,f). \tag{3.17}$$

In view of Lemma 2.1, noting that

$$\begin{aligned} N_2(r,0;G) &\leq N_{k+2}(r,0;f^n)+k\overline{N}(r,\infty;f)+S(r,f) \\ &\leq (k+2)\overline{N}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f) \end{aligned}$$

and using (3.16)–(3.17) in (3.15), we obtain

$$T(r,F)\leq (k+3)\ \overline{N}(r,\infty;F)+(k+2)\overline{N}(r,0;f)+N_2(r,0;F)+S(r,f),$$

*i.e.*,

$$(k+3)\Theta(\infty;f)+(k+4)\Theta(0;f)\leq 2k+7-m,$$

which contradicts (1.9).

While $l=1$ (3.17) changes to

$$\begin{aligned} &2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}(r,1;G) \\ \leq\ & N(r,1;G)+\overline{N}_L(r,1;F)+S(r,f). \end{aligned} \tag{3.18}$$

Noting that

$$\overline{N}_L(r,1;F)\leq \frac{1}{2}N(r,0;F'\mid F\neq 0)\leq \frac{1}{2}(\overline{N}(r,0;F)+\overline{N}(r,\infty;F)),$$

using (3.16) and (3.18) in (3.15) and in view of Lemma 2.1 we have

$$T(r,F)\leq \left(k+\frac{7}{2}\right)\ \overline{N}(r,\infty;F)+(k+\frac{9}{2})\overline{N}(r,0;f)+S(r,f),$$

*i.e.*,

$$\left(k+\frac{7}{2}\right)\Theta(\infty;f)+(k+\frac{9}{2})\Theta(0;f)\leq 2k+8-m,$$

which contradicts (1.10).

**Subcase 1.2.** $l=0$.

In this case, $F$ and $G$ share $(1,0)$ except the zeros and poles of $a(z)$.

Here, in view of Lemma 2.2, (3.1) and (3.8), (3.16) changes into

$$\begin{aligned}
(3.19)\quad & \overline{N}(r,1;F)+\overline{N}(r,1;G) \\
\leq\ & N_{E}^{1)}(r,1;F)+\overline{N}_{E}^{(2}(r,1;F)+\overline{N}_{L}(r,1;F)+\overline{N}_{L}(r,1;G)+\overline{N}(r,1;G)+S(r,f) \\
\leq\ & \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}(r,\infty;F)+\overline{N}_{E}^{(2}(r,1;F)+2\overline{N}_{L}(r,1;F) \\
& +2\overline{N}_{L}(r,1;G)+\overline{N}(r,1;G)+\overline{N}_{0}(r,0;F')+\overline{N}_{0}(r,0;G')+S(r,f) \\
\leq\ & \overline{N}(r,\infty;F)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+2\ \overline{N}_{L}(r,1;F) \\
& +\overline{N}_{L}(r,1;G)+N(r,1;G)+\overline{N}_{0}(r,0;F')+\overline{N}_{0}(r,0;G')+S(r,f) \\
\leq\ & 4\ \overline{N}(r,\infty;F)+N_{2}(r,0;F)+\overline{N}(r,0;F)+N_{2}(r,0;G)+T(r,G) \\
& +\overline{N}_{0}(r,0;F')+\overline{N}_{0}(r,0;G')+S(r,f).
\end{aligned}$$

So using (3.19) in (3.15) we get in view of Lemma 2.1 that

$$T(r,F)\leq(2k+6)\ \overline{N}(r,\infty;f)+(2k+7)\overline{N}(r,0;f)+S(r,f),$$

*i.e.*,

$$(2k+6)\Theta(\infty;f)+(2k+7)\Theta(0;f)\leq 4k+13-m.$$

This contradicts (1.11).

**Case 2.** Let $H\equiv 0$.

On integration from (2.1) we again get (3.9).

**Subcase 2.1.** Suppose $D\neq 0$. If there exists a pole $z_0$ of f with multiplicities $p$ which is not a pole or a zero of $a(z)$, then $z_0$ is a pole of $F$ with multiplicities $mp$ and a pole of $G$ with multiplicities $np+k$. This contradicts (3.9). So $N(r,\infty;F)=S(r,f)$ and hence, $\Theta(\infty;f)=1$. Also it is clear that $\overline{N}(r,\infty;F)=\overline{N}(r,\infty;G)=S(r,f)$. From (1.9)–(1.11) we know respectively

$$(k+4)\Theta(0;f)>k+4-m, \tag{3.20}$$

$$(k+\frac{9}{2})\Theta(0;f)>k+\frac{9}{2}-m, \tag{3.21}$$

and

$$(2k+7)\Theta(0;f)>2k+7-m. \tag{3.22}$$

Since $D\neq 0$, from (3.9) we get

$$\overline{N}\left(r,1+\frac{1}{D};F\right)=\overline{N}(r,\infty;G)=S(r,f).$$

Suppose $D\neq -1$.

Using the second fundamental theorem for $F$ we get

$$\begin{aligned}
T(r,F) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}(r,1+\frac{1}{D};F) \\
&\leq \overline{N}(r,0;F)+S(r,f).
\end{aligned}$$

*i.e.*,
$$mT(r,f) \leq \overline{N}(r,0;f) + S(r,f) \leq mT(r,f) + S(r,f).$$

So, we have $mT(r,f) = \overline{N}(r,0;f)$ and so $\Theta(0;f) = 1-m$, which contradicts (3.20)–(3.22).

If $D = -1$, then we again get (3.13) and from which we know $\overline{N}(r,0;F) = \overline{N}(r,\infty;G) = S(r,f)$ and hence, $\overline{N}(r,0;f) = S(r,f)$. If $C \neq -1$ proceeding in the same way as done in the proof of theorem 1.1 we can get a contradiction.

So $C = -1$ and we get from (3.13) that $FG \equiv 1$, which implies $\left[\frac{(f^n)^{(k)}}{f^n}\right] = \frac{a^2}{f^{n+m}}$.

In view of the first fundamental theorem, we get from above
$$(n+m)T(r,f) \leq k[\overline{N}(r,\infty;f) + \overline{N}(r,0;f)] + S(r,f) = S(r,f),$$
which is impossible.

**Subcase 2.2.** $D = 0$ and so from (3.9) we get
$$G - 1 \equiv C\ (F-1).$$

If $C \neq 1$, then
$$G \equiv C\ \left(F - 1 + \frac{1}{C}\right)$$
and
$$\overline{N}(r,0;G) = \overline{N}\left(r, 1 - \frac{1}{C}; F\right).$$

By the second fundamental theorem and Lemma 2.1 for $p = 1$ and Lemma 2.3 we have
$$\begin{aligned}
& mT(r,f) + S(r,f) = T(r,F) \\
\leq\ & \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}\left(r, 1 - \frac{1}{C}; F\right) + S(r,G) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,f) \\
\leq\ & (k+2)\overline{N}(r,0;f) + (k+1)\overline{N}(r,\infty;f) + S(r,f).
\end{aligned}$$

Hence,
$$(k+1)\Theta(\infty;f) + (k+2)\Theta(0;f) \leq 2k + 3 - m.$$

So, it follows that
$$\begin{aligned}
& (k+3)\Theta(\infty;f) + (k+4)\Theta(0;f) \\
\leq\ & 2\Theta(\infty;f) + (k+1)\Theta(\infty;f) + (k+2)\Theta(0;f) + \Theta(0;f) \\
\leq\ & 2k + 7 - m,
\end{aligned}$$
$$(k + \frac{7}{2})\Theta(\infty;f) + (k + \frac{9}{2})\ \Theta(0;f) \leq 2k + 8 - m,$$

and

$$(2k+6)\Theta(\infty; f) + (2k+7)\Theta(0; f) \leq 4k+13-m.$$

This contradicts (1.9)–(1.11). Hence, $C = 1$ and so $F \equiv G$, that is $f^m \equiv (f^n)^{(k)}$. This completes the proof of the theorem. $\square$

*Proof of Theorem 1.2.* Let $F = \frac{f^m}{a}$ and $G = \frac{(f^n)^{(k)}}{a}$.

Here, we observe that $\overline{N}(r, 0; F \mid\geq 2) \leq \overline{N}(r, 0; f) + S(r, f)$.

When $H \not\equiv 0$ we follow the proof of Theorem 1.1 while for $H \equiv 0$ we follow the proof of Theorem 1.3. So, we omit the detailed proof. $\square$

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