SOME RESULTS ON PRIME RADICAL IN ORDERED Γ-SEMIGROUPS

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The aim of this paper is to obtain and establish some important results in ordered Γ-semigroups extending those for ordered semigroups concerning prime, weakly prime, semiprime and weakly semiprime ideals, characterization of semilattice congruences by means of prime ideals, characterization of an arbitrary nonempty intersection of prime ideals. We also characterize the weakly prime ideals by m-system. We introduce the notion of prime radical in ordered Γ-semigroups and give the prime radical theorems of ordered Γ-semigroups.

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1. INTRODUCTION AND PRELIMINARIES

In 1981, Sen [23] introduced the concept and notion of the Γ - semigroup as a generalization of semigroup and of ternary semigroup. Many classical notions and results of the theory of semigroups have been extended and generalized to Γ - semigroups. Weakly prime, prime ideals in ordered semigroups have been introduced and studied by N. Kehayopulu in [8, 10] extending to ordered semigroups the analogue concepts of rings and their characterizations considered by N.H. McCoy and O. Steinfeld in [20, 21, 25]. Some results on prime, weakly prime ideals of ordered semigroups can be found in [8, 10–15, 26–28] which extend the analogue notions and properties of rings and semigroups. Kwon and Lee [16–18] introduced the concepts of the prime ideals, the weakly prime ideals and the weakly semiprime ideals in ordered Γ - semigroups and gave some characterizations of them extending the analogue results obtained in ordered semigroups considered by Kehayopulu [8]. Our purpose in this paper is mainly to obtain and establish some other results and properties in ordered Γ - semigroups extending those for ordered semigroups concerning prime, weakly prime, semiprime and weakly semiprime ideals, characterization of semilattice congruences by means of prime ideals, characterization of an arbitrary nonempty intersection of prime ideals. We prove that these results remain true for ordered Γ-semigroups as well. We also introduce the concept of m-system in ordered Γ-semigroups and characterize the weakly prime ideals by m-system. We introduce the notion of prime radical in ordered Γ-semigroups and give the prime radical theorems of ordered Γ-semigroups. In this paper, we prove that every ideal of an $\mathcal{N}$-class of an ordered Γ-semigroup does not contain proper prime ideals. As a consequence, every prime ideal of an ordered Γ-semigroup is decomposable into its $\mathcal{N}$-clasess. Establishing an order-preserving bijection between the set of all prime ideals of the $po-\Gamma$-semigroup M and the set of all prime ideals of the $po-\Gamma$-semigroup $S = M/\mathcal{N}$ induced by the complete semilattice congruence $\mathcal{N}$ on M, we extend to ordered Γ-semigroups the II.2.15 Corollary [22] and the analogue result for ordered semigroup [26]. We establish some equivalent statements for the $po-\Gamma$-semigroup $S = M/\mathcal{N}$ to be a chain, extending the analogue result for the semigroup [22, II.3.15.] and for ordered semigroups [26]. Finally, we introduce the notion of radical of an ideal on $po-\Gamma$-semigroups and we give the prime radical theorems of $po-\Gamma$-semigroups analogue to that of ordered semigroups [28]. In this paper, we prove that every weakly semiprime (resp., semiprime) ideal I of an $po-\Gamma$-semigroup M is the intersection of all weakly prime (resp., prime) ideals containing it. In particular, on commutative $po-\Gamma$-semigroups, the radical of an ideal I of M is the intersection of all prime ideals containing it.

We first recall the definition of the Γ-semigroup as a generalization of semigroup and ternary semigroup in another way as follows (cf. [23, 24]):

Definition 1.1. Let M and Γ be two non-empty sets. Any map from $M \times \Gamma \times M \to M$ will be called a Γ-*multiplication* in M and denoted by $(\cdot)_\Gamma$. The result of this multiplication for $a, b \in M$ and $\alpha \in \Gamma$ is denoted by $a\alpha b$. A Γ-*semigroup* M is an ordered pair $(M, (\cdot)_\Gamma)$ where M and Γ are non-empty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on M which satisfies the following property

$$\forall (a, b, c, \alpha, \beta) \in M^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

Example 1.2. Let M be a semigroup and Γ be any nonempty set. Define a mapping $M \times \Gamma \times M \to M$ by $a\gamma b = ab$ for all $a, b \in M$ and $\gamma \in \Gamma$. Then M is a Γ-semigroup.

Example 1.3. Let M be a set of all negative rational numbers. Obviously, M is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} : p$ is prime$\}$. Let $a, b, c \in M$ and $\alpha \in \Gamma$. Now, if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in M$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence, M is a Γ-semigroup.

Example 1.4. Let $M = \{-i, 0, i\}$ and $\Gamma = M$. Then M is a Γ-semigroup under the multiplication over complex numbers while M is not a semigroup under complex number multiplication.

These examples show that every semigroup is a Γ-semigroup. Therefore, Γ-semigroups are a generalization of semigroups.

A Γ-semigroup M is called *commutative* Γ*-semigroup* if for all $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma b = b\gamma a$. A nonempty subset K of a Γ-semigroup M is called a *sub*-Γ*-semigroup* of M if for all $a, b \in K$ and $\gamma \in \Gamma$, $a\gamma b \in K$.

Example 1.5. Let $M = [0, 1]$ and $\Gamma = \{\frac{1}{n} | n$ is a positive integer$\}$. Then M is a Γ-semigroup under usual multiplication. Let $K = [0, 1/2]$. We have that K is a nonemtpy subset of M and $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. Then K is a sub-Γ-semigroup of M.

Definition 1.6. A *po*-Γ*-semigroup* (: *ordered* Γ*-semigroup*) is an ordered set M at the same time a Γ-semigroup such that for all $a, b, c \in M$ and for all $\gamma \in \Gamma$

$$a \leq b \Rightarrow a\gamma c \leq b\gamma c, c\gamma a \leq c\gamma b.$$

Throughout this paper, M stands for an ordered Γ-semigroup. For nonempty subsets A and B of M and a nonempty subset Γ' of Γ, let $A\Gamma' B = \{a\gamma b : a \in A, b \in B$ and $\gamma \in \Gamma'\}$. If $A = \{a\}$, then we also write $\{a\}\Gamma' B$ as $a\Gamma' B$, and similarly if $B = \{b\}$ or $\Gamma' = \{\gamma\}$.

Let T be a sub-Γ-semigroup of M. For $A \subseteq T$ we denote

$$(A]_T = \{t \in T | t \leq a, \text{ for some } a \in A\}$$
$$[A)_T = \{t \in T | t \geq a, \text{ for some } a \in A\}$$

If $T = M$, then we always write $(A]$ (resp., $[A)$) instead of $(A]_M$ (resp. $[A)_M$). Clearly, $A \subseteq (A]_T \subseteq (A]$ and $A \subseteq B$ implies $(A]_T \subseteq (B]_T$ for any nonempty subsets A, B of T. For $A = \{a\}$, we write $(a]$ (resp., $[a)$ instead of $(\{a\}]$ (resp., $[\{a\})$).

Let M be a *po*-Γ-semigroup and A be a nonempty subset of M. Then A is called a *right* (resp., *left*) ideal of M if

1. $A\Gamma M \subseteq A$ (resp. $M\Gamma A \subseteq A$)
2. $a \in A, b \leq a$ for $b \in M \Rightarrow b \in A$

Equivalent Definition.

1. $A\Gamma M \subseteq A$ (resp. $M\Gamma A \subseteq A$).
2. $(A] = A$.

A is called an *ideal* of M if it is right and left ideal of M. A right, left or ideal A of a $po - \Gamma$-semigroup M is called *proper* if $A \neq M$.

Definition 1.7 ([7, 16]). Let M be a $po-\Gamma$-semigroup. An ideal T of M is said to be *prime* if $A\Gamma B \subseteq T$ implies that $A \subseteq T$ or $B \subseteq T$, where $A, B \subseteq M$.

Equivalent Definition. Let M be a $po-\Gamma$-semigroup. An ideal T of M is said to be *prime* if $a\Gamma b \subseteq T$ implies that $a \in T$ or $b \in T(a, b \in M)$.

Definition 1.8 ([9, 19]). Let M be a $po-\Gamma$-semigroup. An ideal T of M is said to be *semiprime* if $A\Gamma A \subseteq T$ implies that $A \subseteq T$, where $A \subseteq M$.

Equivalent Definition. Let M be a $po-\Gamma$-semigroup. An ideal T of M is said to be *semiprime* if $a\Gamma a \subseteq T$ implies that $a \in T(a \in M)$.

Definition 1.9 ([8, 17]). Let M be a $po-\Gamma$-semigroup. An ideal T of M is said to be *weakly prime* if $A\Gamma B \subseteq T$ implies that $A \subseteq T$ or $B \subseteq T$, for all ideals A, B of M.

Definition 1.10 ([8, 17]). Let M be a $po-\Gamma$-semigroup. An ideal T of M is said to be *weakly semiprime* if $A\Gamma A \subseteq T$ implies that $A \subseteq T$, for any ideal A of M.

Definition 1.11 ([16]). Let M be a $po-\Gamma$-semigroup. An equivalence relation $\Re$ on M is called *congruence* if

$$(a, b) \in \Re \Rightarrow (a\gamma c, b\gamma c) \in \Re, (c\gamma a, c\gamma b) \in \Re$$

for all $\gamma \in \Gamma$ and $c \in M$.

Definition 1.12 ([16]). Let M be a $po-\Gamma$-semigroup. A congruence $\Re$ on M is called *semilattice congruence* if

$$(a\gamma a, a) \in \Re \text{ and } (a\gamma b, b\gamma a) \in \Re$$

for all $\gamma \in \Gamma$ and $a, b \in M$.

Definition 1.13. Let M be a $po-\Gamma$-semigroup. A semilattice congruence $\Re$ on M is called *complete* if $a \leq b$ implies $(a, a\gamma b) \in \Re$, $a, b \in M$, $\gamma \in \Gamma$.

Definition 1.14 ([7, 16]). Let M be a $po-\Gamma$-semigroup and F a sub-Γ-semigroup. Then F is called a filter of M if

1. $a, b \in M, a\gamma b \in F\ (\gamma \in \Gamma) \Rightarrow a \in F$ *and* $b \in F$
2. $a \in F, a \leq c\ (c \in M) \Rightarrow c \in F$ *or equivalently* $[F) \subseteq F$.

For $a \in M$ we denote by $N(a)$ the filter of M generated by $a(a \in M)$. We denote by "$\mathcal{N}$" the equivalence relation on M defined by $\mathcal{N} = \{(a, b)|N(a) = N(b)\}$ [7, 16].

For a $po-\Gamma$-semigroup M, from the Theorem 2.7(1)[16], $\mathcal{N}$ is a semilattice congruence, and moreover, is a complete semilattice congruence on M, *i.e.* for $a, b \in M, a \leq b$ implies $(a, a\gamma b) \in \mathcal{N}, \gamma \in \Gamma$. Indeed: Since $N(a) \ni a \leq b$, we have $b \in N(a)$. Since $a, b \in N(a), a\gamma b \in N(a)$ and $N(a\gamma b) \subseteq N(a)$. Since $a\gamma b \in N(a\gamma b)$ we have $a \in N(a\gamma b)$ and $N(a) \subseteq N(a\gamma b)$.

For any $a \in M$, the $\mathcal{N}$-class containing a is denoted by $(a)_{\mathcal{N}}$ and it is clear that it is an ordered sub-Γ-semigroup of M. Indeed: $\emptyset \neq (a)_{\mathcal{N}} \subseteq M (a \in (a)_{\mathcal{N}})$. Let $x, y \in (a)_{\mathcal{N}}$ implies $x\mathcal{N}a$ and $y\mathcal{N}a$, then $x\gamma y\mathcal{N}a\gamma a$, $\gamma \in \Gamma$. Since $a\gamma a\mathcal{N}a$, we have $x\gamma y\mathcal{N}a$. Then $x\gamma y \in (a)_{\mathcal{N}}$. On the set $M/\mathcal{N} = \{(a)_{\mathcal{N}} | a \in M\}$ we define $(a)_{\mathcal{N}}\gamma(b)_{\mathcal{N}} = (a\gamma b)_{\mathcal{N}}, \forall a, b \in M, \gamma \in \Gamma$. It is clear that the set $M/\mathcal{N}$ is a Γ-semigroup. In this set, we define $(a)_{\mathcal{N}} \preceq (b)_{\mathcal{N}}$ if and only if $(a)_{\mathcal{N}} = (a\gamma b)_{\mathcal{N}}, \forall \gamma \in \Gamma$, then it can be easily seen that the set $M/\mathcal{N}$ is an po-Γ-semigroup induced by the complete semilattice congruence $\mathcal{N}$ on M.

Example 1.15. Let us consider the *po* − Γ-semigroup M, where $M = Z_6$ and $\Gamma = \{\overline{1}, \overline{3}\}$. The ideals of M are the sets: $L = \{\overline{0}\}, L_1 = \{\overline{0}, \overline{2}, \overline{4}\}, L_2 = \{\overline{0}, \overline{3}\}, L_3 = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$ and $L_4 = M$.

It can be easily verified that L is not weakly prime. The other ideals L_1, L_2, L_3, L_4 are weakly prime.

Example 1.16. Let us consider the sets M and Γ of Example 1.5 which is an ordered Γ-semigroup under usual multiplication and usual partial order relation. The ideals of M are $I_a = [0, a]$ where $0 \leq a \leq 1$. It can be easily verified that all these ideals are not weakly prime.

Example 1.17. Let $M = \{x, y, z\}$ and $\Gamma = \{\alpha, \beta\}$ with the multiplication defined by the following tables:

α	x	y	z
x	x	y	x
y	x	y	x
z	x	y	x

β	x	y	z
x	x	y	x
y	x	y	y
z	x	y	z

If we define a relation $\leq$ on M as follows:

$$\leq := \{(x, x), (y, y), (z, z), (x, y)\}$$

then it can be easily verified that M is an ordered Γ-semigroup. The ideals of M are $I_1 = \{x, y\}$ and $I_2 = M$. It can be easily seen that I_1, I_2 are weakly prime ideals.

Example 1.18. Let $M = \{\{a, b, c\}, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$ and $\Gamma = \{\{a, b, c\}, \emptyset, \{a\}\}$. If $ABC = A \cap B \cap C$ and $A \leq C \Leftrightarrow A \subseteq C$ for all $A, C \in M$ and $B \in \Gamma$, then M is a po-Γ-semigroup. It can be easily verified that the ideal $\{\{a, b\}, \{a\}, \emptyset\}$ is a weakly prime ideal of M.

2. ON PRIME IDEALS IN ORDERED Γ-SEMIGROUPS

For our results, we need to prove some preliminary results on ordered Γ-semigroups. The following propositions hold true:

Proposition 2.19. *Let M be an ordered Γ-semigroup. M does not contain proper filter if and only if M does not contain proper prime ideals.*

Proof. It is the same as the proof of Remark 2 in [11], we omit it. □

Proposition 2.20. *Let M be an ordered Γ-semigroup, F a filter of M, $a \in F \cap (c)_{\mathcal{N}}(c \in M)$. Then $(c)_{\mathcal{N}} \subseteq F$.*

Proof. It is the same as the proof of Remark 3 in [11], we omit it. □

The following theorem extends to ordered Γ-semigroups, the analogue results proved for ordered semigroups by Kehayopulu and Tsingelis [11].

Theorem 2.21. *Let M be an ordered Γ-semigroup and $c \in M$. If I is an ideal of $(c)_{\mathcal{N}}$, then I does not contain proper prime ideals of I.*

Proof. By Proposition 2.1, it is enough to prove that I does not contain proper filter, *i.e.* I itself, is the only filter of I.

Let us assume that F is a filter of I. We prove that $F = I$. Let $x \in F (F \neq \emptyset), T = \{a \in M | x\gamma x\alpha a \in F, \forall\alpha \in \Gamma \text{ and for some } \gamma \in \Gamma\}$.

1) We have $F = T \cap I$. Indeed: Let $b \in F$. Since $x\gamma x \in F$, we have $x\gamma x\alpha b \in F, \forall\alpha \in \Gamma$ and for some $\gamma \in \Gamma$, *i.e.,* $b \in T$. Since F is a filter of I we have $F \subseteq I$. So, $F \subseteq T \cap I$. Now, let $b \in T \cap I$. Since $b \in T, \exists\gamma \in \Gamma, \forall\alpha \in \Gamma, x\gamma x\alpha b \in F$. Then, since $x\gamma x \in F \subseteq I, b \in I$, F filter of I, we have $b \in F$, that is, $T \cap I \subseteq F$. So, $F = T \cap I$.

2) T is a filter of M. Indeed: $\emptyset \neq T \subseteq M$ (since $(x\gamma)^2 x \in F, \forall\gamma \in \Gamma, x \in T$). If $a, b \in T$, then $a\alpha b \in T, \forall\alpha \in \Gamma$. In fact: Let $a, b \in T$; then $x\gamma x\alpha b \in F, \forall\alpha \in \Gamma$ and for some $\gamma \in \Gamma$. Since $F \subseteq I \subseteq (c)_{\mathcal{N}}(c \in M)$, then $(b\alpha x)_{\mathcal{N}} = (x\gamma x\alpha b)_{\mathcal{N}} = (c)_{\mathcal{N}}$. Hence, $b\alpha x \in (c)_{\mathcal{N}}, \forall\alpha \in \Gamma$ and thus, $b\alpha x\gamma x \in I$. Furthermore, $x\gamma x\alpha(b\beta x\gamma x) = (x\gamma x\alpha b)\beta x\gamma x \in F$ so that $b\beta x\gamma x \in F, \forall\alpha, \beta \in \Gamma$. Similarly, $x\gamma x\alpha a \in F$ implies $x\alpha a \in (c)_{\mathcal{N}}, \forall\alpha \in \Gamma$ and for some $\gamma \in \Gamma$. Since $(x\alpha a\beta b)_{\mathcal{N}} = (x\alpha a)_{\mathcal{N}}\rho(x\beta b)_{\mathcal{N}} = (c)_{\mathcal{N}}, \forall\alpha, \beta, \rho \in \Gamma$, we have $x\alpha a\beta b \in (c)_{\mathcal{N}}$. Consequently, $x\gamma x\alpha a\beta b \in I, \forall\alpha, \beta \in \Gamma$ and for some $\gamma \in \Gamma$. On the other hand, $x\gamma_1 x\alpha a, b\beta x\gamma_2 x \in F, \forall\alpha, \beta \in \Gamma$ and for some $\gamma_1, \gamma_2 \in \Gamma$, implies $(x\gamma_1 x\alpha a)\rho(b\beta x\gamma_2 x) \in F$. Then, also $(x\gamma_1 x\alpha a\rho b)\beta x\gamma_2 x \in F$ so that $x\gamma_1 x\alpha a\rho b \in F, \forall\alpha, \beta, \rho \in \Gamma$. Therefore, $a\alpha b \in T, \forall\alpha \in \Gamma$.

Conversely, if $a, b \in M, a\alpha b \in T, \forall\alpha \in \Gamma$, then $a \in T$ and $b \in T$. Indeed: let $a\alpha b \in T, \forall\alpha \in \Gamma$. Then $x\gamma_1 x\alpha a\beta b \in F, \forall\alpha, \beta \in \Gamma$ and for some $\gamma_1 \in \Gamma$, and thus, also $(x\gamma_1 x\alpha a)\beta(b\rho x\gamma_2 x) = (x\gamma_1 x\alpha a\beta b)\rho x\gamma_2 x) \in F$, for all $\rho \in \Gamma$ and for some $\gamma_2 \in \Gamma$. Moreover, $(x\gamma_1 x\alpha a\beta b) \in F$ implies $x\alpha a, b\rho x \in (c)_{\mathcal{N}}$, and hence, $x\gamma_1 x\alpha a, b\rho x\gamma_2 x \in I$. But since $(x\gamma_1 x\alpha a)\rho(b\beta x\gamma_2 x) \in F$, it follows that $x\gamma_1 x\alpha a, b\beta x\gamma_2 x \in F, \forall \alpha, \beta, \rho \in \Gamma$ and for some $\gamma_1, \gamma_2 \in \Gamma$. As before, since $(x\gamma_1 x\alpha b)\rho x\gamma_2 x = x\gamma_1 x\alpha(b\rho x\gamma_2 x) \in F$ and $x\gamma_1 x\alpha b \in I, \forall \alpha, \rho \in \Gamma$ and for some $\gamma_1, \gamma_2 \in \Gamma$, we have that $x\gamma_1 x\alpha b \in F$. Thus, $a, b \in T$.

Let now $a \in T, S \ni b \geq a$. We prove that $b \in T$. We have $x\gamma x\alpha b \geq x\gamma x\alpha a \in F, \forall \alpha \in \Gamma$ and for some $\gamma \in \Gamma$. Also,

$$x\gamma x\alpha b = x\gamma(x\alpha b), x \in F \subseteq I \subseteq (c)_{\mathcal{N}}$$

and

$$\begin{aligned} x\gamma x\alpha a \in F \subseteq I \subseteq (c)_{\mathcal{N}} \Rightarrow (c)_{\mathcal{N}} &= (x\gamma x\alpha a)_{\mathcal{N}} = (x\gamma x)_{\mathcal{N}}\rho(a)_{\mathcal{N}} = \\ &= (x)_{\mathcal{N}}\rho(a)_{\mathcal{N}} = (x\alpha a)_{\mathcal{N}} \end{aligned}$$

for all $\rho \in \Gamma$. On the other hand, we have for all $\alpha, \beta \in \Gamma$,

$$\begin{aligned} a \leq b \quad &\Rightarrow \quad x\alpha a \leq x\alpha b \Rightarrow (x\alpha a, x\alpha a\beta x\alpha b) \in \mathcal{N} \\ &\Rightarrow \quad (x\alpha a)_{\mathcal{N}} = (x\alpha a\beta x\alpha b)_{\mathcal{N}} = (x)_{\mathcal{N}}\alpha(a)_{\mathcal{N}}\beta(x)_{\mathcal{N}}\alpha(b)_{\mathcal{N}} \\ &\qquad = (x\alpha x)_{\mathcal{N}}\beta(a)_{\mathcal{N}}\alpha(b)_{\mathcal{N}} = (x\alpha x\beta a)_{\mathcal{N}}\alpha(b)_{\mathcal{N}} = (c)_{\mathcal{N}}\alpha(b)_{\mathcal{N}} \\ &\qquad = (x)_{\mathcal{N}}\alpha(b)_{\mathcal{N}} \quad (x \in (c)_{\mathcal{N}}) \\ &\qquad = (x\alpha b)_{\mathcal{N}}. \end{aligned}$$

Hence, $x\alpha b \in (x\alpha b)_{\mathcal{N}} = (x\alpha a)_{\mathcal{N}} = (c)_{\mathcal{N}}$. Since I is an ideal of $(c)_{\mathcal{N}}$, we have $x\gamma(x\alpha b) \in I\Gamma(c)_{\mathcal{N}} \subseteq I$, and $x\gamma x\alpha b \in I$. Then, since F is a filter of I, we have $x\gamma x\alpha b \in F$, and so $b \in T$. Since T is a filter of M, $x \in T, x \in (c)_{\mathcal{N}}$, by Proposition 2.2, $(c)_{\mathcal{N}} \subseteq T$. Thus, we have

$$I \supseteq F = T \cap I \supseteq (c)_{\mathcal{N}} \cap I = I$$

i.e., $F = I$. □

COROLLARY 2.22. *Every prime ideal I of an ordered Γ-semigroup is a union of $\mathcal{N}$-classes of M.*

Proof. Let $a \in I$ and $t \in (a)_{\mathcal{N}}$. Since $(a)_{\mathcal{N}}$ is an ideal of $(a)_{\mathcal{N}}$, by Theorem 2.3, $(a)_{\mathcal{N}}$ does not contain proper prime ideal. $(a)_{\mathcal{N}} \cap I$ is a prime ideal of $(a)_{\mathcal{N}}$. Indeed, for all $\gamma \in \Gamma$:

$$\emptyset \neq (a)_{\mathcal{N}} \cap I \subseteq (a)_{\mathcal{N}} \quad (a \in (a)_{\mathcal{N}}, a \in I)$$

$$\begin{aligned} (a)_{\mathcal{N}}\gamma((a)_{\mathcal{N}} \cap I) &\subseteq (a)_{\mathcal{N}}\gamma(a)_{\mathcal{N}} \cap (a)_{\mathcal{N}}\gamma I = (a\gamma a)_{\mathcal{N}} \cap (a)_{\mathcal{N}}\gamma I \\ &= (a)_{\mathcal{N}} \cap (a)_{\mathcal{N}}\gamma I \subseteq (a)_{\mathcal{N}} \cap M\gamma I \subseteq (a)_{\mathcal{N}} \cap I \\ ((a)_{\mathcal{N}} \cap I)\gamma(a)_{\mathcal{N}} &\subseteq (a)_{\mathcal{N}}\gamma(a)_{\mathcal{N}} \cap I\gamma(a)_{\mathcal{N}} \subseteq (a)_{\mathcal{N}} \cap I\gamma M \subseteq (a)_{\mathcal{N}} \cap I \end{aligned}$$

Let $x \in (a)_{\mathcal{N}} \cap I, (a)_{\mathcal{N}} \ni y \leq x$. Since $y \leq x \in I$, I an ideal of M, $y \in I$. Thus, $y \in (a)_{\mathcal{N}} \cap I$. So, $(a)_{\mathcal{N}} \cap I$ is a prime ideal of $(a)_{\mathcal{N}}$.

Since $(a)_{\mathcal{N}} \cap I$ is a prime ideal of $(a)_{\mathcal{N}}$, then we get $(a)_{\mathcal{N}} = (a)_{\mathcal{N}} \cap I$ and $t \in I$.

Let $b, c \in (a)_{\mathcal{N}}, b\alpha c \in (a)_{\mathcal{N}} \cap I, \alpha \in \Gamma$. Since $b\alpha c \in I$ and I is a prime ideal of M, we have $b \in I$ or $c \in I$. Hence, $b \in (a)_{\mathcal{N}} \cap I$ or $c \in (a)_{\mathcal{N}} \cap I$. $\square$

The following corollary give us the relationship between the prime ideals of the ordered Γ-semigroup M and the induced ordered Γ-semigroup $S = M/\mathcal{N}$ extending the analogue result for semigroup (II.2.15 Corollary [22]) and for ordered semigroup ([26, Proposition 2.1]).

Corollary 2.23. *Let $\mathcal{I}$ and $\mathcal{J}$ be the sets of all prime ideals of ordered Γ-semigroup M and $S = M/\mathcal{N}$, respectively. Then, there exists an order-preserving bijection between $\mathcal{I}$ and $\mathcal{J}$.*

Proof. Let M be an ordered Γ-semigroup and I a prime ideal of M. We consider the set $J = \{(a)_{\mathcal{N}} \in S | a \in I\}$. We show that J is a prime ideal of S. Indeed: Let $(a)_{\mathcal{N}} \in J$. Then, for any $(b)_{\mathcal{N}} \in S$, we have $a \in I$ and $b \in M$. Since I is an ideal of M, we have $a\alpha b \in I, \alpha \in \Gamma$. This implies $(a)_{\mathcal{N}}\gamma(b)_{\mathcal{N}} = (a\gamma b)_{\mathcal{N}} \in J, \forall \gamma \in \Gamma$. Similarly, we also have for all $\gamma \in \Gamma$, $(b)_{\mathcal{N}}\gamma(a)_{\mathcal{N}} \in J$. Let $(x)_{\mathcal{N}} \in J$ and $(y)_{\mathcal{N}} \in S$ with $(y)_{\mathcal{N}} \preceq (x)_{\mathcal{N}}$. Then it is clear that $x \in I$ and $(y)_{\mathcal{N}} = (y\alpha x)_{\mathcal{N}}, \alpha \in \Gamma$. Since I is an ideal of M, we have $y\alpha x \in I$. Consequently, we take $(y)_{\mathcal{N}} \in J$, and so J is an ideal of S. Moreover, if $(z)_{\mathcal{N}}, (t)_{\mathcal{N}} \in S$, and $(z)_{\mathcal{N}}\gamma(t)_{\mathcal{N}} \in J, \forall \gamma \in \Gamma$, then $(z\gamma t)_{\mathcal{N}} = (z)_{\mathcal{N}}\gamma(t)_{\mathcal{N}} \in J$, and so $z\gamma t \in I$. Since I is a prime ideal, we have $z \in I$ or $t \in I$. This implies $(z)_{\mathcal{N}} \in J$ or $(t)_{\mathcal{N}} \in J$. Hence, J is a prime ideal of S.

On the other hand, let J be a prime ideal of S. We consider the set $I = \{a \in M | (a)_{\mathcal{N}} \in J\}$. We show that I is a prime ideal of M. Indeed, if $a \in I$ and $b \in M$, then $(a)_{\mathcal{N}} \in J$ and $(b)_{\mathcal{N}} \in S$. Since J is an ideal of S, we have $(a\gamma b)_{\mathcal{N}} = (a)_{\mathcal{N}}\gamma(b)_{\mathcal{N}} \in J$ and $(b\gamma a)_{\mathcal{N}} = (b)_{\mathcal{N}}\gamma(a)_{\mathcal{N}} \in J, \forall \gamma \in \Gamma$. Consequently, by definition, we have $a\gamma b, b\gamma a \in I$. Moreover, if $x \in I$ and $y \in M$ with $y \leq x$, then we have $(x)_{\mathcal{N}} \in J$ and $(y, y\gamma x) \in \mathcal{N}$ since $\mathcal{N}$ is a complete semilattice congruence as we proved in Section 1. From $(y, y\gamma x) \in \mathcal{N}$ it follows that $(y)_{\mathcal{N}} = (y\gamma x)_{\mathcal{N}}, \forall \gamma \in \Gamma$. This implies $(y)_{\mathcal{N}} \preceq (x)_{\mathcal{N}}$. By $(x)_{\mathcal{N}} \in J$ and $(y)_{\mathcal{N}} \preceq (x)_{\mathcal{N}}$, since J is an ideal of S, we obtain $(y)_{\mathcal{N}} \in J$. It follows that $y \in I$ and so I is an ideal of M. Now, let $z, t \in M$ such that $z\gamma t \in I, \forall \gamma \in \Gamma$. By definition we have $(z)_{\mathcal{N}}\gamma(t)_{\mathcal{N}} = (z\gamma t)_{\mathcal{N}} \in J$. This implies $(z)_{\mathcal{N}} \in J$ or $(t)_{\mathcal{N}} \in J$ since J is a prime ideal. Hence, we have $z \in I$ or $t \in I$, that is, I is a prime ideal of M.

We define a map $\Phi : \mathcal{I} \to \mathcal{J}$ by $\Phi(I) = J = \{(a)_{\mathcal{N}} \in S | a \in I\}$, for any $I \in \mathcal{I}$. From the above, it is clear that Φ is a surjection. We prove that Φ is injective. Indeed:

$$J_1 = \varPhi(I_1) = \{(a)_{\mathcal{N}} \in S | a \in I_1\} \text{ and } J_2 = \varPhi(I_2) = \{(a)_{\mathcal{N}} \in S | a \in I_2\},$$

for any $I_1, I_2 \in \mathcal{I}$. If $J_1 = J_2$, then for any $a \in I_1$, we have $(a)_{\mathcal{N}} \in J_1 = J_2$. This shows that $a \in I_2$ and hence, $I_1 \subseteq I_2$. Similarly, we have $I_2 \subseteq I_1$. Thus, $I_1 = I_2$ and $\varPhi$ is injective.

We prove now that $\varPhi$ preserves the ordering. Indeed, if $I_1 \subseteq I_2$, then we have

$$\{(a)_{\mathcal{N}} \in S | a \in I_1\} \subseteq \{(a)_{\mathcal{N}} \in S | a \in I_2\},$$

that is, $\varPhi(I_1) \subseteq \varPhi(I_2)$. So, we proved that $\varPhi$ is an order-preserving bijection from $\mathcal{I}$ into $\mathcal{J}$. $\square$

The following theorem gives us some necessary and sufficient conditions for the induced ordered Γ-semigroup $S = M/\mathcal{N}$ by $\mathcal{N}$ to be a chain extending the analogue result for ordered semigroups ([26], Theorem 2.2).

Theorem 2.24. *Let M be an ordered Γ-semigroup and $S = M/\mathcal{N}$ the induced ordered Γ-semigroup by $\mathcal{N}$. Then, the following conditions are equivalent:*

1. *S is a chain;*
2. *the set of prime ideals of M is a chain under inclusion;*
3. *every non-empty intersection of prime ideals of M is a prime ideal of M;*
4. *$\forall a, b \in M, \{a, b\} \cap N(a) \cap N(b) \neq \emptyset$.*

Proof. (1) ⇒ (2). Let I, J be prime ideals of M. If there exist $a, b \in M$ such that $a \in I \backslash J$ and $b \in J \backslash I$, then $a\Gamma b \subseteq I \cap J$ because I and J are ideals of M. By (1) we assume that $(a)_{\mathcal{N}} \preceq (b)_{\mathcal{N}}$, that is, $(a)_{\mathcal{N}} = (a\gamma b)_{\mathcal{N}}, \forall \gamma \in \Gamma$. By $a\Gamma b \subseteq J$ and Corollary 2.4, we have $(a\gamma b)_{\mathcal{N}} \subseteq J$. Consequently, we have $a \in (a)_{\mathcal{N}} = (a\gamma b)_{\mathcal{N}} \subseteq J, \forall \gamma \in \Gamma$, which is impossible because $a \in I \backslash J$. Therefore, $I \subseteq J$ or $J \subseteq I$.

(2) ⇒ (3). Let we suppose that $I = \bigcap_{\alpha \in A} I_\alpha \neq \emptyset$, where I_α are prime ideals of M and A is an index set. Let $a \in I, b \in M$. Then $a \in I_\alpha, \forall \alpha \in A$, and $a\Gamma b, b\Gamma a \subseteq I$. Let $c \in M$ such that $c \leq d$. Since $d \in I_\alpha$ and I_α is an ideal of M for all $\alpha \in A$, we have $c \in I_\alpha, \forall \alpha \in A$ and hence, $c \in I$. Therefore, I is an ideal of M.

Let us show now that I is prime. First, we show that each $M \backslash I_\alpha$ is either a sub-Γ-semigroup of M or is empty. Suppose that $M \backslash I_\alpha \neq \emptyset$. Then, let $a, b \in M \backslash I_\alpha$. If $a\Gamma b \nsubseteq M \backslash I_\alpha$, then $a\Gamma b \subseteq I_\alpha$. Since I_α is prime, we have $a \in I_\alpha$ or $b \in I_\alpha$, which is impossible. Thus, $a\Gamma b \subseteq M \backslash I_\alpha$, that is, $M \backslash I_\alpha$ is a sub-Γ-semigroup of M.

Now, we show that $M\backslash I$ is a sub-Γ-semigroup of M or is empty. By (2) it follows that $\{I_\alpha\}_{\alpha\in A}$ is a chain under inclusion. Suppose that $M\backslash I \neq \emptyset$. Let $x, y \in M\backslash I$. Since

$$M\backslash I = M\backslash \bigcap_{\alpha\in A} I_\alpha = \bigcup_{\alpha\in A}(M\backslash I_\alpha),$$

we have $x \in M\backslash I_\beta$ for some $\beta \in A$ and $y \in M\backslash I_\gamma$ for some $\gamma \in A$. Since $\{I_\alpha\}_{\alpha\in A}$ is a chain, then we may suppose that $I_\beta \subseteq I_\gamma$. Then $M\backslash I_\gamma \subseteq M\backslash I_\beta$, and so, $y \in M\backslash I_\beta$. Since $M\backslash I_\beta$ is a sub-Γ-semigroup of M, we have $x\Gamma y \subseteq M\backslash I_\beta$. Hence, $x\Gamma y \subseteq \bigcup_{\alpha\in A}(M\backslash I_\alpha) = M\backslash I$, that is, $M\backslash I$ is a sub-Γ-semigroup of M.

Let us show now that I is prime. If $M\backslash I = \emptyset$, then $I = M$, and so I is prime. Let us assume that $M\backslash I \neq \emptyset$ and let $s, t \in M$ such that $s\gamma t \in I, \forall\gamma \in \Gamma$. If $s \notin I$ and $t \notin I$, then $s, t \in M\backslash I$. Thus, since $M\backslash I$ is a sub-Γ-semigroup of M, we have $s\Gamma t \in M\backslash I$ which is impossible. Therefore, $s \in I$ or $t \in I$, and hence, I is a prime ideal.

(3) $\Rightarrow$ (4). If there exist $a, b \in M$ such that $\{a, b\} \cap N(a) \cap N(b) = \emptyset$, then $a \notin N(b)$ and $b \notin N(a)$. Since $N(a)$ and $N(b)$ are filters of M, then we have $a\Gamma b \nsubseteq N(a)$ and $a\Gamma b \nsubseteq N(b)$. Hence, we have

$$a\Gamma b \subseteq (M\backslash N(a)) \cap (M\backslash N(b)) \text{ and } a, b \notin (M\backslash N(a)) \cap (M\backslash N(b)).$$

By Lemma [16], we have that $M\backslash N(a)$ and $M\backslash N(b)$ are both prime ideals. So, we have that the non-empty intersection of prime ideals $M\backslash N(a)$ and $M\backslash N(b)$ is not prime, which contradicts the condition (3).

(4) $\Rightarrow$ (1). Let us suppose that the ordered Γ-semigroup $S = M/\mathcal{N}$ is not a chain. Then for some $a, b \in M$, we have $(b)_\mathcal{N} \npreceq (a)_\mathcal{N}$ and $(b)_\mathcal{N} \nsucceq (a)_\mathcal{N}$, that is, $(a\gamma b)_\mathcal{N} \neq (a)_\mathcal{N}$ and $(a\gamma b)_\mathcal{N} \neq (b)_\mathcal{N}, \forall\gamma \in \Gamma$. However, these imply that $(a\gamma b, a) \nsubseteq \mathcal{N}$ and $(a\gamma b, b) \nsubseteq \mathcal{N}$. So, by the definition of $\mathcal{N}$, we have $N(a\gamma b) \neq N(a)$ and $N(a\gamma b) \neq N(b)$. Consequently, $b \notin N(a)$, for if otherwise, $b \in N(a)$, then we have $a\gamma b \in N(a), \forall\gamma \in \Gamma$. It implies $N(a\gamma b) = N(a)$ which is a contradiction. Similarly, we have $a \notin N(b)$. This shows that $\{a, b\} \cap N(a) \cap N(b) = \emptyset$ which contradicts the condition (4). Thus, S is a chain. $\square$

3. ON PRIME RADICALS FOR ORDERED Γ-SEMIGROUPS

In 1994, Dutta and Adhikari introduced the concept of m-system in Γ-semigroups [1]. We introduce here the concepts of m-systems and n-systems in ordered Γ-semigroups as an extending form of the analogue concepts in ordered semigroups defined by Wu and Xie [27] and Kehayopulu [15]. In this section, we characterize the weakly prime and semiprime ideals of ordered Γ-semigroups by

m-system and n-system. We also introduce the notion of radical of an ideal on $po-\Gamma$-semigroups and we give the prime radical theorems of $po-\Gamma$-semigroups analogue to that of ordered semigroups [28].

Definition 3.25. Let T be a nonempty subset of an ordered Γ-semigroup M. T is called an $m-system$ if

$$\forall a, b \in T, \exists x \in M, (a\Gamma x\Gamma b] \cap T \neq \emptyset.$$

T is called $n-system$ of M if

$$\forall a \in T, \exists x \in M, (a\Gamma x\Gamma a] \cap T \neq \emptyset.$$

LEMMA 3.26 ([15]). *Let M be an ordered Γ-semigroup and I an ideal of M. Then the following statements are true:*

1. *I is weakly prime if and only if either $M\backslash I = \emptyset$ or the set $M\backslash I$ is an m-system.*
2. *I is weakly semiprime if and only if either $M\backslash I = \emptyset$ or the set $M\backslash I$ is an n-system.*

Proof. 1). Let $M\backslash I \neq \emptyset$. Let us suppose that, there exist $a, b \in M\backslash I$ such that $(a\Gamma M\Gamma b] \cap M\backslash I = \emptyset$. Then $a, b \notin I$ and so $J(a), J(b) \nsubseteq I$. By Lemma 1.4 ((5), (7)) [3] (cf.[8]), we have $(a\Gamma M\Gamma b] \subseteq (J(a)\Gamma J(b)] \subseteq J(a)\Gamma J(b) \subseteq I$. It is impossible, since I is weakly prime. Therefore, $M\backslash I$ is a m-system.

Conversely, let $M\backslash I$ is a m-system. Let $U, V \nsubseteq I$ where U, V are two ideals of M. Then there exist $c \in U, d \in V$ such that $c \notin I, d \notin I$. Since $c, d \in M\backslash I$ and $M\backslash I$ is an m-system, there exists $x \in M$ such that $(c\Gamma x\Gamma d] \cap M\backslash I \neq \emptyset$. Since $c \in U$, then $c \in J(U)$ so, $c \in M^1\Gamma U\Gamma M^1$; similarly, $d \in M^1\Gamma V\Gamma M^1$. If $U\Gamma V \subseteq I$, then we have $(c\Gamma x\Gamma d] \subseteq (M^1\Gamma U\Gamma(M^1\Gamma x\Gamma M^1)\Gamma V\Gamma M^1] \subseteq I$ since I is an ideal and hence, $(c\Gamma x\Gamma d] \subseteq I \cap M\backslash I$. This is impossible. By contraposition we conclude that I is weakly prime.

2). The proof is similar to 1). $\square$

We prove now the following results on ordered Γ-semigroups which extend the analogue results on ordered semigroups [28].

LEMMA 3.27. *Let M be an ordered Γ-semigroup, I an ideal of M and T a m-system such that $I \cap T = \emptyset$. Then there exists an m-system T^* maximal relative to the properties: $I \cap T^* = \emptyset, T \subseteq T^*$.*

Proof. Let $\mathcal{T}$ be the partially ordered set of m-system N such that $T \subseteq N$ and $I \cap N = \emptyset$. Then $\mathcal{T} \neq \emptyset$ since $T \in \mathcal{T}$. Let $\mathcal{C}$ be a chain in $\mathcal{T}$ and let $D = \bigcup_{C\in\mathcal{C}} C$. If $a, b \in D$, then there exists $C \in \mathcal{C}$ such that $a, b \in C$, so $(a\Gamma x\Gamma b] \cap C \neq \emptyset$ for some $x \in M$. Hence, $(a\Gamma x\Gamma b] \cap D \neq \emptyset$, that is, D is a m-system satisfying that $T \subseteq D$ and $I \cap D = \emptyset$. Hence, $D \in \mathcal{T}$ and by Zorn's Lemma, $\mathcal{T}$ contains a maximal element T^*. $\square$

Lemma 3.28. *Let M be an ordered Γ-semigroup, I an ideal of M, T a m-system such that $I \cap T = \emptyset$ and let T^* be any m-system of M maximal relative to the properties: $I \cap T^* = \emptyset, T \subseteq T^*$. Then $M \backslash T^*$ is a minimal weakly prime ideal of M containing I.*

Proof. Let $\mathcal{P}$ be the partially ordered set of all ideals P of M such that $I \subseteq P$ and $T^* \cap P = \emptyset$. Similarly, as mentioned above (see Lemma 3.3), by Zorn's Lemma, $\mathcal{P}$ contains a maximal element Q.

Let $U, V \nsubseteq Q$ where U, V are two ideals of M. Let $A = J(U) \cup Q, B = J(V) \cup Q$. We see that A and B are ideals with property $Q \subset A, Q \subset B$. Hence, $I \subseteq A, I \subseteq B$, and by maximality of Q, we must have $A, B \notin \mathcal{P}$. Thus, $T^* \cap A \neq \emptyset, T^* \cap B \neq \emptyset$. Let $c \in T^* \cap A, d \in T^* \cap B$. Consequently, since T^* is an m-system, there exists $x \in M$ such that $(c\Gamma x\Gamma d] \cap T^* \neq \emptyset$. Since $c \in T^* \cap (J(U) \cup Q)$ and $T^* \cap Q = \emptyset$, we must have $c \in J(U)$ so that $c \in M^1\Gamma U\Gamma M^1$; similarly $d \in M^1\Gamma V\Gamma M^1$. If $U\Gamma V \subseteq Q$, then $(c\Gamma x\Gamma d] \subseteq (M^1\Gamma U\Gamma(M^1\Gamma x\Gamma M^1)\Gamma V\Gamma M^1] \subseteq Q$, since Q is an ideal, and hence, $(c\Gamma x\Gamma d] \subseteq T^* \cap Q$, contradicting the fact that $T^* \cap Q = \emptyset$. By contraposition we conclude that Q is a weakly prime ideal of M and thus, $Q' = M\backslash Q$ is an m-system if it is nonempty. Since $Q \supseteq I$ and $Q \cap T^* = \emptyset$, it follows that $Q' \cap I = \emptyset$ and $T^* \subseteq Q'$. By maximality of T^*, we must have $Q' = T^*$ so that $M\backslash T^* = Q$.

Let J be a weakly prime ideal of M such that $Q \supset J \supset I$. Then $M\backslash J \supset M\backslash Q = T^*$ contradicting maximality of T^* relative to the properties of $T \subseteq T^*, T^* \cap I = \emptyset$, since $M\backslash J$ is a m-system such that $(M\backslash J) \supset M$ and $(M\backslash J) \cap I = \emptyset$. Therefore, $Q = M\backslash T^*$ is a minimal weakly prime ideal of M containing I. $\square$

The following lemma characterizes the semiprime ideals.

Lemma 3.29. *Let M be an ordered Γ-semigroup and I be a semiprime ideal of M. Then*

1. *If $a\Gamma b \subseteq I$, then $a\Gamma x\Gamma b \subseteq I, \forall x \in M$.*
2. *If $a\Gamma b^n \subseteq I$, then $a\Gamma b \subseteq I, n \in N$.*
3. *If $a_1\Gamma a_2\Gamma ...\Gamma a_n \subseteq I$, then $a_{1\pi}\Gamma a_{2\pi}\Gamma ...\Gamma a_{n\pi} \subseteq I$ for any permutation π of $\{1, 2, ..., n\}$.*

Proof. 1) If $a\Gamma b \subseteq I$, then $(b\Gamma a)^2 = b\Gamma(a\Gamma b)\Gamma a \subseteq I$ and thus, $b\Gamma a \subseteq I$. Let $x \in M$. Then $(a\Gamma x\Gamma b)^2 = a\Gamma x\Gamma(b\Gamma a)\Gamma x\Gamma b \subseteq I$ and thus, $a\Gamma x\Gamma b \subseteq I$.

2) Let us prove the statement for $n = 2$, that is, let $a\Gamma b^2 \subseteq I$. Then $(a\Gamma b)\Gamma b \subseteq I$. Applying 1) for $x = a$, we have $(a\Gamma b)\Gamma(a\Gamma b) \subseteq I$, that is, $(a\Gamma b)^2 \subseteq I$. Then we have $a\Gamma b \subseteq I$ since I is semiprime. By induction we can prove that if $a\Gamma b^n \subseteq I$, then $a\Gamma b \subseteq I, n \in N$.

3) If $a\Gamma b \subseteq I$, then $(b\Gamma a)^2 = b\Gamma(a\Gamma b)\Gamma a \subseteq I$ and thus, $b\Gamma a \subseteq I$.

Let us assume that $a\Gamma b\Gamma c \subseteq I$. Then we see successively that each of the following sets is a subset of I: $b\Gamma c\Gamma a, (b\Gamma c)\Gamma(a\Gamma c), (a\Gamma c)\Gamma(b\Gamma c)$, $(b\Gamma a\Gamma c\Gamma b)\Gamma c$, $(c\Gamma b)\Gamma(a\Gamma c\Gamma b)$, $(a\Gamma c\Gamma b)^2, a\Gamma c\Gamma b, b\Gamma a\Gamma c, c\Gamma b\Gamma a$. Furthermore, the assumption implies $c\Gamma a\Gamma b \subseteq I$. Hence, the assertion of the lemma is valid for $n = 2, 3$.

In order to complete the proof, we only need to show that $a\Gamma x\Gamma y\Gamma b \subseteq I$ implies $a\Gamma y\Gamma x\Gamma b \subseteq I$. Let us assume that $a\Gamma x\Gamma y\Gamma b \subseteq I$. Then we see successively that each of the following sets is a subset of I: $(x\Gamma y\Gamma b)\Gamma a, b\Gamma(x\Gamma y\Gamma b\Gamma a)$, $(b\Gamma a)\Gamma(y\Gamma x\Gamma y)$, $(b\Gamma a)\Gamma(y\Gamma x)^2$, $(y\Gamma x)\Gamma(b\Gamma a)\Gamma(y\Gamma x)$, $(b\Gamma a\Gamma y\Gamma x)^2, b\Gamma(a\Gamma y\Gamma x)$, $a\Gamma y\Gamma x\Gamma b$. □

LEMMA 3.30. *Let M be an ordered Γ-semigroup, S be a m-system and I a semiprime ideal of M, $I \cap S = \emptyset$. Then there exists a maximal semiprime ideal P with respect to the properties of $I \subseteq P, P \cap S = \emptyset$. Furthermore, the following statements are true:*

1. *If $a \notin P$, we denote $P'_a = \{b \in M | a\gamma b \in P, \forall \gamma \in \Gamma\}$, then $P'_a = P$.*
2. *P is a prime ideal of M.*

Proof. The existence of such semiprime ideal P of M is easily seen by Zorn's Lemma.

(1) It is clear that $P \subseteq P'_a$. We have also $P'_a \subseteq P$. Indeed:

(A) If $b \in P'_a, y \in M, y \leq b$, then $a\gamma y \leq a\gamma b \in P, \forall \gamma \in \Gamma$. Thus, $a\gamma y \subseteq P, \forall \gamma \in \Gamma$, that is, $y \in P'_a$.

(B) Let $x \in P'_a, y \in M$. Then $a\Gamma x \subseteq P$. It implies $a\Gamma x\Gamma y \subseteq P$, that is, $x\Gamma y \subseteq P'_a$. By Lemma 3.5(3), we have $a\Gamma y\Gamma x \subseteq P$, that is, $y\Gamma x \subseteq P'_a$.

(C) Let $x\gamma x \in P'_a, \forall \gamma \in \Gamma$, for any $x \in M$. Then $a\alpha x\gamma x \in P, \forall \alpha, \gamma \in \Gamma$, by Lemma 3.5(2), we have $a\alpha x \in P, \forall \alpha \in \Gamma$, that is, $x \in P'_a$. Therefore, we have that P'_a is a semiprime ideal of M. On the other hand,

(D) If $a \in S$, then $P'_a \cap S = \emptyset$. Indeed: if $b \in S \cap P'_a$, then $a\Gamma b \subseteq P$ and there exists $x \in M$ such that $(a\Gamma x\Gamma b] \cap S \neq \emptyset$. Furthermore, by Lemma 3.5(1), we have $a\Gamma x\Gamma b \subseteq P$, impossible. By hypothesis, $P'_a = P$.

(E) If $a \notin S$, then also $P'_a \cap S = \emptyset$. Indeed: if $c \in P'_a \cap S$, then $a\Gamma c \subseteq P$, thus, $c\Gamma a \subseteq P$ by Lemma 3.5(3). Therefore $a \in P'_c$. Since $P \cap S = \emptyset$, we have $c \notin P$. By (D), $P'_c = P$, thus, $a \in P$, which is impossible. By hypothesis, $P'_a = P$.

(2) By (1), if $a\Gamma b \subseteq P, a \notin P$, then $b \in P'_a = P$, that is, P is prime. □

It is easy to see that the nonempty intersection of weakly prime ideals of an ordered Γ-semigroup M is a weakly semiprime ideal. The following theorem gives the converse [28].

THEOREM 3.31. *Every weakly semiprime ideal I of an ordered Γ-semigroup M is the intersection of weakly prime ideals containing I.*

Proof. Let $\{I_\alpha\}_{\alpha\in A}$ are all weakly prime ideals of M containing I. Then $I \subseteq \bigcap_{\alpha\in A} I_\alpha$. If $d_0 \in M\backslash I$, we denote $D = [d_0)$, then there exists $x_0 \in M$ such that $(d_0\Gamma x_0\Gamma d_0] \cap (M\backslash I) \neq \emptyset$ since I is weakly semiprime (Lemma 3.2). Thus, there exists $d \in (d_0\Gamma x_0\Gamma d_0]$ such that $d \in M\backslash I$, that is, $d \notin I$. Therefore $d_0\Gamma x_0\Gamma d_0 \nsubseteq I$. It follows that $(d_0\Gamma x_0\Gamma d_0] \cap I \neq \emptyset$. Let we denote $D_1 = [d_0\Gamma x_0\Gamma d_0)$, $d_1 = d_0\beta_0 x_0\gamma_0 d_0 \notin I$, $\beta_0, \gamma_0 \in \Gamma$. Then, by induction we denote $S = \bigcup\limits_{n=0}^{\infty} D_n$, where $D_n = [d_n), d_n = d_{n-1}\beta_{n-1}x_{n-1}\gamma_{n-1}d_{n-1}, d_i \notin I(i = 1, 2, 3, ...), \beta_{n-1}, \gamma_{n-1} \in \Gamma$. It is easily seen that $S \cap I = \emptyset$. We have that S is a m-system of M. Indeed: Let $a, b \in S$. Then there exist D_i, D_j such that $a \in D_i, b \in D_j$, thus,

$$a \geq d_i = d_{i-1}\beta_{i-1}x_{i-1}\gamma_{i-1}d_{i-1}, b \geq d_j = d_{j-1}\beta_{j-1}x_{j-1}\gamma_{j-1}d_{j-1}$$

where $\beta_{i-1}, \beta_{j-1}, \gamma_{i-1}, \gamma_{j-1} \in \Gamma$.

1) If $i = j$, then $a\beta_i x_i\gamma_i b \geq d_i\beta_i x_i\gamma_i d_i = d_{i+1}$. Thus, $[a\Gamma x_i\Gamma b) \subseteq D_{j+1} \subseteq S$.

2) If $j > i$, then

$$\begin{aligned} d_j &= d_{j-1}\beta_{j-1}x_{j-1}\gamma_{j-1}d_{j-1} \\ &= d_{j-2}\beta_{j-2}(x_{j-2}\gamma_{j-2}d_{j-2}\beta_{j-1}x_{j-1}\gamma_{j-1}d_{j-2}\beta_{j-2}x_{j-2})\gamma_{j-2}d_{j-2} \\ &= ... = d_i\beta_i k\gamma_i d_i \end{aligned}$$

for some $k \in M, \beta_i, \gamma_i \in \Gamma$. Thus,

$$a\beta_i(k\gamma_i d_i\beta_j x_j)\gamma_j b \geq d_i\beta_i k\gamma_i d_i\beta_j x_j\gamma_j d_j = d_j\beta_j x_j\gamma_j d_j = d_{j+1}.$$

Whence,

$$[a\beta_i(k\gamma_i d_i\beta_j x_j)\gamma_j b) \subseteq D_{j+1} \subseteq S.$$

3) If $j < i$, similar to 2), we can also prove that there exists $h \in M$ such that $[a\Gamma h\Gamma b) \subseteq D_{i+1} \subseteq S$.

By 1), 2) and 3), we conclude that for any $a, b \in S$, there exists $x \in M$ such that $(a\Gamma x\Gamma b] \cap S \neq \emptyset$. By Defenition 3.1, we have S is a m-system of M. By Lemmas 3.3, 3.4, there exists a maximal m-system S^* relative to the properties: $I \cap S^* = \emptyset, S \subseteq S^*$, and $M\backslash S^*$ is a minimal weakly prime ideal of M containing I. Since $d_0 \in S^*$, we have $d_0 \notin M\backslash S^*$. Thus, $d_0 \notin \bigcap_{\alpha\in A} I_\alpha$. Consequently, $\bigcap_{\alpha\in A} I_\alpha \subseteq I$. □

THEOREM 3.32. *The following conditions on an ideal I of an ordered Γ-semigroup M are equivalent:*

1. *I is the intersection of prime ideals containing it.*
2. *I is the intersection of minimal prime ideals containing it.*
3. *I is the union of $\mathcal{N}$-classes.*
4. *I is semiprime.*

Proof. 1) $\Rightarrow$ 2). Let $I = \bigcap_{\alpha \in A} I_\alpha$ where each I_α is a prime ideal. Let fix $\alpha \in A$ and let $\Im$ be the partially ordered set of all prime ideals J of M for which $I \subseteq J \subseteq I_\alpha$. Then $\Im \neq \emptyset$ since $I_\alpha \in \Im$. Let $\mathcal{C}$ be a chain in $\Im$ and let $A = \bigcap_{C \in \mathcal{C}} C$. Then $I \subseteq A \subseteq I_\alpha$ and A is an ideal of M. Since the partially ordered set $\{M \backslash C | C \in \mathcal{C}\}$ forms a chain, it follows that $M \backslash A = \bigcup_{C \in \mathcal{C}} (M \backslash C)$ is an ordered sub-Γ-semigroup of M or is empty, which shows that A is a prime ideal. Hence, $A \in \Im$ and the Minimal Principle assures the existence of a minimal element, say J_α in $\Im$. But then, J_α is also minimal relative to the property of being a prime ideal containing I. In addition

$$I \subseteq \bigcap_{\alpha \in A} J_\alpha \subseteq \bigcap_{\alpha \in A} I_\alpha = I$$

so $I = \bigcap\limits_{\alpha \in A} J_\alpha$.

2) $\Rightarrow$ 3). This follows by Corollary 2.4.

3) $\Rightarrow$ 4). Let $I = \bigcup \{(x)_\mathcal{N} | x \in I\}$ and $a\Gamma a \subseteq I$. Then $a \in (a)_\mathcal{N} = (a\gamma a)_\mathcal{N} \subseteq I, \forall \gamma \in \Gamma$ so that $a \in I$.

4) $\Rightarrow$ 1). Let $\{I_\alpha\}_{\alpha \in A}$ is the set of all prime ideals containing I. Then $I \subseteq \bigcap_{\alpha \in A} I_\alpha$. If $a \notin I$, by the proof of Theorem 3.5, we can construct a m-system S such that $a \in S, S \cap I = \emptyset$. By Lemma 3.6, there exists a prime ideal P such that $a \notin P$ and $I \subseteq P$. Therefore, $a \notin \bigcap_{\alpha \in A} I_\alpha$. Consequently, $\bigcap_{\alpha \in A} I_\alpha \subseteq I$. Thus, $I = \bigcap_{\alpha \in A} I_\alpha$. □

Definition 3.33. Let M be an ordered Γ-semigroup and I an ideal of M. Then the subset of M

$$\{x \in M | (x\gamma)^{n-1} x \in I, \text{ for all } \gamma \in \Gamma \text{ and for some } n \in N\}$$

is called the *radical* of I and is denoted by $\sqrt{I}$.

The following theorem is an immediate consequence of Definition 3.8.

Theorem 3.34. *Let M be an commutative ordered Γ-semigroup, I an ideal of M. Then $\sqrt{I}$ is an ideal in M and $I \subset \sqrt{I}$.*

It is clear that $(\sqrt{I}] = \sqrt{I}$.

Lemma 3.35. *Let M be an commutative ordered Γ-semigroup. Let I be a semiprime ideal of M. Then $I = \sqrt{I}$.*

Proof. Theorem 3.10 implies $I \subset \sqrt{I}$. Let $x \in \sqrt{I}$, that is, there exists an $n \in N$ such that $(x\gamma)^{n-1} x \in I, \forall \gamma \in \Gamma$. Theorem 3.8 implies that I is the intersection of prime ideals containing it. Let $\{P_i\}_{i \in A}$ be the collection of all prime ideals in M that contain I. We have $I = \cap \{P_i\}_{i \in A}$. Since $(x\gamma)^{n-1} \in I$, then for every $i \in A$, $(x\gamma)^{n-1} x \in P_i$. Since P_i, $i \in A$ are prime ideals, it follows that $x \in P_i$, for all $i \in A$. Thus, $x \in \cap \{P_i\}_{i \in A} = I$. □

Corollary 3.36. *Let M be an commutative ordered Γ-semigroup and I be proper ideal in M. Then $\sqrt{I}$ is semiprime.*

Proof. Theorem 3.10 implies that the radical of I is a proper ideal in M and $\sqrt{I} \subset \sqrt{\sqrt{I}}$. Let $x \in \sqrt{\sqrt{I}}$. Definition 3.9 implies that there exists an $n \in N$ such that $(x\gamma)^{n-1}x \in \sqrt{I}$. Moreover, Definition 3.9 now implies that there exists an $m \in N$ such that $((x\gamma)^{n-1}x)\gamma)^{m-1}x \in I$. Thus, $(x\gamma)^{nm-1}x \in I$ and Definition 3.0 implies $x \in \sqrt{I}$. □

For any ideal I of an ordered Γ-semigroup M, we write

$$I^* = \text{The intersection of all prime ideals containing } I.$$

Theorem 3.37. *Let M be an commutative ordered Γ-semigroup and I be proper ideal in M. Then $\sqrt{I} = I^*$.*

Proof. Let $x \in M$ such that $x \in \sqrt{I}$, that is, for some positive integer n it is valid that $x^n \in I$. Let P be a prime ideal in M that contains I. Since P is prime and $x^n \in P$, it follows that $x \in P$. Since P was an arbitrary prime ideal in M containing I, it follows that $x \in I^*$. Conversely, let us assume there exists $x \in I^*$ such that for some $\gamma \in \Gamma, (x\gamma)^{n-1}x \notin I$ for all positive integer n. Choose one such x and let $S = \{x, x\gamma x, (x\gamma)^2 x, ..., (x\gamma)^n x, ...\}$. Since the m-system S is a non-empty disjoint from I, that is, $S \cap I = \emptyset$, by Lemma 3.3 and Lemma 3.4, there exists a prime ideal $Q \supseteq I$ such that $Q \cap S = \emptyset$. This implies $x \notin Q$. Thus, $x \notin I^*$, a contradiction. □

Another independent proof of the first part of the above proof is given as follows:

Corollary 3.12 implies that $\sqrt{I}$ is semiprime. By Theorem 3.8, $\sqrt{I}$ is the intersection of all prime ideals containing it. Furthermore, if a prime ideal P contains I, then for any $x \in \sqrt{I}, (x\gamma)^{n-1}x \in I \subseteq P$. Since P is prime, we have $x \in P$, that is, P contains $\sqrt{I}$. Therefore, $\sqrt{I}$ is the intersection of all prime ideals containing I.

Corollary 3.38. *Let M be an commutative ordered Γ-semigroup and A, B be proper ideal in M. Then*

1. *If $A \subset B$, then $\sqrt{A} \subset \sqrt{B}$;*
2. *If $A \cap B \neq \emptyset$, then $\sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.*

Proof. (1). If $x \in \sqrt{A}$, then there exists an $n \in N$ such that $(x\gamma)^{n-1}x \in A \subset B$. Hence, $x \in \sqrt{B}$.

(2). Since $A \cap B \neq \emptyset$, it is clear that $A \cap B$ is an ideal in M. Let $x \in \sqrt{A \cap B}$. Then there exists an $n \in N$ such that $(x\gamma)^{n-1}x \in A \cap B$. Therefore, $(x\gamma)^{n-1}x \in A$ and $(x\gamma)^{n-1}x \in B$ and it follows that $x \in \sqrt{A}$ and

$x \in \sqrt{B}$. Hence, $x \in \sqrt{A} \cap \sqrt{B}$. Consequently, $x \in \sqrt{A} \cap \sqrt{B}$ implies that there exist $n, m \in N$ such that $(x\gamma)^{n-1}x \in A$ and $(x\gamma)^{m-1}x \in B$. Clearly, $(x\gamma)^{nm-1}x \in A \cap B$. Thus, $x \in \sqrt{A \cap B}$. $\square$

COROLLARY 3.39. *If I is a prime ideal in a commutative ordered Γ-semigroup M, then I is semiprime.*

Proof. Theorem 3.10 implies $I \subset \sqrt{I}$. Let $\{P_i\}_{i\in A}$ be the collection of all prime ideals in M that contain I. Clearly, $I \in \{P_i\}_{i\in A}$ and $\sqrt{I} = \cap_{i\in A} P_i \subset I$. Thus, $I = \sqrt{I}$. Lemma 3.11 implies I is semiprime. $\square$

When the ordered Γ-semigroup M is not commmutative, the radical $\sqrt{(I)}$ of the ideal I, in general, is not an ideal of M and therefore, it is not intersection of all prime ideals containing I. The following example shows this.

Example 3.40. Let $M = \{a, b, c, d, e\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$. Then it can be easily verified that M is a non-commutative Γ-semigroup with the Γ-multiplication defined by

α	a	b	c	d	e
a	b	b	a	b	d
b	b	b	b	b	b
c	b	b	c	b	e
d	b	b	b	b	b
e	b	b	b	b	b

β	a	b	c	d	e
a	b	b	b	b	b
b	b	b	b	b	b
c	b	b	b	b	b
d	a	b	b	d	b
e	c	b	b	e	b

γ	a	b	c	d	e
a	a	b	b	d	b
b	b	b	b	b	b
c	c	b	b	e	b
d	b	b	b	b	b
e	b	b	b	b	b

It is easy verified that $I = \{b\}$ is an ideal of M. Obviously, the radical $\sqrt{I} = \{b, e\}$ is not an ideal of M.

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