UNIFORM TREATMENT OF JENSEN TYPE INEQUALITIES

J. PEČARIĆ, I. PERIĆ and M. RODIĆ LIPANOVIĆ

Communicated by the former editorial board

The main purpose of this paper is to give the uniform treatment of the Jensen type inequalities. Using the Green function, we give some new conditions for Jensen's inequality to hold. Integral and discrete case are presented. In the discrete version we get the generalization of the results from [9] and [10], and in integral version the generalization of the results from [19]. Using the same method we also give adequate results for the converses of the Jensen inequality. As a consequence, also the results concerning the Hermite-Hadamard inequalities are presented.

AMS 2010 Subject Classification: 26D15.

Key words: Green function, Jensen inequality, Hermite-Hadamard inequality.

1. INTRODUCTION

The well-known Jensen inequality asserts that for function f holds

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1.1}$$

if f is convex function on interval $I \subseteq \mathbb{R}$, where p_i are positive real numbers and $x_i \in I$ $(i = 1, ..., n)$, while $P_n = \sum_{i=1}^{n} p_i$.

The idea of considering the quotient of differences of the left and the right side of the Jensen inequality (1.1) for different functions can be seen in [11, p. 9], where authors gave an estimation of that quotient, assuming that p_i and x_i are as above and $P_n = 1$. Their result is a discrete version of a result previously given in [1].

Theorem 1.1 ([11], p. 9). *Let $p_i > 0$ $(i = 1, ..., n)$ with $P_n = 1$, and $x_i \in I$ $(i = 1, ..., n)$ are not all equal. Let $f, g : I \to \mathbb{R}$ be twice differentiable functions such that*

$$0 \leq m \leq f''(x) \leq M \quad \textit{and} \quad 0 < k \leq g''(x) \leq K \quad \textit{for all } x \in I.$$

Then

$$\frac{m}{K} \leq \frac{\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)}{\sum_{i=1}^{n} p_i g(x_i) - g\left(\sum_{i=1}^{n} p_i x_i\right)} \leq \frac{M}{k}.$$

Furthermore, A. McD. Mercer gave in [9] and [10] the following two mean-value theorems of the Lagrange and Cauchy type, using the same conditions on p_i and x_i.

Theorem 1.2. *Let I be a compact real interval and $f, g : I \to \mathbb{R}$. Let $x_i \in I$ and $p_i > 0$ $(i = 1, ..., n)$ such that $P_n = 1$.*

(i) *If $f \in C^2(I)$, then*

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) = \frac{1}{2} f''(\xi) \left(\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right) \tag{1.2}$$

holds for some $\xi \in I$.

(ii) *If $f, g \in C^2(I)$, then*

$$\frac{\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)}{\sum_{i=1}^{n} p_i g(x_i) - g\left(\sum_{i=1}^{n} p_i x_i\right)} = \frac{f''(\xi)}{g''(\xi)} \tag{1.3}$$

holds for some $\xi \in I$, provided that the denominator of the left-hand side is non-zero.

Remark 1.1. We use $f \in C^2(I)$ to denote that function f has continuous second derivative on I.

Having in mind the integral Jensen inequality, the authors in [19] gave similar results in integral form.

Theorem 1.3 ([19]). *Let I be a compact real interval and $\varphi, \psi : I \to \mathbb{R}$. Let h be an integrable function with respect to a normalized weight ω on $[a, b] \subset \mathbb{R}$ such that the range of h is a subset of I.*

(i) *If $\varphi \in C^2(I)$, then*

$$\begin{aligned} &\int_a^b \varphi\left(h(x)\right)\omega(x)\mathrm{d}x - \varphi\left(\int_a^b h(x)\omega(x)\mathrm{d}x\right) \\ &= \frac{1}{2}\varphi''(\xi)\left[\int_a^b \left(h(x)\right)^2 \omega(x)\mathrm{d}x - \left(\int_a^b h(x)\omega(x)\mathrm{d}x\right)^2\right] \end{aligned} \tag{1.4}$$

holds for some $\xi \in I$.

(ii) *If $\varphi, \psi \in C^2(I)$, then*

$$\frac{\int_a^b \varphi\left(h(x)\right)\omega(x)\mathrm{d}x - \varphi\left(\int_a^b h(x)\omega(x)\mathrm{d}x\right)}{\int_a^b \psi\left(h(x)\right)\omega(x)\mathrm{d}x - \psi\left(\int_a^b h(x)\omega(x)\mathrm{d}x\right)} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \tag{1.5}$$

holds for some $\xi \in I$, provided that the denominator of the left-hand side is non-zero.

The aim of our paper is to give the unified treatment of inequalities of the Jensen type and the type of converses of the Jensen inequality, using the Green function. We begin with the integral version of the results for the Jensen-type inequalities, for real Stieltjes measure. As a consequence we get the results of type (1.4) and (1.5) for such measure. In discrete case, given in the next section, we get as a consequence the Mercer-like results (Theorem 1.2). We consider the case when p_i are real numbers and, using the Green function, give the conditions for equalities of type (1.2) and (1.3) to hold. Furthermore, using the properties of the Green function and using the well-known theorems like Jensen's or Jensen-Steffensen's inequality, we give more explicit conditions for equalities of type (1.2), (1.3), (1.4), (1.5) to hold.

We do similar for converses of the Jensen inequality.

In special case, from these results we get the conditions, using the Green function, on a real Stieltjes measure $d\lambda$ such that the Hermite-Hadamard inequality remains valid. It's known that the Hermite-Hadamard inequality holds when the measure is positive, that is when λ is increasing function, but here we allow that λ is not necessarily increasing.

2. MAIN RESULTS

Consider the Green function G defined on $[\alpha,\beta]\times[\alpha,\beta]$ by

$$G(t,s)=\begin{cases}\frac{(t-\beta)(s-\alpha)}{\beta-\alpha} & \text{for } \alpha\le s\le t,\\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha} & \text{for } t\le s\le \beta.\end{cases} \tag{2.1}$$

The function G is convex under s, it is symmetric, so it is also convex under t. The function G is continuous under s and continuous under t.

For any function $\varphi:[\alpha,\beta]\to\mathbb{R}$, $\varphi\in C^2([\alpha,\beta])$, we can easily show by integrating by parts that the following is valid

$$\varphi(x)=\frac{\beta-x}{\beta-\alpha}\varphi(\alpha)+\frac{x-\alpha}{\beta-\alpha}\varphi(\beta)+\int_\alpha^\beta G(x,s)\varphi''(s)\mathrm{d}s, \tag{2.2}$$

where the function G is defined as above in (2.1) (see also [22]).

Using this, we now derive several interesting results concerning the Jensen type inequalities.

The following theorem gives us the conditions on the real Stieltjes measure $d\lambda$, such that $\lambda(a)\neq\lambda(b)$, that for continuous convex function φ the Jensen inequality holds. Here, we also allow that the measure can be negative.

THEOREM 2.1. *Let $g : [a,b] \to \mathbb{R}$ be continuous function and $[\alpha,\beta]$ interval such that the image of g is a subset of $[\alpha,\beta]$. Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and such that $\lambda(a) \neq \lambda(b)$ and $\overline{g} \in [\alpha,\beta]$, where*

$$\overline{g} = \frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}.$$

Then the following two statements are equivalent:

(1) *For every continuous convex function $\varphi : [\alpha,\beta] \to \mathbb{R}$ holds*

$$\varphi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) \leq \frac{\int_a^b \varphi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}. \tag{2.3}$$

(2) *For all $s \in [\alpha,\beta]$ holds*

$$G\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}, s\right) \leq \frac{\int_a^b G\left(g(x),s\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}, \tag{2.4}$$

where the function $G : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ is defined in (2.1).

Furthermore, the statements (1) *and* (2) *are also equivalent if we change the sign of inequality in both* (2.3) *and* (2.4).

Proof. (1) $\Rightarrow$ (2) : Let (1) hold. As the function $G(\cdot,s)$ ($s \in [\alpha,\beta]$) is also continuous and convex, it follows that also for this function (2.3) holds, *i.e.* it holds (2.4).

(2) $\Rightarrow$ (1) : Let (2) hold. We know that we can represent every function $\varphi : [\alpha,\beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha,\beta])$, in the form (2.2), where the function G is defined in (2.1). By easy calculation, using (2.2), we can easily get that

$$\begin{aligned}&\varphi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \varphi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \\ &\quad = \int_\alpha^\beta \left[G\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}, s\right) - \frac{\int_a^b G\left(g(x),s\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]\varphi''(s)\mathrm{d}s.\end{aligned} \tag{2.5}$$

If the function φ is also convex, then $\varphi''(s) \geq 0$ for all $s \in [\alpha,\beta]$. So, if for every $s \in [\alpha,\beta]$ holds (2.4), then it follows that for every convex function $\varphi : [\alpha,\beta] \to \mathbb{R}$, with $\varphi \in C^2([\alpha,\beta])$, inequality (2.3) holds.

At the end, note that it is not necessary to demand the existence of the second derivative of the function φ (see [20], p. 172). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials.

The last part of our theorem can be proved analogously. □

Remark 2.1. Let the conditions from the previous theorem hold. Then the following two statements are equivalent:

(1′) For every continuous concave function $\varphi : [\alpha, \beta] \to \mathbb{R}$ the reverse inequality in (2.3) holds.

(2′) For all $s \in [\alpha, \beta]$ inequality (2.4) holds, where the function G is defined in (2.1).

Moreover, the statements (1′) and (2′) are also equivalent if we change the sign of inequality in both statements (1′) and (2′).

Remark 2.2. If the function λ in Theorem 2.1 is increasing and bounded, with $\lambda(a) \neq \lambda(b)$, then inequality (2.3) becomes Jensen's integral inequality.

The fact that for function $\varphi : [\alpha, \beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha, \beta])$, equality (2.5) holds, will be very useful in deriving our forthcoming results.

In the following two theorems we give the conditions on the real Stieltjes measure $d\lambda$, with $\lambda(a) \neq \lambda(b)$, so that for the functions of the class C^2, the equalities of type (1.4) and (1.5) hold.

THEOREM 2.2. *Let $g : [a, b] \to \mathbb{R}$ be continuous function and $\varphi : [\alpha, \beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha, \beta])$, where the image of g is a subset of $[\alpha, \beta]$. Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a) \neq \lambda(b)$ and $\overline{g} \in [\alpha, \beta]$. If for all $s \in [\alpha, \beta]$ the inequality (2.4) holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in (2.4) holds, then there exists some $\xi \in [\alpha, \beta]$ such that*

$$\varphi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \varphi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} = \frac{1}{2}\varphi''(\xi)\left[\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)^2 - \frac{\int_a^b \left(g(x)\right)^2\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]. \tag{2.6}$$

Proof. Following the assumptions of our theorem, we have that the function φ'' is continuous and

$$G\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}, s\right) - \frac{\int_a^b G\left(g(x), s\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}$$

doesn't change it's positivity on $[\alpha, \beta]$. For our function φ equality (2.5) holds, and now, applying the integral mean-value theorem, we get that there exists some $\xi \in [\alpha, \beta]$ such that

$$\varphi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \varphi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}$$

$$= \varphi''(\xi)\int_\alpha^\beta \left[G\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}, s\right) - \frac{\int_a^b G\left(g(x), s\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]\mathrm{d}s. \tag{2.7}$$

It can be easily checked that it holds

$$\int_\alpha^\beta G(t,s)\mathrm{d}s = \int_\alpha^t \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}\mathrm{d}s + \int_t^\beta \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}\mathrm{d}s$$

$$= \frac{t-\beta}{\beta-\alpha}\int_\alpha^t (s-\alpha)\mathrm{d}s + \frac{t-\alpha}{\beta-\alpha}\int_t^\beta (s-\beta)\mathrm{d}s$$

$$= \frac{1}{2}(t-\alpha)(t-\beta). \tag{2.8}$$

Calculating the integral on the right side in (2.7) we get

$$\varphi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \varphi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}$$

$$= \varphi''(\xi)\left[\int_\alpha^\beta G\left(\overline{g}, s\right)\mathrm{d}s - \frac{1}{\int_a^b \mathrm{d}\lambda(x)}\int_a^b\int_\alpha^\beta G\left(g(x), s\right)\,\mathrm{d}s\,\mathrm{d}\lambda(x)\right]$$

$$= \varphi''(\xi)\left[\frac{1}{2}(\overline{g}-\alpha)(\overline{g}-\beta) - \frac{1}{\int_a^b \mathrm{d}\lambda(x)}\int_a^b \frac{1}{2}(g(x)-\alpha)(g(x)-\beta)\,\mathrm{d}\lambda(x)\right]$$

$$= \frac{1}{2}\varphi''(\xi)\left[\overline{g}^{\,2} - \frac{\int_a^b \left(g(x)\right)^2\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right],$$

which concludes our proof. □

THEOREM 2.3. *Let $g : [a,b] \to \mathbb{R}$ be continuous function and $\varphi, \psi : [\alpha,\beta] \to \mathbb{R}$, $\varphi,\psi \in C^2([\alpha,\beta])$, where the image of g is a subset of $[\alpha,\beta]$. Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, such that $\lambda(a) \neq \lambda(b)$ and $\overline{g} \in [\alpha,\beta]$. If for all $s \in [\alpha,\beta]$ inequality* (2.4) *holds or if for all $s \in [\alpha,\beta]$ the reverse inequality in* (2.4) *holds, then there exists some $\xi \in [\alpha,\beta]$ such that*

$$\frac{\varphi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \varphi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}}{\psi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \psi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \tag{2.9}$$

provided that the denominator of the left-hand side is nonzero.

Proof. Consider the function χ defined by

$$\chi(t) = \left[\psi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \psi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]\cdot\varphi(t)$$
$$-\left[\varphi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \varphi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]\cdot\psi(t).$$

Function χ is the linear combination of functions φ and ψ, so it is also defined on the segment $[\alpha,\beta]$, it is continuous and χ'' is continuous on $[\alpha,\beta]$. Now we can apply Theorem 2.2 on function χ and it follows that there exists some $\xi\in[\alpha,\beta]$ such that the following is valid

$$\chi\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right) - \frac{\int_a^b \chi\left(g(x)\right)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}$$
$$=\frac{1}{2}\chi''(\xi)\left[\left(\frac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right)^2 - \frac{\int_a^b \left(g(x)\right)^2\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right]. \tag{2.10}$$

After a short calculation we get that the left-hand side of this equation equals zero. The term in the square brackets on the right-hand side of (2.10) is different from 0, because otherwise, from Theorem 2.2 applied on the function ψ, we would have that the denominator on the left-hand side of (2.9) equals zero, which contradicts our assumption from our theorem (Theorem 2.3). It follows that

$$\chi''(\xi) = 0,$$

and the assertion of our theorem follows directly. □

Remark 2.3. The general method for deriving this kind of results as in Theorem 2.3, is given in [19].

Remark 2.4. Note also that setting the function ψ as $\psi(x) = x^2$ in Theorem 2.3, we get the statement of Theorem 2.2.

As a consequence of the previous two theorems, we shall now give some further results where we give explicit conditions on the functions g and λ, for (2.6) and (2.9) to hold.

Corollary 2.1. *Let $g:[a,b]\to\mathbb{R}$ be continuous function and $\varphi,\psi:[\alpha,\beta]\to\mathbb{R}$, where the image of g is a subset of $[\alpha,\beta]$. Let $\lambda:[a,b]\to\mathbb{R}$ be increasing and bounded function such that $\lambda(a)\neq\lambda(b)$.*

(i) *If $\varphi\in C^2([\alpha,\beta])$, then there exists some $\xi\in[\alpha,\beta]$ such that* (2.6) *holds.*

(ii) *If* $\varphi, \psi \in C^2([\alpha, \beta])$, *then there exists some* $\xi \in [\alpha, \beta]$ *such that* (2.9) *holds.*

Proof. The function g is continuous, the function $G : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ defined in (2.1) is continuous and convex, λ is increasing, bounded and such that $\lambda(a) \neq \lambda(b)$, so by the integral Jensen inequality we have that for all $s \in [\alpha, \beta]$ the inequality (2.4) holds. Applying Theorem 2.2 (respectively Theorem 2.3) we get the statement (i) (respectively (ii)) of this corollary. □

Corollary 2.2. *Let* $g : [a, b] \to \mathbb{R}$ *be continuous and monotonic function,* $\varphi, \psi : [\alpha, \beta] \to \mathbb{R}$, *where the image of* g *is a subset of* $[\alpha, \beta]$. *Let* $\lambda : [a, b] \to \mathbb{R}$ *be continuous function or the function of bounded variation, such that* $\lambda(a) \leq \lambda(x) \leq \lambda(b)$ *for all* $x \in [a, b]$, *and* $\lambda(b) > \lambda(a)$.

(i) *If* $\varphi \in C^2([\alpha, \beta])$, *then there exists some* $\xi \in [\alpha, \beta]$ *such that* (2.6) *holds.*

(ii) *If* $\varphi, \psi \in C^2([\alpha, \beta])$ *then there exists some* $\xi \in [\alpha, \beta]$ *such that* (2.9) *holds.*

Proof. The function G is continuous and convex, so under the conditions on g and λ from this corollary, we can apply the integral Jensen-Steffensen inequality. We get that for all $s \in [\alpha, \beta]$ the inequality (2.4) holds. Now the statement (i) (respectively (ii)) of this corollary follows directly from Theorem 2.2 (respectively Theorem 2.3). □

Analogous results can be derived using the Boas generalization of the Jensen-Steffensen inequality (the Jensen-Boas inequality, see [3] or [20], p. 59), the Brunk generalization of the Jensen-Steffensen inequality (the Jensen-Brunk inequality, see [4] or [20], p. 60) or the generalization of the Jensen-Steffensen inequality given in [13] (see also [20], p. 62), the reverse Jensen inequality given in [14] (see also [20], p. 84), the reverse Jensen-Steffensen inequality given in [14] (see also [20], p. 84), the reverse Jensen-Brunk inequality given in [14] (see also [20], p. 85) or the reverse Jensen-Boas inequality given in [14] (see also [20], p. 86).

3. DISCRETE CASE

In this section, we give the results for discrete case. The proofs are similar to those in the integral case given in Section 2, so we give these results here without the proofs.

In discrete Jensen's inequality we have that p_i $(i = 1, ..., n)$ are positive real numbers. Here, we give the generalization of that result, allowing that p_i can also be negative, with the sum different from 0, but with a supplementary demand on p_i, x_i given using the Green function $G : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ defined in (2.1).

For p_i, x_i $(i = 1, ..., n)$ we shall use the common notation: $P_k = \sum_{i=1}^{k} p_i$, $\overline{P_k} = P_n - P_{k-1}$ $(k = 1, ..., n)$, and $\overline{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$.

We already know from the previous section that we can represent any function $f : [\alpha, \beta] \to \mathbb{R}$, $f \in C^2([\alpha, \beta])$, in the form (2.2), where the function G is defined in (2.1), and by some calculation it's easy to show that the following holds:

$$f(\overline{x}) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) = \int_\alpha^\beta \left(G(\overline{x}, s) - \frac{1}{P_n}\sum_{i=1}^{n} p_i G(x_i, s) \right) f''(s)\mathrm{d}s.$$

We have the following result.

THEOREM 3.1. *Let $x_i \in [a, b] \subseteq [\alpha, \beta]$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$, be such that $P_n \neq 0$ and $\overline{x} \in [\alpha, \beta]$. Then the following two statements are equivalent:*

(1) *For every continuous convex function $f : [\alpha, \beta] \to \mathbb{R}$ holds*

$$f(\overline{x}) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \tag{3.1}$$

(2) *For all $s \in [\alpha, \beta]$ holds*

$$G(\overline{x}, s) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i G(x_i, s), \tag{3.2}$$

where the function $G : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ is defined in (2.1).

Moreover, the statements (1) *and* (2) *are also equivalent if we change the sign of inequality in both inequalities, in* (3.1) *and in* (3.2).

Remark 3.1. Let the conditions from the previous theorem hold. Then the following two statements are equivalent:

(1′) For every continuous concave function $f : [\alpha, \beta] \to \mathbb{R}$ the reverse inequality in (3.1) holds.

(2′) For all $s \in [\alpha, \beta]$ inequality (3.2) holds, where the function G is defined in (2.1).

Moreover, the statements (1′) and (2′) are also equivalent if we change the sign of inequality in both statements (1′) and (2′).

Remark 3.2. Note that in the case when all $p_i > 0$ $(i = 1, ..., n)$, inequality (3.1) becomes discrete Jensen's inequality (1.1).

In the following theorem the generalization of Mercer's result (1.2) from [9] is given.

THEOREM 3.2. *Let $x_i \in [a, b] \subseteq [\alpha, \beta]$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$, be such that $P_n \neq 0$ and $\overline{x} \in [\alpha, \beta]$, and let $f : [\alpha, \beta] \to \mathbb{R}$, $f \in C^2([\alpha, \beta])$. If for all*

$s \in [\alpha, \beta]$ the inequality (3.2) *holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in* (3.2) *holds, then there exists some $\xi \in [\alpha, \beta]$ such that*

$$f(\overline{x}) - \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) = \frac{1}{2} f''(\xi) \left[\overline{x}^2 - \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i^2 \right]. \tag{3.3}$$

In the following theorem it is given the generalization of Mercer's result (1.3) from [10].

THEOREM 3.3. *Let $x_i \in [a,b] \subseteq [\alpha, \beta]$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$, be such that $P_n \neq 0$ and $\overline{x} \in [\alpha, \beta]$, and let $f, g : [\alpha, \beta] \to \mathbb{R}$, $f, g \in C^2([\alpha, \beta])$. If for all $s \in [\alpha, \beta]$ the inequality* (3.2) *holds or if for all $s \in [\alpha, \beta]$ the reverse inequality in* (3.2) *holds, then there exists some $\xi \in [\alpha, \beta]$ such that*

$$\frac{f(\overline{x}) - \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i)}{g(\overline{x}) - \frac{1}{P_n} \sum_{i=1}^{n} p_i g(x_i)} = \frac{f''(\xi)}{g''(\xi)} \tag{3.4}$$

provided that the denominator of the left-hand side is nonzero.

As a consequence of the previous two theorems, we now give further results in which we give explicit conditions on p_i, x_i $(i = 1, ..., n)$ for (3.3) and (3.4) to hold, where using the properties of the function G we can skip the supplementary conditions on that function.

COROLLARY 3.1. *Let $x_i \in [a,b]$, $p_i \in \mathbb{R}^+$ $(i = 1, ..., n)$, and let $f, g : [a,b] \to \mathbb{R}$.*

(i) *If $f \in C^2([a,b])$, then there exists some $\xi \in [a,b]$ such that* (3.3) *holds.*
(ii) *If $f, g \in C^2([a,b])$, then there exists some $\xi \in [a,b]$ such that* (3.4) *holds.*

Proof. Note that $p_i > 0$ $(i = 1, ..., n)$ implies that $\overline{x} \in [a,b]$. So without loss of generality, we can narrow the interval we are looking from $[\alpha, \beta]$ to $[a,b]$ (that is we set $\alpha = a$, $\beta = b$). The function G is convex, so by the Jensen inequality we have that for all $s \in [a,b]$ inequality (3.2) holds. Now applying Theorem 3.2 and Theorem 3.3 we get the statements of this corollary. □

Remark 3.3. With an extra condition that $P_n = 1$, we get the result (1.2) (see also [9]) and (1.3) (see also [10]).

COROLLARY 3.2. *Let $(x_1, ..., x_n)$ be monotonic $n-$tuple, $x_i \in [a,b]$ $(i = 1, ..., n)$, and let $(p_1, ..., p_n)$ be real $n-$tuple such that*

$$0 \leq P_k \leq P_n \ \ (k = 1, ..., n), \quad P_n > 0,$$

and let $f, g : [a,b] \to \mathbb{R}$.

(i) *If $f \in C^2([a,b])$, then there exists some $\xi \in [a,b]$ such that* (3.3) *holds.*
(ii) *If $f, g \in C^2([a,b])$, then there exists some $\xi \in [a,b]$ such that* (3.4) *holds.*

Proof. Under the conditions on x_i, p_i $(i = 1, ..., n)$, from this corollary, we have that $\overline{x} \in [a, b]$ (see the proof of the Jensen-Steffensen inequality in, for example, [20], p. 57). Suppose that $x_1 \geq x_2 \geq \ldots \geq x_n$. It holds

$$P_n(x_1 - \overline{x}) = \sum_{i=2}^{n} p_i(x_1 - x_i) = \sum_{j=2}^{n} (x_{j-1} - x_j)(P_n - P_{j-1}) \geq 0$$

so it follows that $x_1 \geq \overline{x}$. Furthermore, it is

$$P_n(\overline{x} - x_n) = \sum_{i=1}^{n-1} p_i(x_i - x_n) = \sum_{j=1}^{n-1} (x_j - x_{j+1}) P_j \geq 0,$$

so $\overline{x} \geq x_n$. Therefore, as well as in the proof of the previous corollary, without loss of generality we can narrow the interval we are looking from $[\alpha, \beta]$ to $[a, b]$ (that is we set $\alpha = a$, $\beta = b$). By the Jensen-Steffensen inequality we have that for the convex function G the inequality (3.2) holds for all $s \in [a, b]$. Now the statements of this corollary follow directly from Theorem 3.2 and Theorem 3.3. □

Analogous results can be derived using the reverse Jensen inequality (see for example [5], p. 45), the reverse Jensen-Steffensen inequality (see for example [20], p. 83), or the Jensen-Petrović inequality (see [2] or [15], p. 69 or [20], p. 157) which is equivalent to the Jensen-Steffensen and the reverse Jensen-Steffensen inequality together. Also, similar analogous results can be derived using the result from [21] (see also [12]) where are considered the $n-$tuples $(x_1, ..., x_n)$ and $(p_1, ..., p_n)$ where n is an odd number.

4. CONVERSES OF THE JENSEN INEQUALITY

In [18] (see also [16] and [17]) the following result concerning converses of the Jensen inequality is given.

THEOREM A (Corollary 12, [18]). *Let f be the convex function on $[M_1, M_2]$ and $g : [a, b] \to [M_1, M_2]$ integrable with respect to the measure μ on $[a, b]$. If μ is such that $\int_a^b \mathrm{d}\mu(t) = 1$ and if for all $s \in [M_1, M_2]$ holds*

$$\int_a^b \widetilde{G}\,(g(t), s)\,\mathrm{d}\mu(t) \leq 0, \tag{4.1}$$

where the function $\widetilde{G}$ is defined on $[M_1, M_2] \times [M_1, M_2]$ by

$$\widetilde{G}(t, s) = \begin{cases} \dfrac{(t - M_2)(s - M_1)}{M_2 - M_1}, & \text{for } s \leq t, \\[2ex] \dfrac{(s - M_2)(t - M_1)}{M_2 - M_1}, & \text{for } t \leq s, \end{cases}$$

then the following inequality holds

(4.2)
$$\int_a^b f(g(t))\,\mathrm{d}\mu(t) \le \frac{M_2 - \int_a^b g(t)\mathrm{d}\mu(t)}{M_2 - M_1} f(M_1) + \frac{\int_a^b g(t)\mathrm{d}\mu(t) - M_1}{M_2 - M_1} f(M_2).$$

The reverse inequality in (4.1) *implies the reverse inequality in* (4.2).

In [18] Theorem A is proved using the interpolation polynomials (Hermite interpolating polynomial, as well as Lidstone interpolating polynomial).

In the following theorem, we give the version of this result for some signed measures, in which we weaken the conditions on the function G, and we use the similar method as in Section 2 of this paper.

Theorem 4.1. *Let $g : [a,b] \to \mathbb{R}$ be continuous function and $[\alpha,\beta]$ be an interval such that the image of g is a subset of $[\alpha,\beta]$. Let $m, M \in [\alpha,\beta]$ $(m \neq M)$ be such that $m \le g(t) \le M$ for all $t \in [a,b]$. Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a) \neq \lambda(b)$. Then the following two statements are equivalent:*

(1) *For every continuous convex function $\varphi : [\alpha,\beta] \to \mathbb{R}$*

$$\frac{\int_a^b \varphi(g(x))\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \le \frac{M - \overline{g}}{M - m}\varphi(m) + \frac{\overline{g} - m}{M - m}\varphi(M) \tag{4.3}$$

holds, where $\overline{g} = \dfrac{\int_a^b g(x)\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}$.

(2) *For all $s \in [\alpha,\beta]$ holds*

$$\frac{\int_a^b G(g(x),s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \le \frac{M - \overline{g}}{M - m}G(m,s) + \frac{\overline{g} - m}{M - m}G(M,s) \tag{4.4}$$

where the function $G : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ is defined in (2.1).

Furthermore, the statements (1) *and* (2) *are also equivalent if we change the sign of inequality in both* (4.3) *and* (4.4).

Proof. The idea of the proof is very similar to the proof of the Theorem 2.1.

(1) $\Rightarrow$ (2) : Let (1) hold. As the function $G(\cdot, s)$ $(s \in [\alpha,\beta])$ is also continuous and convex, it follows that also for this function (4.3) holds, *i.e.* it holds (4.4).

(2) $\Rightarrow$ (1) : Let (2) hold. We know that we can represent every function $\varphi : [\alpha,\beta] \to \mathbb{R}$, $\varphi \in C^2([\alpha,\beta])$, in the form (2.2), where the function G is

defined in (2.1). By easy calculation, using (2.2), we can easily get that

$$\begin{aligned} &\frac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\varphi(m) - \frac{\overline{g}-m}{M-m}\varphi(M) \\ &= \int_\alpha^\beta \left[\frac{\int_a^b G\left(g(x), s\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m} G(m,s) - \frac{\overline{g}-m}{M-m} G(M,s) \right] \varphi''(s)\mathrm{d}s. \end{aligned} \tag{4.5}$$

If the function φ is also convex, then $\varphi''(s) \geq 0$ for all $s \in [\alpha, \beta]$. So, if for every $s \in [\alpha, \beta]$ holds (4.4), then it follows that for every convex function $\varphi : [\alpha, \beta] \to \mathbb{R}$, with $\varphi \in C^2([\alpha, \beta])$, inequality (4.3) holds.

At the end, as in the proof of Theorem 2.1, note that it is not necessary to demand the existence of the second derivative of the function φ (see [20], p. 172). The differentiability condition can be directly eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials.

The last part of our theorem can be proved analogously. □

Remark 4.1. Let the conditions from the previous theorem hold. Then the following two statements are equivalent:

(1′) For every continuous concave function $\varphi : [\alpha, \beta] \to \mathbb{R}$ the reverse inequality in (4.3) holds.

(2′) For all $s \in [\alpha, \beta]$ inequality (4.4) holds, where the function G is defined in (2.1).

Moreover, the statements (1′) and (2′) are also equivalent if we change the sign of inequality in both statements (1′) and (2′).

Remark 4.2. Note that in all the results in this section we allow that the mean value $\overline{g}$ goes out of the interval $[\alpha, \beta]$, while in the results from previous sections we demanded that $\overline{g} \in [\alpha, \beta]$.

Setting $m = \alpha$ and $M = \beta$ in Theorem 4.1, we get the following corollary and with this also the result from the above mentioned Theorem A for the real Stieltjes measure $d\lambda$.

Corollary 4.1. *Let $g : [a, b] \to \mathbb{R}$ be continuous function and $[\alpha, \beta]$ be an interval such that the image of g is a subset of $[\alpha, \beta]$. Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a) \neq \lambda(b)$. Then the following two statements are equivalent:*

(1) *For every continuous convex function $\varphi : [\alpha, \beta] \to \mathbb{R}$ holds*

$$\frac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq \frac{\beta-\overline{g}}{\beta-\alpha}\varphi(\alpha) + \frac{\overline{g}-\alpha}{\beta-\alpha}\varphi(\beta). \tag{4.6}$$

(2) *For all* $s \in [\alpha, \beta]$ *holds*

$$\frac{\int_a^b G\left(g(x), s\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq 0, \tag{4.7}$$

where the function $G : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ *is defined in* (2.1).

Furthermore, the statements (1) *and* (2) *are also equivalent if we change the sign of inequality in both* (4.6) *and* (4.7).

For the converses of Jensen's inequality we can also give the results of type (1.4) and (1.5). We give these results in the following two theorems.

THEOREM 4.2. *Let* $g : [a, b] \to \mathbb{R}$ *be continuous function and* $\varphi : [\alpha, \beta] \to \mathbb{R}$, $\varphi \in C^2\left([\alpha, \beta]\right)$, *where the image of* g *is a subset of* $[\alpha, \beta]$. *Let* $m, M \in [\alpha, \beta]$ ($m \neq M$) *be such that* $m \leq g(t) \leq M$ *for all* $t \in [a, b]$. *Let* $\lambda : [a, b] \to \mathbb{R}$ *be continuous function or the function of bounded variation, and* $\lambda(a) \neq \lambda(b)$. *Let for every* $s \in [\alpha, \beta]$ *hold* (4.4) *or let for every* $s \in [\alpha, \beta]$ *hold the reverse inequality in* (4.4). *Then there exists some* $\xi \in [\alpha, \beta]$ *such that the following holds*

$$\begin{aligned} &\frac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M - \overline{g}}{M - m}\varphi(m) - \frac{\overline{g} - m}{M - m}\varphi(M) \\ &= \frac{1}{2}\varphi''(\xi) \left[\frac{\int_a^b \left(g(x)\right)^2 \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M - \overline{g}}{M - m} \cdot m^2 - \frac{\overline{g} - m}{M - m} \cdot M^2\right]. \end{aligned} \tag{4.8}$$

Proof. The idea of the proof is very similar to the proof of the Theorem 2.2. Following the assumptions of our theorem, we have that the function φ'' is continuous and that

$$\frac{\int_a^b G\left(g(x), s\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M - \overline{g}}{M - m}G(m, s) - \frac{\overline{g} - m}{M - m}G(M, s)$$

doesn't change it's positivity on $[\alpha, \beta]$. For our function φ equality (4.5) holds, and now, applying the integral mean-value theorem we get that there exists some $\xi \in [\alpha, \beta]$ such that

$$\begin{aligned} &\frac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M - \overline{g}}{M - m}\varphi(m) - \frac{\overline{g} - m}{M - m}\varphi(M) \\ &= \varphi''(\xi) \int_\alpha^\beta \left[\frac{\int_a^b G\left(g(x), s\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M - \overline{g}}{M - m}G(m, s) - \frac{\overline{g} - m}{M - m}G(M, s)\right] \mathrm{d}s. \end{aligned} \tag{4.9}$$

Calculating the integral on the right side in (4.9) and using (2.8), we get the statement (4.8) of our theorem. □

Remark 4.3. Note that (4.8) can also be expressed as

$$\frac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\varphi(m) - \frac{\overline{g}-m}{M-m}\varphi(M) = \frac{1}{2}\varphi''(\xi)\left[\frac{\int_a^b \left(g(x)\right)^2 \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \overline{g}(M+m) + Mm\right].$$

THEOREM 4.3. *Let* $g : [a,b] \to \mathbb{R}$ *be continuous function and* $\varphi, \psi : [\alpha,\beta] \to \mathbb{R}$, $\varphi, \psi \in C^2([\alpha,\beta])$, *where the image of* g *is a subset of* $[\alpha,\beta]$. *Let* $m, M \in [\alpha,\beta]$ $(m \neq M)$ *be such that* $m \leq g(t) \leq M$ *for all* $t \in [a,b]$. *Let* $\lambda : [a,b] \to \mathbb{R}$ *be continuous function or the function of bounded variation, and* $\lambda(a) \neq \lambda(b)$. *Let for every* $s \in [\alpha,\beta]$ *hold* (4.4) *or let for every* $s \in [\alpha,\beta]$ *hold the reverse inequality in* (4.4). *Then there exists some* $\xi \in [\alpha,\beta]$ *such that the following holds*

$$\frac{\dfrac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \dfrac{M-\overline{g}}{M-m}\varphi(m) - \dfrac{\overline{g}-m}{M-m}\varphi(M)}{\dfrac{\int_a^b \psi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \dfrac{M-\overline{g}}{M-m}\psi(m) - \dfrac{\overline{g}-m}{M-m}\psi(M)} = \frac{\varphi''(\xi)}{\psi''(\xi)}, \tag{4.10}$$

provided that the denominator of the left-hand side of (4.10) *is nonzero.*

Proof. The idea of the proof is very similar to the proof of the Theorem 2.3. Again, we define the function χ as a linear combination of functions φ and ψ.

$$\chi(t) = \left[\frac{\int_a^b \psi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\psi(m) - \frac{\overline{g}-m}{M-m}\psi(M)\right] \cdot \varphi(t) - \left[\frac{\int_a^b \varphi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\varphi(m) - \frac{\overline{g}-m}{M-m}\varphi(M)\right] \cdot \psi(t).$$

Function χ is also defined on the segment $[\alpha,\beta]$, it is continuous and χ'' is continuous on $[\alpha,\beta]$. Now, we can apply Theorem 4.2 on function χ and it follows that there exists some $\xi \in [\alpha,\beta]$ such that the following is valid

$$\frac{\int_a^b \chi\left(g(x)\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\chi(m) - \frac{\overline{g}-m}{M-m}\chi(M) = \frac{1}{2}\chi''(\xi)\left[\frac{\int_a^b \left(g(x)\right)^2 \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{M-\overline{g}}{M-m}\cdot m^2 - \frac{\overline{g}-m}{M-m}\cdot M^2\right].$$

After a short calculation we get that the left-hand side of this equation equals to zero. The term in the square brackets on the right-hand side is different from 0, because otherwise, from Theorem 4.2 applied on the function ψ, we would have that the denominator on the left-hand side of (4.10) equals to 0. It follows that

$$\chi''(\xi) = 0.$$

Now the assertion of our theorem follows directly. □

5. DISCRETE FORM OF THE CONVERSES OF THE JENSEN INEQUALITY

In this section, we give the results for converses of Jensen's inequality in discrete case. The proofs are similar to those in the integral case given in the previous section, so we give these results here without the proofs.

As we can represent any function $f : [\alpha, \beta] \to \mathbb{R}$, $f \in C^2([\alpha, \beta])$, in the form (2.2), where the function G is defined in (2.1), by some calculation it's easy to show that the following holds:

$$\begin{aligned} &\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - \frac{b-\overline{x}}{b-a} f(a) - \frac{\overline{x}-a}{b-a} f(b) \\ &\qquad = \int_{\alpha}^{\beta} \left(\frac{1}{P_n}\sum_{i=1}^{n} p_i G(x_i, s) - \frac{b-\overline{x}}{b-a} G(a, s) - \frac{\overline{x}-a}{b-a} G(b, s) \right) f''(s)\mathrm{d}s. \end{aligned}$$

We have the following result.

THEOREM 5.1. *Let $x_i \in [a, b] \subseteq [\alpha, \beta]$, $a \neq b$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$ be such that $P_n \neq 0$. Then the following two statements are equivalent:*

(1) *For every continuous convex function $f : [\alpha, \beta] \to \mathbb{R}$ holds*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \leq \frac{b-\overline{x}}{b-a} f(a) + \frac{\overline{x}-a}{b-a} f(b). \tag{5.1}$$

(2) *For all $s \in [\alpha, \beta]$ holds*

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i G(x_i, s) \leq \frac{b-\overline{x}}{b-a} G(a, s) + \frac{\overline{x}-a}{b-a} G(b, s) \tag{5.2}$$

where the function $G : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ is defined in (2.1).

Moreover, the statements (1) *and* (2) *are also equivalent if we change the sign of inequality in both inequalities, in* (5.1) *and in* (5.2).

Remark 5.1. If we set that all $p_i \in \mathbb{R}$ $(i = 1, ..., n)$ are positive, then (5.1) becomes classical converse of the Jensen inequality (see for example [15], p. 48).

Remark 5.2. Let the conditions from the previous theorem hold. Then the following two statements are equivalent:

(1′) For every continuous concave function $\varphi : [\alpha, \beta] \to \mathbb{R}$ the reverse inequality in (5.1) holds.

(2′) For all $s \in [\alpha, \beta]$ inequality (5.2) holds, where the function G is defined in (2.1).

Moreover, the statements (1′) and (2′) are also equivalent if we change the sign of inequality in both statements (1′) and (2′).

Setting $a = \alpha, b = \beta$ in Theorem 5.1 we get the following result.

Corollary 5.1. *Let $x_i \in [\alpha, \beta]$, $\alpha \neq \beta$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$ be such that $P_n \neq 0$. Then the following two statements are equivalent:*

(1) *For every continuous convex function $f : [\alpha, \beta] \to \mathbb{R}$ holds*

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leq \frac{\beta - \overline{x}}{\beta - \alpha} f(\alpha) + \frac{\overline{x} - \alpha}{\beta - \alpha} f(\beta). \tag{5.3}$$

(2) *For all $s \in [\alpha, \beta]$ holds*

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i G(x_i, s) \leq 0 \tag{5.4}$$

where the function $G : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ is defined in (2.1).

Moreover, the statements (1) *and* (2) *are also equivalent if we change the sign of inequality in both inequalities, in* (5.3) *and in* (5.4).

Now follow the results of the type (1.2) and (1.3) for the converses of the Jensen inequality in discrete case.

Theorem 5.2. *Let $x_i \in [a, b] \subseteq [\alpha, \beta]$, $a \neq b$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$ be such that $P_n \neq 0$ and let $f : [\alpha, \beta] \to \mathbb{R}$, $f \in C^2([\alpha, \beta])$. Let for every $s \in [\alpha, \beta]$ hold* (5.2) *or let for every $s \in [\alpha, \beta]$ the reverse inequality in* (5.2) *hold. Then there exists some $\xi \in [\alpha, \beta]$ such that the following is valid*

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) - \frac{b - \overline{x}}{b - a} f(a) - \frac{\overline{x} - a}{b - a} f(b)$$
$$= \frac{1}{2} f''(\xi) \left[\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i^2 - \frac{b - \overline{x}}{b - a} a^2 - \frac{\overline{x} - a}{b - a} b^2 \right]. \tag{5.5}$$

Remark 5.3. Note that (5.5) can also be expressed as

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - \frac{b-\overline{x}}{b-a} f(a) - \frac{\overline{x}-a}{b-a} f(b) = \frac{1}{2} f''(\xi) \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i^2 - \overline{x}(a+b) + ab \right].$$

THEOREM 5.3. *Let $x_i \in [a,b] \subseteq [\alpha,\beta]$, $a \neq b$, $p_i \in \mathbb{R}$ $(i = 1, ..., n)$ be such that $P_n \neq 0$ and let $f, g : [\alpha,\beta] \to \mathbb{R}$, $f, g \in C^2([\alpha,\beta])$. Let for every $s \in [\alpha,\beta]$ hold* (5.2) *or let for every $s \in [\alpha,\beta]$ the reverse inequality in* (5.2) *hold. Then there exists some $\xi \in [\alpha,\beta]$ such that the following is valid*

$$\frac{\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - \frac{b-\overline{x}}{b-a} f(a) - \frac{\overline{x}-a}{b-a} f(b)}{\frac{1}{P_n}\sum_{i=1}^{n} p_i g(x_i) - \frac{b-\overline{x}}{b-a} g(a) - \frac{\overline{x}-a}{b-a} g(b)} = \frac{f''(\xi)}{g''(\xi)}$$

provided that the denominator of the left-hand side is nonzero.

6. THE HERMITE-HADAMARD INEQUALITY

The classical Hermite-Hadamard inequality states that for a convex function $f : [a,b] \to \mathbb{R}$ the following estimation holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{6.1}$$

Its weighted form is proved by L. Fejér in [6]: if $f : [a,b] \to \mathbb{R}$ is a convex function and $p : [a,b] \to \mathbb{R}$ nonnegative integrable function, symmetric with respect to the middle point $(a+b)/2$, then the following estimation holds

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,\mathrm{d}x \leq \int_a^b f(x)p(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\int_a^b p(x)\,\mathrm{d}x.$$

A.M. Fink in [7] discussed on the generalization of (6.1) (separately looking its left and right side inequality) considering certain signed measures. In their recent paper [8], authors gave complete characterization for the right side inequality ([8], Theorem 1), which in fact, as special case, follows from the result already given in [16].

As a consequence of our results given in Section 2 and 4, here we give the complete characterization for the left and the right side of the generalized Hermite-Hadamard inequality for the real Stieltjes measure.

As a consequence of our results given in Section 2, follow the results for the left-side inequality of the generalized Hermite-Hadamard inequality.

Corollary 6.1. *Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a) \neq \lambda(b)$. Let $[\alpha, \beta] \subseteq \mathbb{R}$ be such that $[a, b] \subseteq [\alpha, \beta]$ and $\overline{x} \in [\alpha, \beta]$, where*

$$\overline{x} = \frac{\int_a^b x \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}.$$

Then the following two statements are equivalent:

(1) *For every continuous convex function $f : [\alpha, \beta] \to \mathbb{R}$ holds*

$$f\left(\overline{x}\right) \leq \frac{\int_a^b f\left(x\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}. \tag{6.2}$$

(2) *For all $s \in [\alpha, \beta]$ holds*

$$G\left(\overline{x}, s\right) \leq \frac{\int_a^b G\left(x, s\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}, \tag{6.3}$$

where the function $G : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ is defined in (2.1).

Furthermore, the statements (1) *and* (2) *are also equivalent if we change the sign of inequality in both* (6.2) *and* (6.3).

Analogous remark as Remark 2.1 here also holds.

Note that for the left-side inequality of the generalized Hermite-Hadamard inequality it is necessary to demand that $\overline{x} \in [\alpha, \beta]$.

Remark 6.1. For $\lambda(x) = x$, it is $\int_a^b \mathrm{d}\lambda(x) = b - a$ and $\overline{x} = \frac{a+b}{2}$, so (6.2) becomes the left inequality in the classical Hermite-Hadamard inequality (6.1).

For the left-hand side of the generalized Hermite-Hadamard inequality we can also derive adequate mean-value theorems of the Lagrange and Cauchy type.

Corollary 6.2. *Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, $\lambda(a) \neq \lambda(b)$, and $f : [\alpha, \beta] \to \mathbb{R}$, $f \in C^2([\alpha, \beta])$, where $[a, b] \subseteq [\alpha, \beta]$ and $\overline{x} \in [\alpha, \beta]$. Let for every $s \in [\alpha, \beta]$ hold* (6.3) *or let for every $s \in [\alpha, \beta]$ the reverse inequality in* (6.3) *hold. Then there exists some $\xi \in [\alpha, \beta]$ such that the following is valid*

$$f\left(\overline{x}\right) - \frac{\int_a^b f\left(x\right) \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} = \frac{1}{2}\varphi''(\xi) \left[\overline{x}^2 - \frac{\int_a^b x^2 \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}\right].$$

It's easy to see that for $\lambda(x) = x$, the condition (6.3) is always fulfilled. It is $\overline{x} = \frac{a+b}{2} \in [a, b]$, so we can narrow the interval we are looking from $[\alpha, \beta]$ to $[a, b]$. In that case, the previous corollary gives us that for any function $f \in C^2([a, b])$ there exists some $\xi \in [a, b]$ such that the following is valid

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t = \frac{1}{12} f''(\xi)\left(a^2 + 4ab + b^2\right).$$

Corollary 6.3. *Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, $\lambda(a) \neq \lambda(b)$, and $f, g : [\alpha, \beta] \to \mathbb{R}$, $f, g \in C^2([\alpha, \beta])$, where $[a, b] \subseteq [\alpha, \beta]$ and $\overline{x} \in [\alpha, \beta]$. Let for every $s \in [\alpha, \beta]$ hold* (6.3) *or let for every $s \in [\alpha, \beta]$ the reverse inequality in* (6.3) *hold. Then there exists some $\xi \in [\alpha, \beta]$ such that the following is valid*

$$\frac{f(\overline{x}) - \dfrac{\int_a^b f(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}}{g(\overline{x}) - \dfrac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)}} = \frac{f''(\xi)}{g''(\xi)},$$

provided that the denominator of the left-hand side is nonzero.

Similarly, from the results given in the fourth section we get the results for the right-side inequality of the generalized Hermite-Hadamard inequality. Here, we allow that the mean value $\overline{x}$ goes outside of the interval $[\alpha, \beta]$.

Corollary 6.4. *Let $\lambda : [a, b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a) \neq \lambda(b)$. Let $[\alpha, \beta] \subseteq \mathbb{R}$ be such that $[a, b] \subseteq [\alpha, \beta]$. Then the following two statements are equivalent:*

(1) *For every continuous convex function $f : [\alpha, \beta] \to \mathbb{R}$ holds*

$$\frac{\int_a^b f(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq \frac{b-\overline{x}}{b-a} f(a) + \frac{\overline{x}-a}{b-a} f(b). \tag{6.4}$$

(2) *For all $s \in [\alpha, \beta]$ holds*

$$\frac{\int_a^b G(x, s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq \frac{b-\overline{x}}{b-a} G(a, s) + \frac{\overline{x}-a}{b-a} G(b, s) \tag{6.5}$$

where the function $G : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ is defined in (2.1).

Furthermore, the statements (1) *and* (2) *are also equivalent if we change the sign of inequality in both* (6.4) *and* (6.5).

Of course, here also analogous remark as Remark 4.1 holds.

Furthermore, setting $\alpha = a$ and $\beta = b$ in previous corollary, we get that the right side of the inequality (6.5) equals to zero, so (6.5) becomes

$$\frac{\int_a^b G(x,s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq 0.$$

Remark 6.2. For $\lambda(x) = x$ inequality (6.4) becomes the right-side inequality in the classical Hermite-Hadamard inequality (6.1).

Also, for the right-side of the generalized Hermite-Hadamard inequality, we can derive adequate results of the Lagrange and Cauchy type.

Corollary 6.5. *Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, $\lambda(a) \neq \lambda(b)$, and $f : [\alpha,\beta] \to \mathbb{R}$, $f \in C^2([\alpha,\beta])$, where $[a,b] \subseteq [\alpha,\beta]$. Let for every $s \in [\alpha,\beta]$ hold* (6.5) *or let for every $s \in [\alpha,\beta]$ the reverse inequality in* (6.5) *hold. Then there exists some $\xi \in [\alpha,\beta]$ such that the following is valid*

$$\begin{aligned} &\frac{\int_a^b f(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{b-\overline{x}}{b-a} f(a) - \frac{\overline{x}-a}{b-a} f(b) \\ &\qquad = \frac{1}{2} f''(\xi) \left[\frac{\int_a^b x^2 \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{b-\overline{x}}{b-a} \cdot a^2 - \frac{\overline{x}-a}{b-a} \cdot b^2 \right]. \end{aligned} \tag{6.6}$$

Note that (6.6) can also be expressed as

$$\begin{aligned} &\frac{\int_a^b f(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \frac{b-\overline{x}}{b-a} f(a) - \frac{\overline{x}-a}{b-a} f(b) \\ &\qquad = \frac{1}{2} f''(\xi) \left[\frac{\int_a^b x^2 \mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \overline{x}(a+b) + ab \right]. \end{aligned}$$

For $\lambda(x) = x$ the condition (6.5) is always fulfilled. We can also narrow the interval we are looking from $[\alpha,\beta]$ to $[a,b]$. In that case, the previous corollary gives us that for any function $f \in C^2([a,b])$ there exists some $\xi \in [a,b]$ such that the following is valid

$$\frac{1}{b-a} \int_a^b f(t)\mathrm{d}t - \frac{f(a)+f(b)}{2} = -\frac{1}{12} f''(\xi)\,(a-b)^2 .$$

Corollary 6.6. *Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, $\lambda(a) \neq \lambda(b)$, and $f,g : [\alpha,\beta] \to \mathbb{R}$, $f,g \in C^2([\alpha,\beta])$, where $[a,b] \subseteq [\alpha,\beta]$. Let for every $s \in [\alpha,\beta]$ hold* (6.5) *or let for every $s \in [\alpha,\beta]$ the reverse inequality in* (6.5) *hold. Then there exists some $\xi \in [\alpha,\beta]$ such that* *the following is valid*

$$\frac{\dfrac{\int_a^b f(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \dfrac{b-\overline{x}}{b-a}f(a) - \dfrac{\overline{x}-a}{b-a}f(b)}{\dfrac{\int_a^b g(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} - \dfrac{b-\overline{x}}{b-a}g(a) - \dfrac{\overline{x}-a}{b-a}g(b)} = \frac{f''(\xi)}{g''(\xi)}, \tag{6.7}$$

provided that the denominator of the left-hand side of (6.7) *is nonzero.*

At the end, as a consequence of the Corollary 6.1 and Corollary 6.4, we get the necessary and sufficient conditions on a real Stieltjes measure, given using the Green function G, that for any convex function the generalization of the Hermite-Hadamard inequality holds.

Corollary 6.7. *Let $\lambda : [a,b] \to \mathbb{R}$ be continuous function or the function of bounded variation, and $\lambda(a) \neq \lambda(b)$. Let $[\alpha,\beta] \subseteq \mathbb{R}$ be such that $[a,b] \subseteq [\alpha,\beta]$ and $\overline{x} \in [\alpha,\beta]$. Then the following two statements are equivalent:*

(1) *For every continuous convex function $f : [\alpha,\beta] \to \mathbb{R}$ holds*

$$f(\overline{x}) \leq \frac{\int_a^b f(x)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq \frac{b-\overline{x}}{b-a}f(a) + \frac{\overline{x}-a}{b-a}f(b). \tag{6.8}$$

(2) *For all $s \in [\alpha,\beta]$ holds*

$$G(\overline{x},s) \leq \frac{\int_a^b G(x,s)\,\mathrm{d}\lambda(x)}{\int_a^b \mathrm{d}\lambda(x)} \leq \frac{b-\overline{x}}{b-a}G(a,s) + \frac{\overline{x}-a}{b-a}G(b,s), \tag{6.9}$$

where the function $G : [\alpha,\beta] \times [\alpha,\beta] \to \mathbb{R}$ is defined in (2.1).

Furthermore, the statements (1) *and* (2) *are also equivalent if we change the sign of inequality in both* (6.8) *and* (6.9).

Acknowledgments. The present investigation was supported by the Croatian Ministry of Science, Education and Sports under the Research Grants 117-1170889-0888 and 058-1170889-1050.

REFERENCES

[1] H. Alzer, *Eine Doppelungleichung für Integrale.* Anz. Österreich Akad. Wiss. Math-Natur. Kl. **127** (1990), 37–40.
[2] R.E. Barlow, A.W. Marshall and F. Proschan, *Some inequalities for starshaped and convex functions.* Pacific J. Math. **29** (1969), 19–42.
[3] R.P. Boas, *The Jensen-Steffensen inequality.* Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. **302–319** (1970), 1–8.
[4] H.D. Brunk, *On an inequality for convex functions.* Proc. Amer. Math. Soc. **7** (1956), 817–824.
[5] P.S. Bullen, *Means and Their Inequalities.* Kluwer Academic Publ., Dordrecht/Boston/London, 2003.

[6] L. Fejér, *Uber die Fourierreihen, II.* Math. Naturwiss. Anz. Ungar. Akad. Wiss. **24** (1906), 369–390.
[7] A.M. Fink, *A best possible Hadamard inequality.* Math. Inequal. Appl. **1** (1998), 223–230.
[8] A. Florea and C.P. Niculescu, *A Hermite-Hadamard inequality for convex-concave symmetric functions.* Bull. Soc. Sci. Math. Roum. **50** (98), *2* (2007), 149–156.
[9] A. McD. Mercer, *An "error term" for the Ky Fan inequality.* J. Math. Anal. Appl. **220** (1998), 774–777.
[10] A. McD. Mercer, *Some new inequalities involving elementary mean values.* J. Math. Anal. Appl. **229** (1999), 677–681.
[11] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and new Inequalities in Analysis.* Kluwer Academic Publ., Dordrecht/Boston/London, 1993.
[12] J.E. Pečarić, *On an inequality of P.M. Vasić and R.R. Janić.* Publ. Inst. Math. (Beograd) (N. S.) **28**(42) (1980), 145–149.
[13] J.E. Pečarić, *A new proof of the Jensen-Steffensen inequality.* Mathematica **23**(46) (1981), 73–77.
[14] J.E. Pečarić, *On variants of Jensen's inequality.* Glas. Mat. Ser. III **18**(38) (1983), 281–289.
[15] J.E. Pečarić, *Konveksne funkcije: Nejednakosti.* Naučna knjiga, Beograd, 1987.
[16] J. Pečarić, V. Čuljak and A.M. Fink, *On some inequalities for convex function of higher order.* Nonlinear Stud. **6** (1999), *2*, 131–140.
[17] J. Pečarić, V. Čuljak and M. Rogina, *On some inequalities for convex function of higher order II.* Nonlinear Anal. **45** (2001), 281–294.
[18] J. Pečarić and V. Čuljak, *Interpolation polynomials and inequalities for convex functions of higher order.* Math. Inequal. Appl. **5** (2002), 369–386.
[19] J.E. Pečarić, I. Perić and H.M. Srivastava, *A family of the Cauchy type mean-value theorems.* J. Math. Anal. Appl. **306** (2005), 730–739.
[20] J.E.Pečarić, F.Proschan and Y.L.Tong, *Convex Functions, Partial Orderings, and Statistical Applications.* Mathematics in Science and Engineering **187**, Academic Press, 1992.
[21] P.M. Vasić and R.R. Janić, *An inequality for convex functions.* Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. **302–319** (1970), 39–42.
[22] D.V. Widder, *Completely convex function and Lidstone series.* Trans. Amer. Math. Soc. **51** (1942), 387–398.

Received 5 January 2012

University of Zagreb,
Faculty of Textile Technology,
Prilaz baruna Filipovića 28a,
10000 Zagreb, Croatia
pecaric@hazu.hr

University of Zagreb,
Faculty of Food Technology and Biotechnology,
Department of mathematics,
Pierottijeva 6,
10000 Zagreb, Croatia
iperic@pbf.hr

University of Zagreb,
Faculty of Textile Technology,
Prilaz baruna Filipovića 28a,
10000 Zagreb, Croatia
mrodic@ttf.hr