AN ITERATIVE METHOD TO SOLVE SYMMETRIC POSITIVE DEFINITE MATRIX EQUATIONS

DAVOD KHOJASTEH SALKUYEH and FATEMEH PANJEH ALI BEIK

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Let $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ be a symmetric positive definite linear operator. In this paper, we propose an iterative algorithm to solve the general matrix equation $\mathcal{A}(X) = C$ which includes the Lyapunov matrix equation and Sylvester matrix equation as special cases. It is proved that the sequence of the approximate solutions, obtained by the presented algorithm, satisfies an optimality condition. More precisely, the application of the proposed method is investigated for solving the Sylvester matrix equation $\mathcal{S}(X) = C$, where $\mathcal{S}$ is the well-known Sylvester operator $\mathcal{S}(X) = AX + XB$. In order to illustrate the effectiveness of the algorithm, two numerical examples are given.

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1. INTRODUCTION

Linear matrix equations play a fundamental role in many areas, such as control theory, system theory, stability theory and some other fields of pure and applied mathematics (see [5, 6, 18, 19]). For instance, the solution of the Sylvester matrix equation $AX + XB = C$ can be utilized to parameterize the feedback gains in pole assignment problem for linear system (for more details see [1]). The Lyapunov matrix equation $AX + XA^T = -C$ has important applications in stability analysis of linear systems [20].

In the literature, the problem of finding a solution to several linear matrix equations has been investigated widely, for more details see [2, 4, 7, 8, 10, 12, 17] and the references therein. For example, Xie *et al.* [14] have developed a gradient based and a least squares based iterative algorithms for solving matrix equation $AXB + CX^TD = F$. In [4], an iterative projection method onto matrix Krylov subspace has been presented for solving the linear matrix equation

$$\sum_{i=1}^{q} A_i X B_i = C, \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{p\times p}, i = 1, 2, \ldots, p$, and $C, X \in \mathbb{R}^{n\times p}$. Li and Wang [13] have proposed a gradient based iterative method for finding the minimal norm least squares solution to the linear matrix equation (1).

More recently, Li *et al.* [12] have presented a new iterative method to solve matrix equation $AX = B$, where A is a symmetric positive definite (SPD) matrix. As shown, the sequence of approximate solutions produced by the proposed method satisfies an optimality property. This fact inspires us to extend the method to a new iterative method for solving more general linear matrix equations. To do so, we consider the linear operator $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$, and focus on the linear matrix equations of the following form:

$$\mathcal{A}(X) = C, \tag{2}$$

where $C \in \mathbb{R}^{p\times q}$ is given and $X \in \mathbb{R}^{m\times n}$ is an unknown matrix. Evidently, many investigated linear matrix equations in the literature are included in Eq. (2).

The rest of this paper is organized as follows. In Section 2, we define a new inner product and recall some principles and definitions which are utilized throughout this work. In Section 3, an iterative algorithm is presented for solving Eq. (2). Moreover, it is proved that the sequence of approximate solutions computed by the algorithm satisfy an optimality property. In Section 4, we demonstrate how the algorithm can be used for solving the Sylvester matrix equation $AX + XB = C$. Furthermore, it is shown that the algorithm is convergent for solving Sylvester matrix equation. Section 5 is devoted for reviewing two well-known iterative algorithms which will be numerically compared with our method in the section of numerical experiments. In Section 6, some numerical examples are presented to investigate the validity of the theoretical results, established in this work, and show that the proposed method is effective and feasible for computing the solution of the Sylvester matrix equation $AX + XB = C$. Finally, the paper is ended with a brief conclusion in Section 7.

2. PRELIMINARIES

In this section, we review some concepts which are used in the next sections. Furthermore, we introduce a new inner product which is utilized for presenting an iterative method for solving (2).

The notation A^T is used to denote the transpose of a matrix A. For two matrices X and Y in $\mathbb{R}^{m\times n}$, we define the inner product $\langle X, Y\rangle = trace(Y^T X)$ where $trace(Y^T X)$ stands for the trace of the matrix $Y^T X$. The induced matrix norm is the well-known Frobenius norm, *i.e.*, $\|X\|_F^2 = \langle X, X\rangle$. For the matrix

$Z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^{m\times n}$, the vector $vec(Z) \in \mathbb{R}^{mn}$ is obtained by stacking the columns of the matrix Z, *i.e.*, $vec(Z) = (z_1^T, z_2^T, \ldots, z_m^T)^T$. If $z = vec(Z)$, then we denote $Z = unvec(z)$. In this paper, the following relations will be used

$$vec(AXB) = (B^T \otimes A)vec(X),$$
$$(vec(A))^T vec(B) = trace(A^T B),$$

where $\otimes$ denotes the Kronecker product operator (see [3, 11]).

Definition 2.1. Let $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^{p\times q}$ be a linear operator. The linear operator $\mathcal{A}^* : \mathbb{R}^{p\times q} \to \mathbb{R}^{m\times n}$ is said to be the transpose of $\mathcal{A}$ if

$$\langle \mathcal{A}(X), Y\rangle = \langle X, \mathcal{A}^*(Y)\rangle, \quad \text{for all} \quad X \in \mathbb{R}^{m\times n},\ Y \in \mathbb{R}^{p\times q}.$$

$\mathcal{A}$ is said to be symmetric if $\mathcal{A} = \mathcal{A}^*$.

Definition 2.2. The linear operator $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is said to be symmetric positive definite (SPD), if

1) $\mathcal{A} = \mathcal{A}^*$,
2) $\langle \mathcal{A}(X), X\rangle > 0, \quad \forall X \in \mathbb{R}^{m\times n}$.

Now, we introduce a new inner product denoted by $\langle ., .\rangle_{\mathcal{A}}$. Assume that $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is a SPD linear operator. For two matrices X and Y in $\mathbb{R}^{m\times n}$, the inner product $< X, Y >_{\mathcal{A}}$ is defined such that

$$\langle X, Y\rangle_{\mathcal{A}} = \langle \mathcal{A}(X), Y\rangle, \quad \forall X, Y \in \mathbb{R}^{m\times n}.$$

The corresponding induced norm is represented by $\|.\|_{\mathcal{A}}$ and defined such that $\|X\|_{\mathcal{A}}^2 = \langle \mathcal{A}(X), X\rangle$, for $X \in \mathbb{R}^{m\times n}$. It is not difficult to see that $< ., . >_{\mathcal{A}}$ is an inner product.

Definition 2.3. Let the linear operator $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ be SPD. The set of matrices $\{M_1, M_2, \ldots, M_p\}$, where $M_i \in \mathbb{R}^{m\times n}$, $i = 1, \ldots, p$, is called $\mathcal{A}$-orthonormal if

$$\langle M_i, M_j\rangle_{\mathcal{A}} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

3. THE PROPOSED ALGORITHM

This section is devoted to presenting an iterative algorithm to compute the solution of (2) when $\mathcal{A}$ is SPD.

Assume that $\{M_1, M_2, \ldots, M_p\}$ is a set of $\mathcal{A}$-orthonormal matrices in $\mathbb{R}^{m\times n}$. Let $\mathcal{M}_p = span\{M_1, M_2, \ldots, M_p\}$ and assume that X_k is the kth approximate solution to (2). For computing the next approximate solution X_{k+1}, we solve the following minimization problem

$$\min_{X\in X_k+\mathcal{M}_p} \|E(X)\|_{\mathcal{A}}, \tag{3}$$

where $E(X) = X^* - X$ and X^* is the exact solution of (2). As $X_{k+1} \in X_k + \mathcal{M}_p$, we have:

$$X_{k+1} = X_k + \sum_{j=1}^{p} \alpha_j^{(k)} M_j, \tag{4}$$

where the scalars $\alpha_j^{(k)} \in \mathbb{R}$, $j = 1, \ldots, p$, are to be determined.

Now, we establish the following useful proposition.

PROPOSITION 3.1. *Suppose that X_{k+1} is computed by (4). Then,*

$$\alpha_j^{(k)} = \langle R_k, M_j \rangle, \qquad j = 1, \ldots, p,$$

if and only if

$$\langle R_{k+1}, M_i \rangle = 0, \qquad i = 1, 2, \ldots, p.$$

Proof. It is not difficult to see that

$$R_{k+1} = R_k - \sum_{j=1}^{p} \alpha_j^{(k)} \mathcal{A}(M_j),$$

where $R_k = C - \mathcal{A}(X_k)$. Hence,

$$\begin{aligned} \langle R_{k+1}, M_i \rangle &= \langle R_k, M_i \rangle - \sum_{j=1}^{p} \alpha_j^{(k)} \langle \mathcal{A}(M_j), M_i \rangle \\ &= \langle R_k, M_i \rangle - \sum_{j=1}^{p} \alpha_j^{(k)} \langle M_j, M_i \rangle_{\mathcal{A}} \\ &= \langle R_k, M_i \rangle - \alpha_i^{(k)}, \end{aligned}$$

which completes the proof. □

THEOREM 3.1. *Assume that X_{k+1} is computed by (4). Then, $\|E_{k+1}\|_{\mathcal{A}}^2$ is minimized if and only if $\alpha_j^{(k)} = \langle R_k, M_j \rangle$, $j = 1, \ldots, p$. In this case, the following statement holds*

$$\|E_{k+1}\|_{\mathcal{A}}^2 = \|E_k\|_{\mathcal{A}}^2 - \sum_{j=1}^{p} (\alpha_j^{(k)})^2, \tag{5}$$

where $E_k = E(X_k)$.

Proof. Suppose that X_{k+1}, computed by (4), is the solution of (3). Let $X \in X_k + \mathcal{M}_p$, hence there exists $\alpha_j \in \mathbb{R}$, $(j = 1, 2, \ldots, p)$ such that

$$X = X_k + \sum_{j=1}^{p} \alpha_j M_j.$$

It is not difficult to see that

$$E(X) = E_k - \sum_{j=1}^{p} \alpha_j M_j, \quad \text{and} \quad \mathcal{A}(E(X)) = \mathcal{A}(E_k) - \sum_{j=1}^{p} \alpha_j \mathcal{A}(M_j).$$

Using the fact that

$$\langle \mathcal{A}(M_j), E_k \rangle = \langle M_j, \mathcal{A}(E_k) \rangle = \langle \mathcal{A}(E_k), M_j \rangle,$$

we derive:

$$\begin{aligned} \|E(X)\|_{\mathcal{A}}^2 &= \langle \mathcal{A}(E(X)), E(X) \rangle \\ &= \langle \mathcal{A}(E_k) - \sum_{j=1}^{p} \alpha_j \mathcal{A}(M_j), E_k - \sum_{i=1}^{p} \alpha_i M_i \rangle \\ &= \langle \mathcal{A}(E_k), E_k \rangle - 2 \sum_{j=1}^{p} \alpha_j \langle R_k, M_j \rangle + \sum_{j=1}^{p} \sum_{i=1}^{p} \alpha_j \alpha_i \langle \mathcal{A}(M_j), M_i \rangle \\ &= \|E_k\|_{\mathcal{A}}^2 - 2 \sum_{j=1}^{p} \alpha_j \langle R_k, M_j \rangle + \sum_{j=1}^{p} \alpha_j^2, \end{aligned} \tag{6}$$

where $R_k = \mathcal{A}(E_k) = C - \mathcal{A}(X_k)$. For X_{k+1} to be a solution of (3), it is necessary that

$$0 = \left(\frac{\partial}{\partial \alpha_j} \|E(X)\|_{\mathcal{A}}^2\right) |_{X=X_k} = -2\langle R_k, M_j \rangle + 2\alpha_j^{(k)},$$

or equivalently,

$$\alpha_j^{(k)} = \langle R_k, M_j \rangle = trace(M_j^T R_k), \quad j = 1, \ldots, p. \tag{7}$$

Conversely, suppose that X_{k+1} is computed by (4) with $\alpha_j^{(k)} = \langle R_k, M_j \rangle$, $j = 1, 2, \ldots, p$. For an arbitrary $X \in X_k + \mathcal{M}_p$, straightforward computations show that

$$\begin{aligned} \|E(X)\|_{\mathcal{A}}^2 &= \|X^* - X\|_{\mathcal{A}}^2 = \|X^* - X_{k+1} - (X - X_{k+1})\|_{\mathcal{A}}^2 \\ &= \|E_{k+1} - \overline{X}\|_{\mathcal{A}}^2 = \langle \mathcal{A}(E_{k+1} - \overline{X}), E_{k+1} - \overline{X} \rangle \\ &= \langle A(E_{k+1}), E_{k+1} \rangle - 2\langle \mathcal{A}(E_{k+1}), \overline{X} \rangle + \langle \mathcal{A}(\overline{X}), \overline{X} \rangle \\ &= \|E_{k+1}\|_{\mathcal{A}}^2 - 2\langle R_{k+1}, \overline{X} \rangle + \|\overline{X}\|_{\mathcal{A}}^2, \end{aligned}$$

where $\overline{X} = X - X_{k+1} \in \mathcal{M}_p$. Since $\overline{X} \in \mathcal{M}_p$, Proposition 3.1 implies that $\langle R_{k+1}, \overline{X} \rangle = 0$. Therefore, we get:

$$\|E(X)\|_{\mathcal{A}}^2 = \|E_{k+1}\|_{\mathcal{A}}^2 + \|\overline{X}\|_{\mathcal{A}}^2,$$

which shows that

$$\|E(X)\|_{\mathcal{A}}^2 \geq \|E_{k+1}\|_{\mathcal{A}}^2.$$

The second part of theorem follows immediately from Eq. (6) by setting

$$X = X_{k+1} = X_k + \sum_{j=1}^{p} \alpha_j^{(k)} M_j,$$

where the values $\alpha_j^{(k)}$, for $i = 1, 2, \ldots, p$, is determined by (7) . □

It is noted that Theorem 3.1 does not guarantee the convergence of the method. However, in the next section, we will show that by providing some mild conditions the convergence of the method is guaranteed for solving the Sylvester matrix equation $AX + XB = C$.

Remark 3.1. Eq. (5) shows that $\|E_{k+1}\|_{\mathcal{A}}^2 \leq \|E_k\|_{\mathcal{A}}^2$. Therefore, $\|E_k\|_{\mathcal{A}}^2 \to \alpha$ as $k \to \infty$ for some $\alpha \in \mathbb{R}$. This shows that $\alpha_j^{(k)} \to 0$, as $k \to \infty$ for $j = 1, \ldots, p$.

Using the above results, we can propose the following algorithm for solving Eq. (2).

Algorithm 1. Prototype for the proposed method

1. Choose an initial guess $X_0 \in \mathbb{R}^{m\times n}$ and set $R_0 = C - \mathcal{A}(X_0)$.
2. For $k = 0, 1, \ldots,$ until convergence, Do
3. Choose a $\mathcal{A}$-orthonormal set of matrices $\{M_1, \ldots, M_p\}$.
4. $\alpha_j^{(k)} := trace(M_j^T R_k)$, $j = 1, \ldots, p$.
5. $X_{k+1} := X_k + \sum_{j=1}^{p} \alpha_j^{(k)} M_j$.
6. Compute $R_{k+1} = C - \mathcal{A}(X_{k+1})$.
7. EndDo

In general, one can use the Gram-Schmidt method to construct an $\mathcal{A}$-orthonormal set of matrices. Unfortunately, in general, it is difficult to construct a simple $\mathcal{A}$-orthonormal set of matrices similar to the one presented in [12]. In the next section, we will propose two strategies for constructing such a set for the Sylvester operator which are easy to employ.

4. APPLICATION TO THE SYLVESTER EQUATION

Consider the Sylvester matrix equation

$$AX + XB = C, \tag{8}$$

where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{m\times n}$ are given matrices, and $X \in \mathbb{R}^{m\times n}$ is unknown. Using the following linear operator,

$$\begin{aligned}&\mathcal{S} : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}\\ &\mathcal{S}(X) = AX + XB,\end{aligned}$$

Eq. (8) can be reformulated as follows:

$$\mathcal{S}(X) = C.$$

THEOREM 4.1. *The Sylvester operator $\mathcal{S}$ is SPD if and only if $A = A^T$, $B = B^T$ and the matrix $\mathcal{T} = I_{n\times n} \otimes A + B^T \otimes I_{m\times m}$ is SPD, where $I_{n\times n}$ and $I_{m\times m}$ are identity matrices of order m and n, respectively.*

Proof. It is not difficult to verify that the Sylvester operator S is symmetric if and only if $A = A^T$ and $B = B^T$. We show that $\mathcal{S}$ is positive definite if and only if $\mathcal{T}$ is positive definite. To do so, let $w \in \mathbb{R}^{mn}$ and $W = unvec(w)$. In this case, we have $w = vec(W)$ and

$$\begin{aligned}w^T\mathcal{T}w &= vec(W)^T(I_{n\times n} \otimes A + B^T \otimes I_{m\times m})vec(W)\\ &= vec(W)^T vec(AW + WB)\\ &= vec(W)^T vec(\mathcal{S}(W))\\ &= trace(W^T\mathcal{S}(W))\\ &= \langle \mathcal{S}(W), W\rangle.\end{aligned}$$

Let $w \neq 0$, hence $W \neq 0$. From the above relation, we have:

$$w^T\mathcal{T}w > 0 \Leftrightarrow \langle \mathcal{S}(W), W\rangle > 0,$$

which completes the proof. □

Remark 4.1. The eigenvalues of $\mathcal{T}$ are of the form $\lambda_i + \mu_j$ where λ_i's and μ_j's are the eigenvalues of A and B, respectively (see [11]). Therefore, $\mathcal{S}$ is SPD if and only if $\mathcal{T}$ is symmetric and $\lambda_i + \mu_j > 0$ for all i, j.

In continuation, we propose two strategies to construct a $\mathcal{S}$-orthonormal basis.

4.1. FIRST STRATEGY FOR CONSTRUCTING A $\mathcal{S}$-ORTHONORMAL BASIS

In this subsection, we propose a way to construct a $\mathcal{S}$-orthonormal set $\{M_1, \ldots, M_p\}$ where $p \leq \min\{m, n\}$. Let $R_k = (r_{ij}^{(k)}) \in \mathbb{R}^{m\times n}$ and set $q = 1$. Let $r_{i_1j_1}$ be the largest entry of R_k in absolute value. We set

$$M_1 = \frac{1}{\|E_{i_1j_1}\|_{\mathcal{S}}} E_{i_1j_1},$$

where $E_{i_1 j_1}$ is an $m \times n$ matrix whose (i_1, j_1) entry is equal to 1 and others are equal to zero. In the qth step, $q = 2, \ldots, p$, we ignore rows $i_1, \ldots, i_{q-1}$ and columns $j_1, \ldots, j_{q-1}$ of R_k and find the largest entry (i_q, j_q) in the remaining matrix. Next, we compute

$$M_q = \frac{1}{\|E_{i_q j_q}\|_{\mathcal{S}}} E_{i_q j_q},$$

where $E_{i_q j_q}$ is an $m \times n$ matrix whose (i_q, j_q) entry is equal to 1 and others are equal to zero. It is not difficult to verify that the set of matrices $\{M_1, \ldots, M_p\}$ is $\mathcal{S}$-orthonormal.

Evidently,

$$\|E_{i_q j_q}\|_{\mathcal{S}} = \sqrt{\langle \mathcal{S}(E_{i_q j_q}), E_{i_q j_q} \rangle} = \sqrt{A_{i_q i_q} + B_{j_q j_q}},$$

and therefore,

$$\alpha_q^{(k)} = trace(M_q^T R_k) = \frac{r_{i_q j_q}}{\sqrt{A_{i_q i_q} + B_{j_q j_q}}}.$$

As observed in the previous section, $\alpha_q^{(k)} \to 0$ as $k \to \infty$. Therefore, we conclude that $R_k \to 0$ as $k \to \infty$ which shows that $X_k \to X^*$.

According to the above results, we see that the Steps 4 and 5 of Algorithm 1 can be written pointwisely as follows:

- $X_{k+1} := X_k$
- For $k = 1, \ldots, p$, Do
- $\quad (X_{k+1})_{i_k j_k} := (X_{k+1})_{i_k j_k} + \dfrac{r_{i_k j_k}}{A_{i_k i_k} + B_{j_k j_k}}$
- End.

This shows that in each iteration of the method only p entries of the current approximate solution of the system are modified.

4.2. SECOND STRATEGY FOR CONSTRUCTING A $\mathcal{S}$-ORTHONORMAL BASIS

When the method applied in this manner, we assume that the relation $p = \min\{m, n\}$ holds. Without loss of generality, we may assume that $p = n$. For $k = 1$ (at first iteration), let

$$(i_q, j_q) = (q, q), \qquad q = 1, \ldots, n.$$

Then, in each iteration of the method, we set $i_q := i_q + 1$, $q = 1, \ldots, n$. If $i_n > m$, then we set $i_n = 1$. In this way, we have $i_\ell \neq i_t$ and $j_\ell \neq j_t$ for $\ell \neq t$. Next, we use the method described in Subsection 4.1 to construct M_k's. After m iterations, all of the entries of the approximate solution X_0 are updated one time. This procedure would be repeated in the next iterations and this guarantees the convergence of the method.

5. REVIEW OF TWO WELL KNOWN METHODS

In the section of the numerical examples, we will compare the numerical results of the proposed method with those of the gradient-based method [13] and the global conjugate gradient (CG) method [9, 16] to solve the sylvester matrix equation. Hence, in this section we briefly review these two methods.

It is not difficult to see that the gradient-based algorithm to solve the SPD Sylvester matrix equation $\mathcal{S}(X) = C$ can be written as

$$X_{k+1} = X_k + \mu(C - \mathcal{S}(X_k)), \qquad k = 0, 1, 2, \ldots,$$

where $\mu \in \mathbb{R}$ and X_0 is a given initial guess. The optimum value of μ is given by

$$\mu_{opt} = \frac{2}{\lambda_{\max} + \lambda_{\min}},$$

where $\lambda_{\max}$ and $\lambda_{\min}$, respectively, are the largest and smallest eigenvalues of $\mathcal{T}$ which is defined in Theorem 2.

It is well-known that the global CG method is a generalization of the CG algorithm to solve linear matrix equations. This method can be described in the following (for more details see [9, 16]).

Algorithm 2. *Global CG*

1. Compute $R_0 = C - \mathcal{S}(X_0)$, $P_0 := R_0$.
2. For $j := 0, 1, \ldots,$ until convergence, Do
3. $\quad \alpha_j := \langle R_j, R_j\rangle / \langle \mathcal{S}(P_j), P_j\rangle$
4. $\quad X_{j+1} := X_j + \alpha_j P_j$
5. $\quad R_{j+1} := R_j - \alpha_j \mathcal{S}(P_j)$
6. $\quad \beta_j := \langle R_{j+1}, R_{j+1}\rangle / \langle R_j, R_j\rangle$
7. $\quad P_{j+1} := R_{j+1} + \beta_j P_j$
8. EndDo

6. NUMERICAL EXPERIMENTS

In this section, we present two numerical examples to demonstrate the efficiency and feasibly of the proposed algorithm. All the numerical experiments given in this section were computed in double precision with some MATLAB codes. In all of the examples, we have utilized the initial guess $X_0 = [X_{ij}^{(0)}] \in \mathbb{R}^{m\times n}$ such that

$$X_{ij}^{(0)} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

The following stopping criterion

$$\delta_k = \frac{\|R^{(k)}\|_F}{\|R^{(0)}\|_F} < \frac{1}{2} \times 10^{-7},$$

was used in both of the examples where $R^{(k)} = C - \mathcal{S}(X_k)$. In what follows, "NMS1", "NMS2", "GB" and "GLCG" stand for the new method with the first strategy presented in Subsections 4.1, the new method with the second strategy presented in Subsections 4.2, the gradient-based method with the optimal parameter μ_{opt} and the global CG, respectively.

Example 6.1. In this example, we consider the Sylvester matrix equation with

$$A = \begin{pmatrix} 1 & 1 & -2 & 2 & 1 \\ 1 & 2 & 0 & -2 & 3 \\ -2 & 0 & 9 & -10 & 5 \\ 2 & -2 & -10 & 40 & 0 \\ 1 & 3 & 5 & 0 & 30 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -2 & 2 & -2 \\ -2 & 17 & 3 & 5 \\ 2 & 3 & 18 & 8 \\ -2 & 5 & 8 & 31 \end{pmatrix},$$

and the right-hand side

$$C = \begin{pmatrix} 5 & 26 & 34 & 45 \\ 6 & 27 & 35 & 46 \\ 4 & 25 & 33 & 44 \\ 32 & 53 & 61 & 72 \\ 41 & 62 & 70 & 81 \end{pmatrix}.$$

It is not difficult to check that the corresponding Sylvester operator $\mathcal{S}$ is SPD. The exact solution of the system is an 5×4 matrix of all ones.

To have a reasonable comparison, the number of iterations of the proposed method is divided by m, since in each iteration of the method only p entries of current approximate solution are modified, whereas in the other methods all of the entries of the solution are modified in each iteration. For this example, the optimal value of μ in the gradient-based method is $\mu_{opt} = 0.0241$. The NMS1, NMS2, GB and GlCG methods converge in 9, 17, 183 and 19 iterations, respectively. As encountered, for this example, the NMS1 is the best among these four methods. For more investigation, the graph of $\log_{10} \delta_k$ are presented in Figure 1.

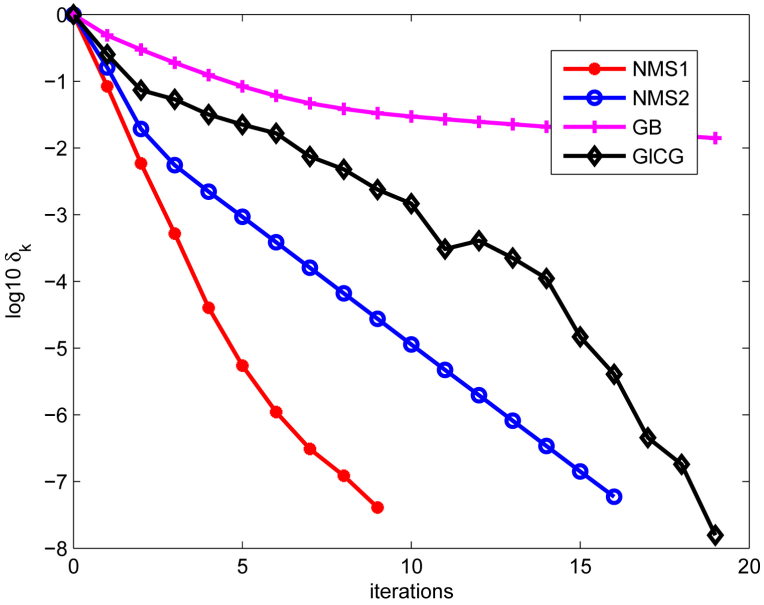

Fig. 1 – $\log_{10}\delta_k$ for Example 6.1.

Example 6.2. Let

$$A = \begin{pmatrix} 4 & 2 & & & -8 \\ 2 & 4 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & 2 & 4 & 2 \\ -8 & & & 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 8 & 1 & & & -0.5 \\ 1 & 8 & 1 & & \\ & 1 & 8 & 1 & \\ & & 1 & 8 & 1 \\ -0.5 & & & 1 & 8 \end{pmatrix},$$

where $A \in \mathbb{R}^{10\times 10}$, $B \in \mathbb{R}^{5\times 5}$ and

$$C = \begin{pmatrix} 6.5 & 8 & \cdots & 8 & 6.5 \\ 16.5 & 18 & \cdots & 18 & 16.5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 16.5 & 18 & \cdots & 18 & 16.5 \\ 6.5 & 8 & \cdots & 8 & 6.5 \end{pmatrix} \in \mathbb{R}^{10\times 5}.$$

The exact solution of the corresponding Sylvester matrix equations is a 10×5 matrix of all ones. All of the assumptions are as the previous example. It is noted that the optimal value of μ in the gradient-based method for this example is $\mu_{opt} = 0.0839$. For this example, the NMS1, NMS2, GB and GICG methods converge in 12, 38, 94 and 21 iterations, respectively. As seen, for this example, the NMS1 is the best among these four methods. For more details, the graph of $\log_{10}\delta_k$ are illustrated in Figure 2.

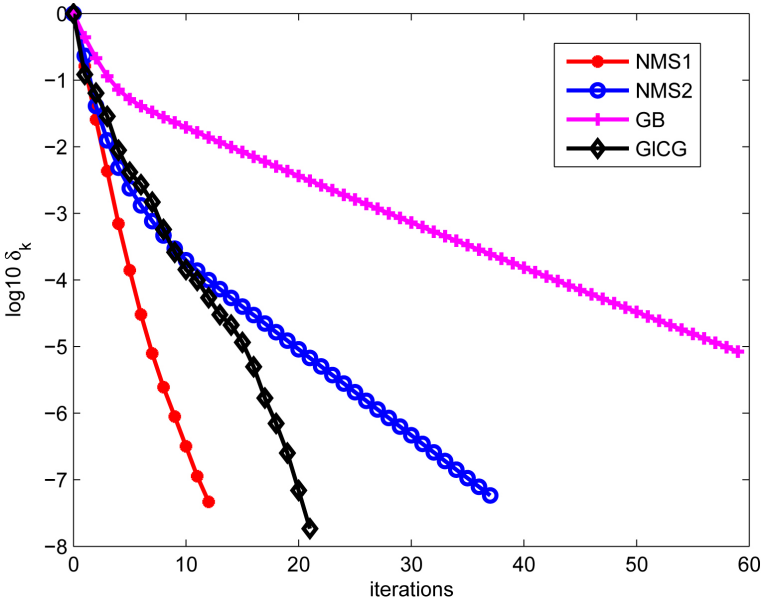

Fig. 1 – $\log_{10} \delta_k$ for Example 6.2.

7. CONCLUSION

We have proposed a useful iterative method for solving symmetric positive definite matrix equations. Moreover, it has been established that, in each iteration, the new approximate solution calculated by the algorithm is the solutions of a minimization problem. The application of the proposed method has been discussed for solving the symmetric positive definite Sylvester equation $AX + XB = C$ with more details. Two numerical examples have been given to illustrate the efficiency of the method. For the presented examples, the new method in conjunction with the proposed method in Subsection 4.1 were the best. Future works may focus on the generalization of the method to general symmetric positive definite matrix equations. One may also focus on making a preconditioner for Sylvester matrix equations.

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University of Guilan,
Faculty of Mathematical Sciences,
Rasht, Iran
khojasteh@guilan.ac.ir,
salkuyeh@gmail.com

Vali-e-Asr University of Rafsanjan,
Department of Mathematics,
Rafsanjan, Iran
f.beik@vru.ac.ir