In this paper, we introduce a new general iterative process for approximating a common fixed point of a semigroup of nonexpansive mappings, with respect to strongly left regular sequence of means defined on an appropriate space of bounded real-valued functions of the semigroups and the set of solutions a system of variational inequality for finite family of inverse strongly monotone mappings in a real Hilbert space. Our result extends and improves the result announced by H. Piri and A.H. Badali [13] and some others.

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Key words: projection, common fixed point, amenable semigroup, iterative process, strong convergence, system of variational inequality.

1. INTRODUCTION

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm are denoted by $\| \cdot \|$, and let $C$ be a nonempty closed convex subset of $H$. A mapping $T: C \to C$ is called nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$, for all $x, y \in C$. By $\text{Fix}(T)$, we denote the set of fixed points of $T$ ($i.e., \text{Fix}(T) = \{ x \in H : Tx = x \}$). It is well known that $\text{Fix}(T)$ is closed and convex. Recall that a self-mapping $f: C \to C$ is a contraction on $C$ if there exists a constant $\alpha \in [0, 1)$ such that $\| f(x) - f(y) \| \leq \alpha \| x - y \|$ for all $x, y \in C$.

Let $B: C \to H$ be a mapping. The variational inequality problem, denoted by $\text{VI}(C, B)$, is to find $x \in C$ such that

\[
\langle Bx, y - x \rangle \geq 0,
\]

for all $y \in C$. The variational inequality problem has been extensively studied in literature. See, for example, [19, 20] and the references therein.

Definition 1.1. Let $B: C \to H$ be a mapping. Then $B$ is called

(i) strongly positive with constant $\overline{\gamma} > 0$, if there is a constant $\overline{\gamma} > 0$ such that

\[
\langle Bx, x \rangle \geq \overline{\gamma} \| x \|^2, \quad \forall x \in C.
\]
(ii) $\eta$–strongly monotone if there exists a positive constant $\eta$ such that
\[ \langle Bx - By, x - y \rangle \geq \eta \| x - y \|^2, \quad \forall x, y \in C, \]

(iii) $k$–Lipschitzian if there exists a positive constant $k$ such that
\[ \| Bx - By \| \leq k \| x - y \|, \quad \forall x, y \in C, \]

(iv) $\beta$–inverse strongly monotone if there exists a positive real number $\beta > 0$ such that
\[ \langle Bx - By, x - y \rangle \geq \beta \| Bx - By \|^2, \quad \forall x, y \in C. \]

It is obvious that any $\beta$–inverse strongly monotone mapping $B$ is $\frac{1}{\beta}$–Lipschitzian.

In 2001 Yamada [18] introduced the following hybrid iterative method for solving the variational inequality
\begin{equation}
(1.2) \quad x_{n+1} = T x_n - \mu \alpha_n F(T x_n), \quad n \in \mathbb{N},
\end{equation}
where $F$ is $k$–Lipschitzian and $\eta$–strongly monotone operator with $k > 0$, $\eta > 0$, $0 < \mu < \frac{2\eta}{k^2}$, then he proved that if $\{\alpha_n\}$ satisfying appropriate conditions, then $\{x_n\}$ generated by (1.2) converges strongly to the unique solution of the variational inequality:
\[ \langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in Fix(T). \]

In 2006, Marino and Xu [12] consider the following iterative method:
\begin{equation}
(1.3) \quad x_{n+1} = (I - \alpha_n A) T x_n + \alpha_n \gamma f(x_n), \quad n \geq 0.
\end{equation}

It is proved that if the sequence $\{\alpha_n\}$ of parameters satisfies the following conditions:

(C1) $\alpha_n \to 0,$

(C2) $\sum_{n=0}^\infty \alpha_n = \infty,$

(C3) either $\sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = 1.$

Then, the sequence $\{x_n\}$ generated by (1.3) converges strongly, as $n \to \infty$, to the unique solution of the variational inequality:
\[ \langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in Fix(T), \]
which is the optimality condition for minimization problem
\[ \min_{x \in Fix(T)} \frac{1}{2} \langle Ax, x \rangle - h(x), \]
where \( h \) is a potential function for \( \gamma f \) \( i.e., \) \( h'(x) = \gamma f(x) \), for all \( x \in H \). Some people also study the application of the iterative method (1.3) \([11, 14]\).

In 2010, Tian \([16]\) combined the iterative (1.3) with the iterative method (1.2) and considered the iterative methods:

\[
x_{n+1} = (I - \mu \alpha_n F)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0,
\]

and he proved that if the sequence \( \{\alpha_n\} \) of parameters satisfies the conditions \( (C_1), (C_2) \) and \( (C_3) \), then the sequences \( \{x_n\} \) generated by (1.4) converges strongly to the unique solution \( x^* \in Fix(T) \) of the variational inequality

\[
\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in Fix(T).
\]

Motivated and inspired by Atsushiba and Takahashi \([2]\), Ceng and Yao \([4]\), Kim \([8]\), Lau \( et \ al. \) \([9]\), Lau \( et \ al. \) \([10]\), Marino and Xu \([12]\), Tian \([16]\), Xu \([17]\) and Yamada \([18]\), Piri and Badali \([13]\) introduced the following iterative algorithm: Let \( x_0 \in C \) and

\[
\begin{align*}
y_n &= \beta_n x_n + (1 - \beta_n)P_C(x_n - \delta_n Bx_n), \\
x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F)T_{\mu_n}y_n, \quad n \geq 0.
\end{align*}
\]

where \( P_C \) is a metric projection of \( H \) onto \( C \), \( B \) is \( \beta \)-inverse strongly monotone, \( \varphi = \{T_t : t \in S\} \) is a nonexpansive semigroup on \( H \) such that the set \( \mathcal{F} = Fix(\varphi) \cap VI(C, B) \neq \emptyset \), \( X \) is a subspace of \( B(S) \) such that \( 1 \in X \) and the mapping \( t \to \langle T_t x, y \rangle \) is an element of \( X \) for each \( x, y \in H \), and \( \{\mu_n\} \) is a sequence of means on \( X \). They prove that if \( \{\mu_n\} \) is left regular and the sequences \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\delta_n\} \) of parameters satisfy appropriate conditions, then the sequences \( \{x_n\} \) and \( \{y_n\} \) generated by (1.5) converge strongly to the unique solution \( x^* \in \mathcal{F} \) of the variational inequalities:

\[
\begin{align*}
\langle (\mu F - \gamma f)x^*, x - x^* \rangle &\geq 0, \quad \forall x \in \mathcal{F}, \\
\langle Bx^*, y - x^* \rangle &\geq 0 \quad \forall y \in C,
\end{align*}
\]

In this paper, motivated and inspired by Atsushiba and Takahashi \([2]\), Ceng and Yao \([4]\), Kim \([8]\), Lau \( et \ al. \) \([9]\), Lau \( et \ al. \) \([10]\), Marino and Xu \([12]\), Katchang Kumam \([7]\), Tian \([16]\), Xu \([17]\), Yamada \([18]\) and Piri and Badali \([13]\), we introduce the following general iterative algorithm: Let \( x_0 \in C \) and

\[
\begin{align*}
y_{m+1,n} &= x_n, \\
y_{i,n} &= \gamma_{i,n}P_C(I - \delta_{i,n} A_i)y_{i+1,n} \\
&\quad + (1 - \gamma_{i,n})P_C(I - \delta_{i+1,n} A_{i+1})y_{i+1,n}, \quad i = 1, 2, \ldots, m, \\
x_{n+1} &= \alpha_n \gamma f(T_{\mu_n}P_C(I - \delta_{1,n} A_1)y_{1,n}) + \beta_n x_n \\
&\quad + ((1 - \beta_n)I - \alpha_n \mu F)T_{\mu_n}P_C(I - \delta_{2,n} A_2)y_{1,n},
\end{align*}
\]

where \( P_C \) is a metric projection of \( H \) onto \( C \), for \( i = 1, 2, \ldots, m + 1 \), \( A_i \) be inverse strongly monotone, \( \varphi = \{T_t : t \in S\} \) is a nonexpansive semigroup on
such that the set $\mathcal{F} = \bigcap_{i=1}^{m+1} VI(C, A_i) \cap Fix(\varphi) \neq \emptyset$, $X$ is a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \langle T_t x, y \rangle$ is an element of $X$ for each $x, y \in H$, and $\{\mu_n\}$ is a sequence of means on $X$. Our purpose in this paper is to introduce this general iterative algorithm for approximating a common element of the set of common fixed point of a semigroup of nonexpansive mappings and the set of solutions of system of variational inequalities for finite family of inverse strongly monotone mapping. We will prove that if $\{\mu_n\}$ is left regular sequence of means and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{i,n}\}_{i=1,n=1}^{m,\infty}$ and $\{\delta_{i,n}\}_{i=1,n=1}^{m,\infty}$ of parameters satisfy appropriate conditions, then the sequences $\{x_n\}$ and $\{y_{i,n}\}_{i=1,n=1}^{m,\infty}$ generated by (1.6) converge strongly to the unique solution $x^* \in \mathcal{F}$ of the system of the variational inequalities:

\[
\begin{align*}
    \langle (\mu F - \gamma f)x^*, x - x^* \rangle &\geq 0, \quad \forall x \in \mathcal{F}, \\
    \langle A_i x^*, y - x^* \rangle &\geq 0, \quad \forall y \in C, \ i = 1, 2, \ldots, m + 1.
\end{align*}
\]

2. PRELIMINARIES

This section collects some lemmas which will be used in the proofs of the main results in the next section.

**Lemma 2.1** ([6]). For a given $x \in H$, $y \in C$,

$$y = P_C x \iff \langle y - x, z - y \rangle \geq 0, \quad \forall z \in C.$$  

It is well known that $P_C$ is a firmly nonexpansive mapping of $H$ onto $C$ and satisfies

\begin{equation}
\| P_C x - P_C y \|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H
\end{equation}

Moreover, $P_C$ is characterized by the following properties: $P_C x \in C$ and for all $x \in H$, $y \in C$,

\begin{equation}
\langle x - P_C x, y - P_C x \rangle \leq 0.
\end{equation}

It is easy to see that (2.2) is equivalent to the following inequality

\begin{equation}
\| x - y \|^2 \geq \| x - P_C x \|^2 + \| y - P_C x \|^2.
\end{equation}

Using Lemma 2.1, one can see that the variational inequality (1.1) is equivalent to a fixed point problem.

It is easy to see that the following is true:

\begin{equation}
u \in VI(C, A) \iff u = P_C (u - \lambda Au), \quad \lambda > 0.
\end{equation}

A set-valued mapping $U : H \to 2^H$ is called monotone if for all $x, y \in H, f \in Ux$ and $g \in Uy$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $U : H \to 2^H$ is maximal if the graph of $G(U)$ of $U$ is not properly contained
in the graph of any other monotone mapping. It is known that a monotone mapping $U$ is maximal if and only if for $(x,f) \in H \times H, \langle x - y, f - g \rangle \geq 0$ for every $(y,g) \in G(U)$ implies that $f \in Ux$. Let $A$ be a monotone mapping of $C$ into $H$ and let $N_C x$ be the normal cone to $C$ at $x \in C$, that is, $N_C x = \{y \in H : \langle z - x, y \rangle \leq 0, \forall z \in C \}$ and define

$$Ux = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset & x \notin C. \end{cases}$$

Then $U$ is the maximal monotone and $0 \in Ux$ if and only if $x \in VI(C,A)$; see [15].

Let $C$ be a nonempty subset of a Hilbert space $H$ and $T : C \to H$ a mapping. Then $T$ is said to be demiclosed at $v \in H$ if, for any sequence $\{x_n\}$ in $C$, the following implication holds:

$$x_n \to u \in C \quad \text{and} \quad Tx_n \to v \quad \text{imply} \quad Tu = v,$$

where $\to$ (resp. $\rightharpoonup$) denotes strong (resp. weak) convergence.

**Lemma 2.2 ([1]).** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and suppose that $T : C \to H$ is nonexpansive. Then, the mapping $I - T$ is demiclosed at zero.

**Lemma 2.3 ([1]).** Let $H$ be a real Hilbert space. Then, for all $x, y \in H$

(i) $\| x - y \|^2 \leq \| x \|^2 + 2 \langle y, x + y \rangle$,

(ii) $\| x - y \|^2 \geq \| x \|^2 + 2 \langle y, x \rangle$.

**Lemma 2.4 ([18]).** Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $E$ and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \lim \inf \alpha_n \leq \lim \sup \alpha_n < 1$. Suppose $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)y_n$ for all integers $n \geq 0$ and

$$\lim \sup_{n \to \infty} (\| y_{n+1} - y_n \| - \| x_{n+1} - x_n \|) \leq 0.$$

Then, $\lim_{n \to \infty} \| y_n - x_n \| = 0$.

**Lemma 2.5 ([17]).** Let $\{a_n\}$ be a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n, \quad n \geq 0,$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{b_n\} \subset (0,1), \sum_{n=0}^{\infty} b_n = \infty$,

(ii) either $\lim \sup_{n \to \infty} c_n \leq 0$ or $\sum_{n=0}^{\infty} |b_n c_n| < \infty$.

Then, $\lim_{n \to \infty} a_n = 0$. 
Lemma 2.6 ([13]). Let $H$ be a real Hilbert space and $F$ be a $k-$Lipschitzian and $\eta-$strongly monotone operator with $k > 0$, $\eta > 0$. Let $0 < \mu < \frac{2\eta}{k^2}$ and $\tau = \mu(\eta - \frac{\mu k^2}{2})$. Then for $t \in (0, \min\{1, \frac{1}{\tau}\})$, $I - t\mu F$ is contraction with constant $1 - t\tau$.

Let $S$ be a semigroup and let $B(S)$ be the space of all bounded real valued functions defined on $S$ with supremum norm. For $s \in S$ and $f \in B(S)$, we define elements $l_sf$ and $r_sf$ in $B(S)$ by

$$(l_sf)(t) = f(st), \quad (r_sf)(t) = f(ts), \quad \forall t \in S.$$ 

Let $X$ be a subspace of $B(S)$ containing $1$ and let $X^*$ be its topological dual. An element $\mu$ of $X^*$ is said to be a mean on $X$ if $\| \mu \| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let $X$ be left invariant (resp. right invariant), i.e., $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean $\mu$ on $X$ is said to be left invariant (resp. right invariant) if $\mu(l_sf) = \mu(f)$ (resp. $\mu(r_sf) = \mu(f)$) for each $s \in S$ and $f \in X$. $X$ is said to be left (resp. right) amenable if $X$ has a left (resp. right) invariant mean. $X$ is amenable if $X$ is both left and right amenable. As is well known, $B(S)$ is amenable when $S$ is a commutative semigroup, see [10]. A net $\{\mu_\alpha\}$ of means on $X$ is said to be strongly left regular if

$$\lim_{\alpha} \| l_s^* \mu_\alpha - \mu_\alpha \| = 0,$$

for each $s \in S$, where $l_s^*$ is the adjoint operator of $l_s$.

Let $S$ be a semigroup and let $C$ be a nonempty closed and convex subset of a reflexive Banach space $E$. A family $\varphi = \{T_t : t \in S\}$ of mapping from $C$ into itself is said to be a nonexpansive semigroup on $C$ if $T_t$ is nonexpansive and $T_{ts} = T_tT_s$ for each $t, s \in S$. By $Fix(\varphi)$ we denote the set of common fixed points of $\varphi$, i.e.

$$Fix(\varphi) = \bigcap_{t \in S} \{x \in C : T_tx = x\}.$$ 

Lemma 2.7 ([10]). Let $S$ be a semigroup and $C$ be a nonempty closed convex subset of a reflexive Banach space $E$. Let $\varphi = \{T_t : t \in S\}$ be a nonexpansive semigroup on $H$ such that $\{T_tx : t \in S\}$ is bounded for some $x \in C$, let $X$ be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \langle T_tx, y^* \rangle$ is an element of $X$ for each $x \in C$ and $y^* \in E^*$, and $\mu$ is a mean on $X$. If we write $T_\mu x$ instead of $\int T_tx d\mu(t)$, then the following hold.

(i) $T_\mu$ is nonexpansive mapping from $C$ into $C$.

(ii) $T_\mu x = x$ for each $x \in Fix(\varphi)$.

(iii) $T_\mu x \in \overline{co}\{T_tx : t \in S\}$ for each $x \in C$. 


Notation. The open ball of radius $r$ centered at 0 is denoted by $B_r$ and for a subset $D$ of $H$, by $\overline{D}$, we denote the closed convex hull of $D$. For $\epsilon > 0$ and a mapping $T: D \to H$, we let $F_\epsilon(T; D)$ be the set of $\epsilon$– approximate fixed points sets of $T$, i.e. $F_\epsilon(T; D) = \{x \in D : \| x - Tx \| \leq \epsilon \}$.

3. MAIN RESULTS

Theorem 3.1. Let $C$ be a nonempty closed convex subset of real Hilbert space $H$, $F: C \to C$ be a $k$–Lipschitzian and $\eta$–strongly monotone operator with $k > 0$, $\eta > 0$, $f: C \to C$ be a contraction with coefficient $0 < \alpha < 1$, $\gamma$ be a number in $(0, \frac{\tau}{\alpha})$, where $\tau = \mu(\eta - \frac{\mu k^2}{2})$ and $0 < \mu < \frac{2\eta}{k^2}$. For $i = 1, 2, \cdots, m+1$, let $A_i: C \to H$ be $\delta_i$–inverse strongly monotone mapping. Let $S$ be a semigroup and $\varphi = \{T_t : t \in S\}$ be a nonexpansive semigroup of $C$ into itself such that $\mathcal{F} = \bigcap_{i=1}^{m+1} VI(C, A_i) \cap Fix(\varphi) \neq \emptyset$. Let $X$ be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T_t x, y \rangle$ is an element of $X$ for each $x \in C$ and $y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on $X$ such that $\sum_{n=1}^{\infty} \| \mu_{n+1} - \mu_n \| < \infty$. Let $\{\alpha_n\}, \{\gamma_{i,n}\}_{i=1}^{m}, \{\beta_n\}$ and $\{\delta_{i,n}\}_{i=1}^{m+1}, \{\epsilon_n\}_{i=1}^{\infty}$ be a sequence in $(0,1)$ satisfy the following conditions:

$(B_1)$ $0 < \liminf_{n \to \infty} \gamma_{i,n} \leq \limsup_{n \to \infty} \gamma_{i,n} < 1$ and $\{\delta_{i,n}\} \subset (0, 2\delta_i)$ $i = 1, 2, \cdots, m+1$,

$(B_2)$ $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n \to \infty} \beta_n = 0$.

$(B_3)$ $\sum_{n=1}^{\infty} | \alpha_{n+1} - \alpha_n | < \infty$, $\sum_{n=1}^{\infty} | \beta_{n+1} - \beta_n | < \infty$

$\sum_{n=1}^{\infty} | \delta_{i,n+1} - \delta_{i,n} | < \infty$, $\sum_{n=1}^{\infty} | \gamma_{i,n+1} - \gamma_{i,n} | < \infty$ $i = 1, 2, \cdots, m+1$.

If $x_0 \in C$ and $\{x_n\}$ and $\{y_{i,n}\}_{i=1}^{m}, \{y_{i,n}\}_{i=1}^{\infty}$ be generated by the iteration algorithm:

$y_{m+1,n} = x_n,$

$y_{i,n} = \gamma_{i,n} P_C(I - \delta_{i,n} A_i) y_{i+1,n} + (1 - \gamma_{i,n}) P_C(I - \delta_{i+1,n} A_{i+1}) y_{i+1,n}, \quad i = 1, 2, \cdots, m,$

$x_{n+1} = \alpha_n \gamma f(T_{\mu_n} P_C(I - \delta_{1,n} A_1) y_{1,n}) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n \mu F) T_{\mu_n} P_C(I - \delta_{2,n} A_2) y_{1,n},$

then, $\{x_n\}$ and $\{y_{i,n}\}_{i=1}^{m}, \{y_{i,n}\}_{i=1}^{\infty}$ converge strongly, as $n \to \infty$, to $x^* \in \mathcal{F}$, which is a unique solution of system of the variational inequalities.

$\begin{cases}
\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0, & \forall x \in \mathcal{F}, \\
\langle A_ix^*, y - x^* \rangle \geq 0, & \forall y \in C, \quad i = 1, 2, \cdots, m+1.
\end{cases}$
Proof. Since \( \{\delta_{i,n}\}_{i=1,n=1}^{m+1,\infty} \) satisfy in condition \((B_1)\) and \(A_i\) is \( \delta_i \)-inverse strongly monotone mapping, for any \( x, y \in C \), we have

\[
\| (I - \delta_{i,n}A_i)x - (I - \delta_{i,n}A_i)y \|^2 \\
= \| (x - y) - \delta_{i,n}(A_i x - A_i y) \|^2 \\
= \| x - y \|^2 - 2\delta_{i,n} \langle x - y, A_i x - A_i y \rangle + \delta^2_{i,n} \| A_i x - A_i y \|^2 \\
\leq \| x - y \|^2 - 2\delta_{i,n}\delta_i \| A_i x - A_i y \|^2 + \delta^2_{i,n} \| A_i x - A_i y \|^2 \\
= \| x - y \|^2 + \delta_{i,n}(\delta_{i,n} - 2\delta_i) \| A_i x - A_i y \|^2 \\
\leq \| x - y \|^2
\]

It follows that

\[(3.1) \quad \| (I - \delta_{i,n}A_i)x - (I - \delta_{i,n}A_i)y \| \leq \| x - y \|, \quad i = 1, 2, \ldots, m + 1.\]

Let \( p \in \mathcal{F} \), in the context of the variational inequality problem the characterization of projection \((2.4)\) implies that \( p = P_C(I - \delta_{i,n}A_i)p \), \( i = 1, 2, \ldots, m + 1 \). Using \((3.1)\), we get

\[
\| y_{i,n} - p \|^2 \\
= \| \gamma_{i,n}P_C(I - \delta_{i,n}A_i)y_{i+1,n} + (1 - \gamma_{i,n})P_C(I - \delta_{i+1,n}A_{i+1})y_{i+1,n} - p \|^2 \\
\leq \gamma_{i,n} \| P_C(I - \delta_{i,n}A_i)y_{i+1,n} - p \|^2 + (1 - \gamma_{i,n}) \| P_C(I - \delta_{i+1,n}A_{i+1})y_{i+1,n} - p \|^2 \\
\leq \gamma_{i,n} \| y_{i+1,n} - p \|^2 + (1 - \gamma_{i,n}) \| y_{i+1,n} - p \|^2 = \| y_{i+1,n} - p \|^2
\]

Which implies that

\[(3.2) \quad \| y_{1,n} - p \| \leq \| y_{2,n} - p \| \leq \cdots \leq \| y_{m,n} - p \| \leq \| y_{m+1,n} - p \| = \| x_n - p \|\]

We claim that \( \{x_n\} \) is bounded. Let \( p \in \mathcal{F} \), using Lemma 2.6, we have

\[
\| x_{n+1} - p \| = \| \alpha_n \gamma f(T_{\mu_n}P_C(I - \delta_{1,n}A_1)y_{1,n}) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n\mu F)T_{\mu_n}P_C(I - \delta_{2,n}A_2)y_{1,n} - p \| \\
= \| (1 - \beta_n)I - \alpha_n\mu F)T_{\mu_n}P_C(I - \delta_{2,n}A_2)y_{1,n} - \alpha_n\gamma f(T_{\mu_n}P_C(I - \delta_{1,n}A_1)y_{1,n}) - \mu F p \| \\
= \| (1 - \beta_n - \alpha_n\tau) \| T_{\mu_n}P_C(I - \delta_{2,n}A_2)y_{1,n} - T_{\mu_n}P_C(I - \delta_{2,n}A_2)p \| + \beta_n \| x_n - p \| + \alpha_n \| \gamma f(T_{\mu_n}P_C(I - \delta_{1,n}A_1)y_{1,n}) - \mu F p \| \\
\leq \| y_{1,n} - p \| + \beta_n \| x_n - p \| + \alpha_n \| \gamma f(T_{\mu_n}P_C(I - \delta_{1,n}A_1)y_{1,n}) - \mu F p \| \\
\leq (1 - \beta_n - \alpha_n\tau) \| y_{1,n} - p \| + \beta_n \| x_n - p \| + \alpha_n \| \gamma f(p) - \mu F p \| \\
\leq (1 - \alpha_n(\tau - \gamma\alpha)) \| x_n - p \| + \alpha_n \| \gamma f(p) - \mu F p \|
\]
By induction,
\[ \| x_n - p \| \leq \max \{ (\tau - \gamma \alpha)^{-1} \| \gamma f(p) - \mu Fp \|, \| x_0 - p \| \} = M_0. \]

Hence, \( \{x_n\} \) is bounded and also \( \{y_{i,n}\}_{i=1,n=1}^{m, \infty} \) and \( \{f(P_C(I-\delta_{1,n}A_1)y_{i,n})\} \) are bounded. Set \( D = \{ y \in H : \| y - p \| \leq M_0 \} \). We remark that \( D \) is \( \varphi \)-invariant bounded closed convex set and \( \{x_n\}_{n=1}^{\infty}, \{y_{i,n}\}_{i=1,n=1}^{m, \infty}, \{P_C(I-\delta_{i,n}A_i)y_{i+1,n}\}_{i=1,n=1}^{m, \infty} \subset D \). Now, we claim that

\[ (3.3) \quad \limsup_{n \to \infty} \sup_{y \in D} \| T_{\mu_n}y - T_T T_{\mu_n}y \| = 0, \quad \forall t \in S. \]

This assertion is proved in [13]. We now show that

\[ (3.4) \quad \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0 \]

From definition of \( \{x_n\} \), we have

\[
\| x_{n+1} - x_n \|
\leq \| \alpha_n \gamma f(T_{\mu_n} P_C(I - \delta_{1,n}A_1)y_{1,n}) + \beta_n x_n \\
+ ((1 - \beta_n) I - \alpha \mu F) T_{\mu_n} P_C(I - \delta_{2,n}A_2)y_{1,n} \\
- \alpha_{n-1} \gamma f(T_{\mu_{n-1}} P_C(I - \delta_{1,n-1}A_1)y_{1,n-1}) + \beta_{n-1} x_{n-1} \\
+ ((1 - \beta_{n-1}) I - \alpha \mu F) T_{\mu_{n-1}} P_C(I - \delta_{2,n-1}A_2)y_{1,n-1} \|
\leq \alpha_n \gamma \| f(T_{\mu_n} P_C(I - \delta_{1,n}A_1)y_{1,n}) - f(T_{\mu_n} P_C(I - \delta_{1,n-1}A_1)y_{1,n-1}) \|
+ (\alpha_n - \alpha_{n-1}) \gamma f(T_{\mu_{n-1}} P_C(I - \delta_{1,n-1}A_1)y_{1,n-1}) + \beta_n [x_n - x_{n-1}]
+ (\beta_n - \beta_{n-1}) x_{n-1} + [(1 - \beta_n) I - \alpha \mu F) T_{\mu_n} P_C(I - \delta_{2,n}A_2)y_{1,n} \\
- ((1 - \beta_n) I - \alpha \mu F) T_{\mu_{n-1}} P_C(I - \delta_{2,n-1}A_2)y_{1,n-1} \\
+ [(1 - \beta_n) I - \alpha \mu F) T_{\mu_{n-1}} P_C(I - \delta_{2,n-1}A_2)y_{1,n-1} \\
- ((1 - \beta_n) I - \alpha \mu F) T_{\mu_{n-1}} P_C(I - \delta_{2,n-1}A_2)y_{1,n-1} \|
\leq \alpha_n \gamma \| f(T_{\mu_n} P_C(I - \delta_{1,n}A_1)y_{1,n}) - T_{\mu_n} P_C(I - \delta_{1,n-1}A_1)y_{1,n-1} \|
+ | \alpha_n - \alpha_{n-1} | \| f(T_{\mu_{n-1}} P_C(I - \delta_{1,n-1}A_1)y_{1,n-1}) \|
+ | \alpha_n - \alpha_{n-1} | \| f(T_{\mu_{n-1}} P_C(I - \delta_{1,n-1}A_1)y_{1,n-1}) \|
+ | \beta_n - \beta_{n-1} | \| x_{n-1} \|
+ (1 - \beta_n - \alpha \tau) \| T_{\mu_n} P_C(I - \delta_{2,n}A_2)y_{1,n} - T_{\mu_{n-1}} P_C(I - \delta_{2,n-1}A_2)y_{1,n-1} \|
+ | \beta_n - \beta_{n-1} | \| T_{\mu_n} P_C(I - \delta_{2,n-1}A_2)y_{1,n-1} \|
+ | \alpha_n - \alpha_{n-1} | \| \mu FT_{\mu_{n-1}} P_C(I - \delta_{2,n-1}A_2)y_{1,n-1} \|

From (2.4) for p \in F, we have

\[
\| T_{\mu_{n-1}} P_C(I - \delta_{i,n-1}A_i)y_{1,n-1} - T_{\mu_n} P_C(I - \delta_{i,n}A_i)y_{1,n} \|
\leq \| T_{\mu_{n-1}} P_C(I - \delta_{i,n-1}A_i)y_{1,n+1} - T_{\mu_{n-1}} P_C(I - \delta_{i,n}A_i)y_{1,n} \|
+ \| T_{\mu_{n+1}} P_C(I - \delta_{i,n}A_i)y_{1,n} - T_{\mu_{n-1}} P_C(I - \delta_{i,n}A_i)y_{1,n} \|

\]
\[ \| y_{i,n+1} - y_{i,n} \| \\
\leq \| \gamma_{i,n+1} u_{i,n+1} + (1 - \gamma_{i,n+1}) v_{i,n+1} - \gamma_{i,n} u_{i,n} - (1 - \gamma_{i,n}) v_{i,n} \| \\
+ \| \gamma_{i,n+1} (u_{i,n+1} - u_{i,n}) + (\gamma_{i,n+1} - \gamma_{i,n}) u_{i,n} + (1 - \gamma_{i,n+1}) (v_{i,n+1} - v_{i,n}) \| \\
= \gamma_{i,n+1} \| u_{i,n+1} - u_{i,n} \| + (1 - \gamma_{i,n+1}) \| v_{i,n+1} - v_{i,n} \| \\
+ \| \gamma_{i,n+1} - \gamma_{i,n} \| \| u_{i,n} \| + \| v_{i,n} \| \\
= \gamma_{i,n+1} \| P_C(I - \delta_{i,n+1} A_i) y_{i+1, n+1} - P_C(I - \delta_{i,n+1} A_i) y_{i+1, n} \| \\
+ P_C(I - \delta_{i,n+1} A_i) y_{i+1, n+1} - P_C(I - \delta_{i,n+1} A_i) y_{i+1, n} \| \\
+ (1 - \gamma_{i,n+1}) \| P_C(I - \delta_{i+1, n+1} A_{i+1}) y_{i+1, n+1} - P_C(I - \delta_{i+1, n+1} A_{i+1}) y_{i+1, n} \| \\
+ P_C(I - \delta_{i+1, n+1} A_{i+1}) y_{i+1, n+1} - P_C(I - \delta_{i+1, n+1} A_{i+1}) y_{i+1, n} \| \\
+ \| \gamma_{i,n+1} - \gamma_{i,n} \| \| u_{i,n} \| + \| v_{i,n} \| \\
\leq \gamma_{i,n+1} \| y_{i+1, n+1} - y_{i+1, n} \| + \gamma_{i,n+1} \| \delta_{i,n+1} - \delta_{i,n} \| \| A_i y_{i+1, n} \| \\
+ (1 - \gamma_{i,n+1}) \| y_{i+1, n+1} - y_{i+1, n} \| \\
+ (1 - \gamma_{i,n+1}) \| \delta_{i+1, n+1} - \delta_{i+1, n} \| \| A_{i+1} y_{i+1, n} \| \\
+ \| \gamma_{i,n+1} - \gamma_{i,n} \| \| u_{i,n} \| + \| v_{i,n} \| \\
= \| y_{i+1, n+1} - y_{i+1, n} \| + \gamma_{i,n+1} \| \delta_{i,n+1} - \delta_{i,n+1} \| \| A_i y_{i+1, n} \| \\
+ (1 - \gamma_{i,n+1}) \| \delta_{i+1, n+1} - \delta_{i+1, n} \| \| A_{i+1} y_{i+1, n} \| \\
+ \| \gamma_{i,n+1} - \gamma_{i,n} \| \| u_{i,n} \| + \| v_{i,n} \| \\
\text{which implies that for some approximate constant } M_i > 0, \\
\| y_{i,n+1} - y_{i,n} \| \leq \| y_{i,n+1} - y_{i,n} \| + \sum_{j=i}^{m} \| \delta_{j,n+1} - \delta_{j,n} \| M_i. \]

From this and (3.2), we get
\[ \| y_{i,n+1} - y_{i,n} \| \leq \| x_{n+1} - x_n \| + \sum_{j=i}^{m} \| \delta_{j,n+1} - \delta_{j,n} \|. \]
(3.5) \[ + |\delta_{j+1,n+1} - \delta_{j+1,n}| + |\gamma_{j,n+1} - \gamma_{j,n}|]M_j. \]

Therefore,
\[
\| x_{n+1} - x_n \| \\
\leq (1 - \beta_n - \alpha_n(\tau - \gamma\alpha))\| x_n - x_{n-1} \| + \sum_{j=1}^{m} \| \delta_{j,n} - \delta_{j,n-1} \|
\]
\[
+ |\delta_{j+1,n} - \delta_{j+1,n-1}| + |\gamma_{j,n} - \gamma_{j,n-1}|]M_j + |\delta_{2,n-1} - \delta_{2,n}|]A_2y_{1,n} \]
\[
+ |\delta_{1,n-1} - \delta_{1,n}|]A_1y_{1,n} + |\mu_{n-1} - 2\mu_n | (|y_{1,n} - p| + |p|) \]
\[
+ |\alpha_n - \alpha_{n-1}| |\gamma f(T_{\mu_{n-1}}P_C(I - \delta_{1,n-1}A_1)y_{1,n-1})| \]
\[
+ \beta_n \| x_n - x_{n-1} \| + |\beta_n - \beta_{n-1}| \| x_{n-1} \|
\]
\[
+ |\beta_{n} - \beta_{n-1}| ||T_{\mu_{n-1}}P_C(I - \delta_{2,n-1}A_2)y_{1,n-1}|| \]
\[
+ |\alpha_n - \alpha_{n-1}| ||\mu FT_{\mu_{n-1}}P_C(I - \delta_{2,n-1}A_2)y_{1,n-1}|| \]
\[
\leq (1 - \alpha_n(\tau - \gamma\alpha)) \| x_n - x_{n-1} \| + \sum_{j=1}^{m} \| \delta_{j,n} - \delta_{j,n-1} \|
\]
\[
+ |\delta_{j+1,n} - \delta_{j+1,n-1}| + |\gamma_{j,n} - \gamma_{j,n-1}|]M_j + |\delta_{1,n-1} - \delta_{2,n}|]A_2y_{1,n} \]
\[
+ |\delta_{2,n-1} - \delta_{2,n}|]A_2y_{1,n} + |\mu_{n-1} - \mu_n | (|y_{1,n} - p| + |p|) \]
\[
+ |\alpha_n - \alpha_{n-1}| ||\gamma f(T_{\mu_{n-1}}P_C(I - \delta_{1,n-1}A_1)y_{1,n-1})|| \]
\[
+ ||\mu FT_{\mu_{n-1}}P_C(I - \delta_{2,n-1}A_2)y_{1,n-1}|| \]
\[
(3.6) \]
\[
+ |\beta_{n} - \beta_{n-1}| ||x_{n-1}|| + ||T_{\mu_{n-1}}P_C(I - \delta_{2,n-1}A_2)y_{1,n-1}|| \]
Thus, it follows from (3.6), condition (B3) and Lemma 2.4 that
\[
\lim_{n \to \infty} (||x_{n+1} - x_n||) = 0.
\]

In this stage, we will show
\[
(3.7) \lim_{n \to \infty} ||x_n - T_{\epsilon}x_n|| = 0, \quad \forall \epsilon \in S.
\]

Let \( t \in S \) and \( \epsilon > 0 \). By [3, Theorem 1.2], there exists \( \delta > 0 \) such that
\[
(3.8) \quad \overline{co}F_\delta(T_{\epsilon}; D) + B_\delta \subset F_\epsilon(T_{\epsilon}; D), \quad \forall \epsilon \in S.
\]

From condition \( B_1 \) and (3.3), there exists \( N_1 \in \mathbb{N} \) such that
\[
\alpha_n \leq \frac{\delta}{2(\tau + \mu k)M_0}, \quad \beta_n \leq \frac{\delta}{4M_0}, \quad T_{\mu_n}P_C(I - \delta_{2,n}A_2)y_{1,n} \in F_\delta(T_{\epsilon}; D), \quad n \geq N_1.
\]

By Lemma 2.6 and (2.4), for \( p \in F \) we have
\[
\alpha_n \| \gamma f(P_C(I - \delta_{1,n}A_1)y_{1,n}) - \mu FT_{\mu_n}P_C(I - \delta_{2,n}A_2)y_{1,n} \|
\]
\[ \leq \alpha_n[\gamma \| f(PC(I - \delta_{1,n}A_1)y_{1,n}) - f(p) \| + \| \gamma f(p) - \mu Fp \| + \mu Fp - \mu FT_{\mu_n}PC(I - \delta_{2,n}A_2)y_{1,n} \|] \]
\[ \leq \alpha_n(\gamma\alpha \| y_{1,n} - p \| + \| \gamma f(p) - \mu Fp \| + \mu k \| y_{1,n} - p \|) \]
\[ \leq \alpha_n(\gamma\alpha M_0 + (\tau - \gamma\alpha)M_0 + \mu kM_0) \]
\[ \leq \alpha_n(\tau + \mu k)M_0 \leq \frac{\delta}{2}, \]

and
\[ \beta_n \| x_n - T_{\mu_n}PC(I - \delta_{2,n}A_2)y_{1,n} \| \]
\[ \leq \beta_n[\| x_n - p \| + \| T_{\mu_n}PC(I - \delta_{2,n}A_2)y_{1,n} - p \|] \]
\[ \leq \beta_n[\| x_n - p \| + \| y_{1,n} - p \|] \]
\[ \leq 2\beta_n \| x_n - p \| \]
\[ \leq 2\beta_nM_0 \leq \frac{\delta}{2}. \]

Therefore, we have
\[ x_{n+1} = T_{\mu_n}PC(I - \delta_{2,n}A_2)y_{1,n} + \alpha_n[\gamma f(PC(I - \delta_{1,n}A_1)y_{1,n}) - \mu FT_{\mu_n}PC(I - \delta_{2,n}A_2)y_{1,n}] + \beta_n[x_n - T_{\mu_n}PC(I - \delta_{2,n}A_2)y_{1,n}] \]
\[ \in F_\delta(T_t; D) + B_{\frac{\delta}{2}} + B_{\frac{\delta}{2}} \subset F_\epsilon(T_t; D), \quad n \geq N_1. \]

This shows that
\[ \| x_n - T_tx_n \| \leq \epsilon, \quad \forall n \geq N_1. \]

Since \( \epsilon > 0 \) is arbitrary, we get (3.7).

Let \( Q = P_\mathcal{F} \). Then \( Q(I - \mu F + \gamma f) \) is a contraction of \( H \) into itself. In fact, we see that
\[ \| Q(I - \mu F + \gamma f)x - Q(I - \mu F + \gamma f)y \| \]
\[ \leq \| (I - \mu F + \gamma f)x - (I - \mu F + \gamma f)y \| \]
\[ \leq \| (I - \mu F)x - (I - \mu F)y \| + \gamma \| f(x) - f(y) \| \]
\[ \leq (1 - \tau) \| x - y \| + \gamma \alpha \| x - y \| \]
and hence, \( Q(I - \mu F + \gamma f) \) is a contraction due to \( (1 - (\tau - \gamma\alpha)) \in (0, 1) \).

Therefore, by Banach’s contraction principle, \( P_\mathcal{F}(I - \mu F + \gamma f) \) has a unique fixed point \( x^* \). Then using (2.4), \( x^* \) is the unique solution of the variational inequality:
\[ \langle (\gamma f - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathcal{F}. \]

We show that
\[ \limsup_{n \to \infty} \langle \gamma f(x^*) - \mu Fx^*, x_n - x^* \rangle \leq 0. \]
We can choose a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) such that
\[
\limsup_{n \to \infty} \langle \gamma f(x^*) - \mu Fx^*, x_n - x^* \rangle = \lim_{j \to \infty} \langle \gamma f(x^*) - \mu Fx^*, x_{n_j} - x^* \rangle.
\]

Because \( \{x_{n_j}\} \) is bounded, therefore \( \{x_{n_j}\} \) has subsequence \( \{x_{n_{j_k}}\} \) such that \( x_{n_{j_k}} \to z \). With no loss of generality, we may assume that \( x_{n_j} \to z \). It follows from (3.7) and Lemma 2.2 that \( z \in Fix(T) \).

In this stage, we will show that
\[
\lim_{n \to \infty} \| u_{i,n} - y_{i+1,n} \| = 0
\]
(3.12)
\[
\lim_{n \to \infty} \| v_{i,n} - y_{i+1,n} \| = 0, \quad i = 1, 2, \cdots, m.
\]

Let \( p \in \mathcal{F} \), from (3.2), Lemma 2.3 and definition of \( \{x_n\} \), we have
\[
\| x_{n+1} - p \|^2
= \| \alpha_n \gamma f(TPC(I - \delta_{1,n}A_1)y_{1,n}) + \beta_n x_n
+ ((1 - \beta_n)I - \alpha_n \mu F)TPC(I - \delta_{2,n}A_2)y_{1,n} - p \|^2
\leq \| ((1 - \beta_n)I - \alpha_n F)TPC(I - \delta_{2,n}A_2)y_{1,n} - ((1 - \beta_n)I - \alpha_n F)p \|^2
+ \beta_n \langle x_n - p, x_{n+1} - p \rangle
+ 2\beta_n \langle x_n - p, x_{n+1} - p \rangle
\leq \langle 1 - \beta_n - \alpha_n \tau \rangle^2 \| y_{1,n} - p \|^2 + 2\beta_n \langle x_n - p, x_{n+1} - p \rangle
+ 2\alpha_n \gamma f(TPC(I - \delta_{1,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle
\leq \gamma_{i,n} \| P_i(I - \delta_{i,n}A_i)y_{i+1,n} + (1 - \gamma_{i,n})P_i(I - \delta_{i+1,n}A_{i+1})y_{i+1,n} - p \|^2
+ 2\beta_n \langle x_n - p, x_{n+1} - p \rangle + 2\alpha_n \gamma f(TPC(I - \delta_{1,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle
\leq \gamma_{i,n} \| P_i(I - \delta_{i,n}A_i)y_{i+1,n} - p \|^2 + (1 - \gamma_{i,n}) \| P_i(I - \delta_{i+1,n}A_{i+1})y_{i+1,n} - p \|^2
+ 2\beta_n \langle x_n - p, x_{n+1} - p \rangle + 2\alpha_n \gamma f(TPC(I - \delta_{1,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle
\leq \gamma_{i,n} \| (y_{i+1,n} - p) - \delta_{i,n}(A_iy_{i+1,n} - A_ip) \|^2 + (1 - \gamma_{i,n}) \| y_{i+1,n} - p \|^2
\]
\[ + 2\beta_n \langle x_n - p, x_{n+1} - p \rangle + 2\alpha_n \langle \gamma f(TP_C(I - \delta_{1,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \]
\[ \leq \gamma_{i,n}[\| y_{i+1,n} - p \|^2 + \delta_{i,n}^2 \| A_iy_{i+1,n} - A_ip \|^2 - 2\delta_{i,n} \langle A_iy_{i+1,n} - A_ip, y_{i+1,n} - p \rangle] \]
\[ + (1 - \gamma_{i,n}) \| y_{i+1,n} - p \|^2 + 2\beta_n \langle x_n - p, x_{n+1} - p \rangle \]
\[ + 2\alpha_n \langle \gamma f(TP_C(I - \delta_{1,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \]
\[ \leq \| y_{i+1,n} - p \|^2 + \delta_{i,n}^2 \| A_iy_{i+1,n} - A_ip \|^2 - 2\delta_{i,n} \delta_i \| A_iy_{i+1,n} - A_ip \|^2 \]
\[ + 2\beta_n \langle x_n - p, x_{n+1} - p \rangle + 2\alpha_n \langle \gamma f(TP_C(I - \delta_{1,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \]
\[ \leq \| x_n - p \|^2 + \delta_{i,n}(\delta_{i,n} - 2\delta_i) \| A_iy_{i+1,n} - A_ip \|^2 \]
\[ + 2\beta_n \langle x_n - p, x_{n+1} - p \rangle + 2\alpha_n \langle \gamma f(TP_C(I - \delta_{1,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \]
which implies that
\[ - \delta_{i,n}(\delta_{i,n} - 2\delta_i) \| A_iy_{i+1,n} - A_ip \|^2 \]
\[ \leq \| x_n - p \|^2 - \| x_{n+1} - p \|^2 + 2\beta_n \langle x_n - p, x_{n+1} - p \rangle \]
\[ + 2\alpha_n \langle \gamma f(TP_C(I - \delta_{1,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \]
\[ \leq [\| x_n - p \| + \| x_{n+1} - p \|] \| x_{n+1} - x_n \| \]
\[ + 2\beta_n \langle x_n - p, x_{n+1} - p \rangle \]
\[ + 2\alpha_n \langle \gamma f(TP_C(I - \delta_{1,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \]
(3.14)

Using (3.4), (3.14) and condition $B_2$, we get
\[ \lim_{n \to \infty} \| A_iy_{i+1,n} - A_ip \| = 0, \quad i = 1, 2, \ldots, m. \]
(3.15)

Repeating the same argument as above, we conclude that
\[ \lim_{n \to \infty} \| A_{i+1}y_{i+1,n} - A_{i+1}p \| = 0, \quad i = 1, 2, \ldots, m. \]
(3.16)

From (2.1), we have
\[ \| u_{i,n} - p \|^2 \]
\[ = \| P_C(I - \delta_{i,n}A_i)y_{i+1,n} - P_C(I - \delta_{i,n}A_ip) \|^2 \]
\[ \leq \langle (I - \delta_{i,n}A_i)y_{i+1,n} - (I - \delta_{i,n}A_ip), u_{i,n} - p \rangle \]
\[ = \frac{1}{2}[\| (I - \delta_{i,n}A_i)y_{i+1,n} - (I - \delta_{i,n}A_ip) \|^2 + \| u_{i,n} - p \|^2 \]
\[ - \| (I - \delta_{i,n}A_i)y_{i+1,n} - (I - \delta_{i,n}A_ip) - (u_{i,n} - p) \|^2] \]
\[ \leq \frac{1}{2}[\| y_{i+1,n} - p \|^2 + \| u_{i,n} - p \|^2 \]
\[ - \| (I - \beta_{i,n}A_i)y_{i+1,n} - (I - \delta_{i,n}A_ip) - (u_{i,n} - p) \|^2] \]
\[ = \frac{1}{2}[\| y_{i+1,n} - p \|^2 + \| u_{i,n} - p \|^2 - \| y_{i+1,n} - u_{i,n} \|^2 \]
\[ + 2\delta_{i,n}\langle y_{i+1,n} - u_{i,n}, A_iy_{i+1,n} - A_ip \rangle - \delta_{i,n}^2 \| A_iy_{i+1,n} - A_ip \|^2]. \]
So, we obtain
\[
\| u_{i,n} - p \|^2 \leq \| y_{i+1,n} - p \|^2 - \| y_{i+1,n} - u_{i,n} \|^2 \\
+ 2\delta_{i,n} \langle y_{i+1,n} - u_{i,n}, A_i y_{i+1,n} - A_i p \rangle \\
- \delta_{i,n}^2 \| A_i y_{i+1,n} - A_i p \|^2, \quad i = 1, 2, \ldots, m.
\]
(3.17)

By using same method as (3.17), we have
\[
\| v_{i,n} - p \|^2 \leq \| y_{i+1,n} - p \|^2 - \| y_{i+1,n} - v_{i,n} \|^2 \\
+ 2\delta_{i+1,n} \langle y_{i+1,n} - v_{i,n}, A_{i+1} y_{i+1,n} - A_{i+1} p \rangle \\
- \delta_{i+1,n}^2 \| A_{i+1} y_{i+1,n} - A_{i+1} p \|^2, \quad i = 1, 2, \ldots, m.
\]
(3.18)

From (3.13), (3.17), (3.18) and Lemma 2.3, we have
\[
\| x_{n+1} - p \|^2 \\
\leq \| y_{i,n} - p \|^2 + 2\beta_n \langle x_n - p, x_{n+1} - p \rangle \\
+ 2\alpha_n \langle \gamma f(TPC(I - \delta_{i,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \\
\leq \| y_{i,n} - p \|^2 + 2\beta_n \| y_{i+1,n} - u_{i,n} \|^2 + 2\delta_{i,n} \langle y_{i+1,n} - u_{i,n}, A_i y_{i+1,n} - A_i p \rangle \\
- \delta_{i,n}^2 \| A_i y_{i+1,n} - A_i p \|^2 + (1 - \gamma_{i,n}) \| y_{i+1,n} - p \|^2 - \| y_{i+1,n} - v_{i,n} \|^2 \\
+ 2\delta_{i+1,n} \langle y_{i+1,n} - v_{i,n}, A_{i+1} y_{i+1,n} - A_{i+1} p \rangle - \delta_{i+1,n}^2 \| A_{i+1} y_{i+1,n} - A_{i+1} p \|^2 \\
+ 2\beta_n \langle x_n - p, x_{n+1} - p \rangle + 2\alpha_n \langle \gamma f(TPC(I - \delta_{i,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \\
\leq \| y_{i+1,n} - p \|^2 + 2\beta_n \| y_{i+1,n} - u_{i,n} \|^2 + 2\delta_{i,n} \langle y_{i+1,n} - u_{i,n}, A_i y_{i+1,n} - A_i p \rangle \\
- \delta_{i,n}^2 \| A_i y_{i+1,n} - A_i p \|^2 + (1 - \gamma_{i,n}) \| y_{i+1,n} - v_{i,n} \|^2 \\
+ 2\delta_{i+1,n} \langle y_{i+1,n} - v_{i,n}, A_{i+1} y_{i+1,n} - A_{i+1} p \rangle - \delta_{i+1,n}^2 \| A_{i+1} y_{i+1,n} - A_{i+1} p \|^2 \\
+ 2\beta_n \langle x_n - p, x_{n+1} - p \rangle + 2\alpha_n \langle \gamma f(TPC(I - \delta_{i,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \\
\leq \| x_n - p \|^2 + 2\beta_n \| y_{i+1,n} - u_{i,n} \|^2 + 2\delta_{i,n} \langle y_{i+1,n} - u_{i,n}, A_i y_{i+1,n} - A_i p \rangle \\
- \delta_{i,n}^2 \| A_i y_{i+1,n} - A_i p \|^2 + (1 - \gamma_{i,n}) \| y_{i+1,n} - v_{i,n} \|^2 \\
+ 2\delta_{i+1,n} \langle y_{i+1,n} - v_{i,n}, A_{i+1} y_{i+1,n} - A_{i+1} p \rangle - \delta_{i+1,n}^2 \| A_{i+1} y_{i+1,n} - A_{i+1} p \|^2 \\
+ 2\beta_n \langle x_n - p, x_{n+1} - p \rangle + 2\alpha_n \langle \gamma f(TPC(I - \delta_{i,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \\
\]
Which implies that
\[
\gamma_{i,n} \| y_{i+1,n} - u_{i,n} \|^2 \\
\leq \| x_n - p \| + \| x_{n+1} - p \| \| x_{n+1} - x_n \|
\]
\[+ \gamma_{i,n}[2\delta_{i,n}\langle y_{i+1,n} - u_{i,n}, A_i y_{i+1,n} - A_ip \rangle]
\]
\[- \delta_{i,n}^2 \parallel A_i y_{i+1,n} - A_ip \parallel^2 + (1 - \gamma_{i,n})[+2\delta_{i+1,n}\langle y_{i+1,n} - v_{i,n},
\]
\[A_{i+1} y_{i+1,n} - A_{i+1}p \rangle - \delta_{i+1,n}^2 \parallel A_{i+1} y_{i+1,n} - A_{i+1}p \parallel^2]
\[+ 2\beta_n\langle x_n - p, x_{n+1} - p \rangle + 2\alpha_n\langle \gamma f(T P_C(I - \delta_{1,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \]

and

\[(1 - \gamma_{i,n}) \parallel y_{i+1,n} - v_{i,n} \parallel^2 \leq \parallel x_n - p \parallel + \parallel x_{n+1} - p \parallel \parallel x_{n+1} - x_n \parallel + \gamma_{i,n}[2\delta_{i,n}\langle y_{i+1,n} - u_{i,n},
\]
\[A_i y_{i+1,n} - A_ip \rangle + \delta_{i,n}^2 \parallel A_i y_{i+1,n} - A_ip \parallel^2 + (1 - \gamma_{i,n})[2\delta_{i+1,n}\langle y_{i+1,n} - v_{i,n}, A_{i+1} y_{i+1,n} - A_{i+1}p \rangle - \delta_{i+1,n}^2 \parallel A_{i+1} y_{i+1,n} - A_{i+1}p \parallel^2]
\[+ 2\beta_n\langle x_n - p, x_{n+1} - p \rangle + 2\alpha_n\langle \gamma f(T P_C(I - \delta_{1,n}A_1)y_{1,n}) - Fp, x_{n+1} - p \rangle \]

Therefore, using condition \(B_2\), (3.4), (3.15) and (3.16) we get (3.12). Simple calculation shows that,

\[(3.19) \quad \parallel y_{i,n} - x_n \parallel \leq \sum_{j=i}^{m} \parallel y_{j,n} - y_{j+1,n} \parallel, \quad i = 1, 2, \ldots, m.\]

From (3.12) and (3.19), we get

\[(3.20) \quad \lim_{n \to \infty} \parallel y_{i,n} - x_n \parallel = 0, \quad i = 1, 2, \ldots, m.\]

Now, let us show that for \(i = 1, 2, \ldots, m+1, z \in VI(C, A_i)\). Let \(U_i: H \to 2^H\) be a set-valued mapping is defined by

\[U_i x = \begin{cases} A_i x + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases} \]

where \(N_C x\) is the normal cone to \(C\) at \(x \in C\). Since \(A_i\) is monotone. Thus, \(U_i\) is maximal monotone see [15]. Let \((x, y) \in G(U_i)\), hence, \(y - A_i x \in N_C x\) and since \(u_{i,n} = P_C(I - \delta_{i,n}A_i)y_{i+1,n}\) therefore, \(\langle x - u_{i,n}, y - A_i x \rangle \geq 0\). On the other hand from (2.3), we have

\[\langle x - u_{i,n}, u_{i,n} - (y_{i+1,n} - \delta_{i,n} A_i y_{i+1,n}) \rangle \geq 0,\]

that is

\[\langle x - u_{i,n}, \frac{u_{i,n} - y_{i+1,n}}{\delta_{i,n}} + A_i y_{i+1,n} \rangle \geq 0.\]

Therefore, we have

\[\langle x - u_{i,n}, y \rangle \geq \langle x - u_{i,n}, A_i x \rangle \geq \langle x - u_{i,n}, A_i x \rangle - \langle x - u_{i,n}, \frac{u_{i,n} - y_{i+1,n}}{\delta_{i,n}} + A_i y_{i+1,n} \rangle.\]
Lemma 2.3, we have
\[
\langle x - u_{i,n}, A_i x - \frac{u_{i,n} - y_{i+1,n}}{\delta_{i,n}} - A_i y_{i+1,n} \rangle
\]
\[
= \langle x - u_{i,n}, A_i x - A_i u_{i,n} \rangle + \langle x - u_{i,n}, A_i u_{i,n} - A_i y_{i+1,n} \rangle
\]
\[
- \langle x - u_{i,n}, \frac{u_{i,n} - y_{i+1,n}}{\delta_{i,n}} \rangle
\]
\[
\geq \langle x - u_{i,n}, A_i u_{i,n} - A_i y_{i+1,n} \rangle - \| x - u_{i,n} \| \frac{u_{i,n} - y_{i+1,n}}{\delta_{i,n}} \|.
\]
(3.21)

Since \( x_{n_j} \to z \) \( A_i \) is \( \frac{1}{\delta_i} \)-Lipschitzian, from (3.12), (3.20) and (3.21) we obtain
\[
\langle x - z, y \rangle \geq 0.
\]

Since \( U_i \) is maximal monotone, we have \( z \in U_i^{-1} 0 \), and hence,
\[
z \in VI(C, A_i), \quad i = 1, 2, \cdots, m.
\]

Repeating the same argument as above for \( v_{i,n} \), we conclude that
\[
z \in VI(C, A_i), \quad i = 2, 3, \cdots, m + 1.
\]

Therefore \( z \in F \) and applying (3.9) and (3.11), we have
\[
\limsup_{n \to \infty} \langle (\gamma f - F) x^*, x_n - x^* \rangle \leq 0.
\]

Finally, we prove that \( x_n \to x^* \) as \( n \to \infty \). Using (2.4), (3.2), and Lemma 2.3, we have
\[
\| x_{n+1} - x^* \|^2
\]
\[
= \| \alpha_n \gamma f(\text{TPC}(I - \delta_{1,n} A_1)y_{1,n}) + \beta_n x_n
\]
\[
+ (1 - \beta_n) I - \alpha_n F \text{TPC}(I - \delta_{2,n} A_2)y_{1,n} - x^* \|^2
\]
\[
= \| \langle 1 - \beta_n \rangle I - \alpha_n F \text{TPC}(I - \delta_{2,n} A_2)y_{1,n}
\]
\[
- (1 - \beta_n) I - \alpha_n F \text{TPC}(I - \delta_{3,n} A_3)x^* \]
\[
+ \alpha_n [x_n - x^*] + \alpha_n [\gamma f(\text{TPC}(I - \delta_{1,n} A_1)y_{1,n}) - Fx^*] \|^2
\]
\[
\leq \| \langle 1 - \beta_n \rangle I - \alpha_n F \text{TPC}(I - \delta_{2,n} A_2)y_{1,n}
\]
\[
- (1 - \beta_n) I - \alpha_n F \text{TPC}(I - \delta_{2,n} A_2)x^* \]
\[
+ \beta_n [x_n - x^*] \|^2 + 2\alpha_n \langle \gamma f(\text{TPC}(I - \delta_{1,n} A_1)y_{1,n}) - Fx^*, x_{n+1} - x^* \rangle
\]
\[
\leq [1 - \beta_n - \alpha_n \tau] \| y_{1,n} - x^* \| + \beta_n \| x_n - x^* \| \|^2
\]
\[
+ 2\alpha_n \langle \gamma f(PC(I - \delta_{1,n} A_1)Ty_{n}) - Fx^*, x_{n+1} - x^* \rangle
\]
\[
\leq \| [(1 - \beta_n - \alpha_n \tau] \| y_{1,n} - x^* \| + \beta_n \| x_n - x^* \| \|^2
\]
+ 2\alpha_n\gamma\langle f(TPC(I - \delta_{1,n}A_1)y_{1,n}) - f(x^*), x_{n+1} - x^* \rangle
+ 2\alpha_n\langle \gamma f(x^*) - Fx^*, x_{n+1} - x^* \rangle
\leq [(1 - \beta_n - \alpha_n\tau) \parallel y_{1,n} - x^* \parallel + \beta_n \parallel x_n - x^* \parallel]^2
+ \alpha_n\gamma\alpha[\parallel y_{1,n} - x^* \parallel^2 + \parallel x_{n+1} - x^* \parallel^2]
+ 2\alpha_n\langle \gamma f(x^*) - Fx^*, x_{n+1} - x^* \rangle
\leq [(1 - \beta_n - \alpha_n\tau) \parallel x_n - x^* \parallel + \beta_n \parallel x_n - x^* \parallel]^2
+ \alpha_n\gamma\alpha[\parallel x_n - x^* \parallel^2 + \parallel x_{n+1} - x^* \parallel^2]
+ 2\alpha_n\langle \gamma f(x^*) - Fx^*, x_{n+1} - x^* \rangle
\leq (1 - \alpha_n\tau)^2 \parallel x_n - x^* \parallel^2 + \alpha_n\gamma\alpha[\parallel x_n - x^* \parallel^2 + \parallel x_{n+1} - x^* \parallel^2]
+ 2\alpha_n\langle \gamma f(x^*) - Fx^*, x_{n+1} - x^* \rangle
\leq 0.\] Consequently, applying Lemma 2.5, we get \limsup_{n\to\infty} c_n \leq 0. Consequently, applying Lemma 2.5, to (3.22), we conclude that \( x_n \to x^* \). Since \parallel y_{i,n} - x^* \parallel \leq \parallel x_n - x^* \parallel \) for \( i = 1, 2, \ldots, m + 1 \), we have \( y_{i,n} \to x^* \). □

By the careful analysis of the proof of Theorem 3.1, we obtain the following theorem. Because its proof is much simpler than the proof of Theorem 3.1, we omit its proof.
Theorem 3.2. Let $C$ be a nonempty closed convex subset of real Hilbert space $H$, $F: C \to C$ be a $k$-Lipschitzian and $\eta$-strongly monotone operator with $k > 0$, $\eta > 0$, $f: C \to C$ be a contraction with coefficient $0 < \alpha < 1$, $\gamma$ be a number in $(0, \frac{\tau}{2})$, where $\tau = \mu(\eta - \frac{\mu k^2}{2})$. Let $S$ be a semigroup, $A: C \to H$ be a $\zeta$-inverse strongly monotone operator and $B: C \to H$ be a $\delta$-inverse strongly monotone. Let $\varphi = \{T_t : t \in S\}$ be a nonexpansive semigroup of $C$ into itself such that $F = \text{Fix}(\varphi) \cap VI(C, A) \cap VI(C, B) \neq \emptyset$, $X$ a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T_t x, y \rangle$ is an element of $X$ for each $x \in C$ and $y \in H$, $\{\mu_n\}$ a left regular sequence of means on $X$ such that $\sum_{n=1}^{\infty} \| \mu_{n+1} - \mu_n \| < \infty$. Let $\{\alpha_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$ and $\{\beta_n\}$ be a sequence in $[0, 1]$. Suppose the following conditions are satisfied:

\begin{align*}
(B_1) & \quad 0 < \lim \inf_{n \to \infty} \gamma_n \leq \lim \sup_{n \to \infty} \gamma_n < 1, \{\zeta_n\} \subset (0, 2\zeta), \{\delta_n\} \subset (0, 2\delta) \\
(B_2) & \quad \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \lim_{n \to \infty} \beta_n = 0, \\
(B_3) & \quad \lim_{n \to \infty} |\beta_{n+1} - \beta_n| = \lim_{n \to \infty} |\delta_{n+1} - \delta_n| = 0, \\
& \quad \lim_{n \to \infty} |\gamma_{n+1} - \gamma_n| = 0.
\end{align*}

If $\{x_n\}$ and $\{y_n\}$ be generated by $x_0 \in C$ and

\begin{align*}
& y_n = \gamma_n PC(x_n - \zeta_n Ax_n) + (1 - \gamma_n) PC(x_n - \delta_n Bx_n), \\
& x_{n+1} = \alpha_n g(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n \mu F) T_{\mu_n} y_n, \quad n \geq 0.
\end{align*}

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly, as $n \to \infty$, to $x^* \in F$, which is a unique solution of the variational inequalities:

\begin{align*}
& \langle (\mu F - \gamma f) x^* , x - x^* \rangle \geq 0, \quad \forall x \in F, \\
& \langle B x^* , y - x^* \rangle \geq 0, \quad \forall y \in C, \\
& \langle A x^* , y - x^* \rangle \geq 0, \quad \forall y \in C.
\end{align*}

Theorem 3.3 ([13]). Let $S$ be a semigroup, $C$ a nonempty closed convex subset of real Hilbert space $H$ and $B: C \to H$ be a $\beta$-inverse strongly monotone. Let $\varphi = \{T_t : t \in S\}$ be a nonexpansive semigroup of $C$ into itself such that $F = \text{Fix}(\varphi) \cap VI(C, B) \neq \emptyset$, $X$ a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T_t x, y \rangle$ is an element of $X$ for each $x \in C$ and $y \in H$, $\{\mu_n\}$ a left regular sequence of means on $X$ such that $\sum_{n=1}^{\infty} \| \mu_{n+1} - \mu_n \| < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ and $\{\delta_n\}$ be a sequence in $[a, b]$, where $0 < a < b < 2\beta$. Suppose the following conditions are satisfied:

\begin{align*}
(B_1) & \quad \lim_{n \to \infty} \alpha_n = 0, \lim_{n \to \infty} \beta_n = 0, \\
(B_2) & \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \\
(B_3) & \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty.
\end{align*}

If $\{x_n\}$ and $\{y_n\}$ be generated by $x_0 \in C$ and

\begin{align*}
& y_n = \beta_n x_n + (1 - \beta_n) PC(x_n - \delta_n Bx_n), \\
& x_{n+1} = \alpha_n g(x_n) + (I - \alpha_n \mu F) T_{\mu_n} y_n, \quad n \geq 0.
\end{align*}
Then, \( \{x_n\} \) and \( \{y_n\} \) converge strongly, as \( n \to \infty \), to \( x^* \in \mathcal{F} \), which is a unique solution of the variational inequalities:

\[
\begin{align*}
&\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0, & \forall x \in \mathcal{F}, \\
&\langle Bx^*, y - x^* \rangle \geq 0 & \forall y \in \mathcal{C}.
\end{align*}
\]

Proof. It suffices to take \( P_C(I - \zeta_n A) = I \) and \( \beta_n = 0 \) for \( n \in \mathbb{N} \) in Theorem 3.2. □

Example 3.4. Let \( C = [0, 1] \times [0, 1] \subset \mathbb{R}^2 \), \( \alpha \in (0, 1) \) and \( k \in \mathbb{R}^+ \). Let \( F \) and \( f \) be a mappings of \( C \) into itself defined by \( Fx = kx \) and \( f(x) = \alpha x \), obviously \( F \) is \( k \)-Lipschitzian and \( k/2 \)-strongly monotone operator and \( f \) is a contraction with coefficient \( 0 < \alpha < 1 \). For \( i = 1, 2, \ldots, m + 1 \), let \( A_i : C \to H \) be a mapping defined by \( A_ix = \frac{1}{i}x \), therefore \( A_i \) is \( i \)-inverse strongly monotone mapping. For every \( t \in \mathbb{R}^+ \), define

\[
T_t : C \to C \\
T_t x = \frac{t}{t + 1} x.
\]

Obviously \( (\mathbb{R}^+, .) \) is a semigroup and \( \varphi = \{T_t : t \in \mathbb{R}^+\} \) is a nonexpansive semigroup of \( C \) into itself such that \( 0 \in \mathcal{F} = \bigcap_{i=1}^{m+1} VI(C, A_i) \cap \text{Fix}(\varphi) \). Let \( \{\lambda_n\} \) be a recursive sequence such that

\[
\lambda_2 = 2, \quad \lambda_{n+1} = \frac{n^2}{n^2 - 1} \lambda_n, \quad n \geq 2,
\]

therefore \( \lambda_n \uparrow^\infty \). Let \( X = C(\mathbb{R}^+) \subset B(\mathbb{R}^+) \), where \( C(\mathbb{R}^+) \) denote the space of all real-valued continuous function on \( \mathbb{R}^+ \) with supremum norm. For every \( f \in \mathbb{R}^+ \), we define

\[
\mu_n(f) = \frac{1}{\lambda_n} \int_0^{\lambda_n} f(t) dt.
\]

In fact for every \( f \in \mathbb{R}^+ \),

\[
|\mu_n(f)| = |\frac{1}{\lambda_n} \int_0^{\lambda_n} f(t) dt | \leq \frac{1}{\lambda_n} \int_0^{\lambda_n} \| f(t) \| dt = \| f \|,
\]

and

\[
\mu_n(1) = \frac{1}{\lambda_n} \int_0^{\lambda_n} 1 dt = 1.
\]

Hence, \( \| \mu_n \| = \mu_n(1) = 1 \), i.e., \( \{\mu_n\} \) is a sequence of means on \( X \). Next, for every \( f \in X \),

\[
|\mu_{n+1}(f) - \mu_n(f)|
\]
\[
\frac{1}{\lambda_{n+1}} \int_0^{\lambda_{n+1}} f(t)\,dt - \frac{1}{\lambda_{n}} \int_0^{\lambda_{n}} f(t)\,dt \leq \frac{1}{\lambda_{n+1}} \int_0^{\lambda_{n+1}} f(t)\,dt - \frac{1}{\lambda_{n}} \int_0^{\lambda_{n}} f(t)\,dt \\
+ \left| \frac{1}{\lambda_{n+1}} \int_0^{\lambda_{n}} f(t)\,dt - \frac{1}{\lambda_{n}} \int_0^{\lambda_{n}} f(t)\,dt \right|
\]

\[
\leq \frac{1}{\lambda_{n+1}} \int_0^{\lambda_{n}} \| f \| \,dt \\
+ \left| \frac{1}{\lambda_{n+1}} - \frac{1}{\lambda_{n}} \right| \int_0^{\lambda_{n}} \| f \| \,dt
\]

\[
= \frac{\lambda_{n+1} - \lambda_{n}}{\lambda_{n+1}} \| f \| + (\frac{1}{\lambda_{n}} - \frac{1}{\lambda_{n+1}}) \lambda_{n} \| f \|
\]

\[
= 2(1 - \frac{\lambda_{n}}{\lambda_{n+1}}) \| f \|
\]

and hence,

\[
\| \mu_{n+1} - \mu_n \| \leq 2(1 - \frac{\lambda_{n}}{\lambda_{n+1}}).
\]

Therefore,

\[
\sum_{n=1}^{\infty} \| \mu_{n+1} - \mu_n \| \leq 2 \sum_{n=1}^{\infty} (1 - \frac{\lambda_{n}}{\lambda_{n+1}}) = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.
\]

Taking \( \alpha_n = \beta_n = \frac{1}{n} \), \( \delta_i = i \) and \( \gamma_{i,n} = \delta_{i,n} = \frac{1}{i} + \frac{1}{n} \), for \( i = 1, 2, \ldots, m+1 \) and \( n \in \mathbb{N} \), all assumption of Theorem 3.1 are established, therefore the sequences \( \{x_n\} \) and \( \{y_{i,n}\}_{i=1}^{m+1}, n=1, \ldots, \infty \) generated by the iteration algorithm:

\[
y_{m+1,n} = x_n,
\]

\[
y_{i,n} = \left(\frac{1}{i} + \frac{1}{n}\right)P_C(I - \frac{1}{i} + \frac{1}{n}A_i)y_{i+1,n} \\
+ \left(1 - \frac{1}{i} + \frac{1}{n}\right)P_C(I - \delta_{i+1,n}A_{i+1})y_{i+1,n}, \quad i = 1, 2, \ldots, m,
\]

\[
x_{n+1} = \frac{1}{n} \gamma f(T_{\mu_n}P_C(I - \delta_{1,n}A_1)y_{1,n}) + \frac{1}{n} x_n \\
+ \left(1 - \frac{1}{n}\right)I - \frac{1}{n} \mu F)T_{\mu_n}P_C(I - \delta_{2,n}A_2)y_{1,n},
\]

are convergent to some \( x^* \in \mathcal{F} \).

Remark 3.1. Theorem 3.1 improves [13, Theorem 3.1] and [16, Theorem 3.2] in the following aspects.
(a) Our iterative process (1.6) is more general than Tian process (1.4) because it can be applied to solving the problem of finding a common element of the set of solutions of systems of variational inequalities.

(b) Our iterative process (1.6) is the extension of Tian process (1.4) from nonexpansive mapping to left amenable semigroup of nonexpansive mappings.

(c) Our iterative process (1.6) is the extension of Piri and Badali iterative process (1.5) from solutions of two variational inequality to solutions of arbitrary finite family of variational inequalities.

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