Recently, there has been a growing interest in the methods addressing volatility in computational finance and econometrics. Peiris et al. [8] have introduced doubly stochastic volatility models with GARCH innovations. Random coefficient autoregressive sequences are special case of doubly stochastic time series. In this paper, we consider some doubly stochastic stationary time series with GARCH and Threshold GARCH errors. Some general properties of process, like variance and kurtosis are derived.

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1. INTRODUCTION

Recently, there has been a growing development in the use of nonlinear volatility models in financial literature. Black-Scholes option pricing model has some contradictory assumptions, such as constant volatility and normally distributed log returns. In many financial time series, particular empirical studies reveal some facts such as stock returns and foreign exchange rates, exhibit leptokurtosis and stochastic volatility. Any distribution can be characterized by a number of features such as mean, variance, skewness and kurtosis. The measure of kurtosis is considered as whether the data are peaked or flat relative to the Normal distribution.

The concept of stochastic volatility for financial time series, the autoregressive conditionally heteroskedastic (ARCH) model was first studied by Engle [4] and its generalization, the GARCH model, by Bollerslev [2]. The GARCH model assume symmetric effects on volatility that is, good and bad news have the same effect on volatility, which is shortfall of these models. The Threshold GARCH (TGARCH) model was introduced by Zakoian [12] and various other nonlinear GARCH extensions have been proposed to capture asymmetric effects [3, 5]. Random coefficient autoregressive (RCA) time series were introduced by Nicholls and Quinn [7] and some of their properties were studied by
In this paper, we derive the kurtosis for various classes of doubly stochastic models with GARCH and TGARCH innovations. In Section 2, some random coefficient autoregressive time series are discussed. In Section 3, some doubly stochastic models with GARCH and TGARCH innovations are introduced and we derive the formula for kurtosis in terms of model parameters.

2. RANDOM COEFFICIENT AUTOREGRESSIVE TIME SERIES

Random coefficient autoregressive sequences represent a special case of doubly stochastic time series. Some random coefficient autoregressive time series are given as follows:

\[(2.1)\]
\[y_t = (\phi + b_t)y_{t-1} + \varepsilon_t, t \in \mathbb{Z}\]

\[(2.2)\]
\[y_t = (\phi + b_t)y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}, t \in \mathbb{Z}\]

\[(2.3)\]
\[y_t = (\phi + b_t + \Phi s_{t-1})y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}, t \in \mathbb{Z}\]

From the equations (2.1)–(2.3) we have Random coefficient autoregressive (RCA-1), Random coefficient autoregressive-moving average (RCA-MA-1), Sign-RCA-MA -1, respectively. The mean, variance and kurtosis for the above processes have been discussed in [8]. The two necessary and sufficient conditions for second order stationarity are given by:

1. \((b_t, \varepsilon_t) \sim \left(\begin{pmatrix}0 \\ 0\end{pmatrix}, \begin{pmatrix}\sigma_b^2 & 0 \\ 0 & \sigma_{\varepsilon}^2\end{pmatrix}\right)\);

2. \(\phi^2 + \sigma_b^2 < 1\).

We write for \(s_t\) as:

\[s_t = \begin{cases} +1 & \text{if } y_t > 0 \\ 0 & \text{if } y_t = 0 \\ -1 & \text{if } y_t < 0 \end{cases}\]

where \(\{b_t\}\) and \(\{\varepsilon_t\}\) are errors in the model respectively, \(\phi, \Phi\) and \(s_t\) are parameters also \(E(s^2_t) = 1\). In order to calculate the kurtosis, \(E(s^4_t) = 1\) [11]. A process of the following form was considered in [10]:

\[(2.4)\]
\[b_{t+1} = ab_t + (1 + b_t)v_{t+1}\]

Recently, Peiris et al. [8] have studied a doubly stochastic model of the following form:

\[(2.5)\]
\[y_t = (\phi + b_t)y_{t-1} + \varepsilon_t\]
Let $y_t$ be a stationary doubly stochastic time series satisfying (2.5) and (2.6). The moments for the model have been recently calculated in [8], where $\varepsilon_t$ is an identically distributed independent sequence of variables with zero mean and variance $\sigma^2_{\varepsilon_t}$.

1. $(\varepsilon_t, \varepsilon_t) \sim N\left((0, 0), \begin{pmatrix} \sigma^2_{\varepsilon} & 0 \\ 0 & \sigma^2_{\varepsilon} \end{pmatrix}\right)$
2. $1 - a^2 - \sigma^2_v < 1$
3. $1 - a^2 - 2\sigma^2_v - \phi^2 + \phi^2 a^2 + \phi^2 \sigma^2_v < 1$

In 2.5 and 2.6 the observed random variable is modeled in two steps. Firstly, the distribution of the observed outcome is represented in a standard way and in the second step, $b_t$ is treated as being itself a random variable.

**Definition 2.1 ([12]).** Let $z_t$ be an independently identically distributed (iid) sequence of random variables such that $E(z_t) = 0$ and $Var(z_t) = 1$ then $\varepsilon_t$ is called a Threshold GARCH$-(p, q)$ process if it satisfies:

$$
\begin{align*}
\sigma_t &= \omega + \sum_{i=1}^{p} \alpha_{i,+} \varepsilon_{t-i}^{+} - \alpha_{i,-} \varepsilon_{t-i}^{-} + \sum_{j=1}^{q} \beta_j \sigma_{t-j} \\
\varepsilon_t &= \sigma_t z_t
\end{align*}
$$

where $\omega, \alpha_{i,+}, \alpha_{i,-}$ and $\beta$ are real numbers and through the coefficients $\alpha_{i,+}$, $\alpha_{i,-}$ the current volatility depends on both the modulus and the sign of the past returns. The positive and negative components of $\varepsilon_t$ are given by

$$
\varepsilon_t^{+} = \max(\varepsilon_t, 0), \quad \varepsilon_t^{-} = \min(\varepsilon_t, 0) \quad \text{and} \quad \varepsilon_t = \varepsilon_t^{+} + \varepsilon_t^{-}.
$$

## 3. Doubly Stochastic Models with Threshold GARCH Errors

In this section, we present our results for doubly stochastic volatility models with Threshold GARCH errors. We consider equations (2.1) to (2.3) and provide the moments of doubly stochastic models with TGARCH$-(1, 1)$.

**Theorem 3.1.** Consider a doubly stochastic model of the following form with TGARCH$-(1, 1)$:

$$
\begin{align*}
y_t &= (\phi + b_t)y_{t-1} + \varepsilon_t \\
b_{t+1} &= ab_t + (1 + b_t)v_{t+1}
\end{align*}
$$
\[
\varepsilon_t = \sigma_t z_t
\]

\[
\sigma_t = \omega + \alpha_1 + \varepsilon_{t-1}^+ - \alpha_1 - \varepsilon_{t-1}^- + \beta_1 \sigma_{t-j}
\]

Then we have the following results:

1. \( E(y_t^2) = \frac{\omega^2 \sigma_z^2 (1 + a_1) (1 - a^2 - \sigma_v^2)}{(1 - a_1)(1 - a_2)(1 - a_2 - 2\sigma_v^2 - \phi^2 + \phi^2 a^2 + \phi^2 \sigma_v^2)} \)

2. \( E(y_t^4) = \frac{f_1}{f_2} \), where

\[
f_1 = 3\sigma_z^4 \omega^4 \frac{(1 + 3a_1 + 5a_2 + 3a_1a_2 + 3a_3 + 5a_1a_3 + 3a_2a_3 + a_1a_2a_3)}{(1 - a_1)(1 - a_2)(1 - a_3)}
\]

\[
+ \frac{6\omega^4 \sigma_z^4 (1 + a_1)^2 [\phi^2 (1 - a^2) + \sigma_v^2 (1 - \phi^2)]}{(1 - a_1)^2 (1 - a_2)^2 (1 - a_2 - 2\sigma_v^2 - \phi^2 + \phi^2 a^2 + \phi^2 \sigma_v^2)}
\]

\[
f_2 = 1 - \phi^4 - 3\sigma_v^4 \left[ \frac{-1 - a^2 - 5\sigma_v^2 + a^3 + a^5 - 16a^3 \sigma_v^2 + 3a \sigma_v^2 - 9a \sigma_v^4}{(-1 + a^3 + 3a \sigma_v^2)(-1 + a^2 + \sigma_v^2)(-1 + 6a^2 \sigma_v^2 + a^4 + 3\sigma_v^4)} \right]
\]

\[
- 6\phi^2 \left[ \frac{\sigma_v^2}{(1 - a^2 - \sigma_v^2)} \right] - 4\phi^6 a \left[ \frac{\sigma_v^4}{(1 - a^3 - 3a \sigma_v^2)(1 - a^2 - \sigma_v^2)} \right]
\]

3. The Kurtosis of the process follows the definition \( K(y) = \frac{E(y_t^4)}{[E(y_t^2)]^2} \).

\[\text{Proof.} \text{ 1. First we calculate the value of } E(y_t^2) \]

\[E(y_t^2) = E[(\phi + b_t)y_{t-1} + \varepsilon_t]^2 = \phi^2 E(y_{t-1}^2) + E(b_t^2) E(y_{t-1}^2) + E(\varepsilon_t^2) \]

\[E(y_t^2) - \phi^2 E(y_{t-1}^2) - E(b_t^2) E(y_{t-1}^2) = E(\sigma_t^2) \sigma_z^2 \]

Using the second order stationarity condition and by putting values of \( E(\sigma_t^2) \) and \( E(b_t^2) \) we get:

\[E(y_t^2) = \frac{E(\sigma_t^2) \sigma_z^2}{(1 - \phi^2 - E(b_t^2))} \]

\[E(y_t^2) = \frac{\omega^2 \sigma_z^2 (1 + a_1)(1 - a^2 - \sigma_v^2)}{(1 - a_1)(1 - a_2)(1 - a_2 - 2\sigma_v^2 - \phi^2 + \phi^2 a^2 + \phi^2 \sigma_v^2)} \]

2. Now we find \( E(y_t^4) \)

\[E(y_t^4) = E[(\phi + b_t)y_{t-1} + \varepsilon_t]^4 \]

\[E(y_t^4) = \phi^4 E(y_{t-1}^4) + E(b_t^4) E(y_{t-1}^4) + E(\varepsilon_t^4) + 6\phi^2 E(b_t^2) E(y_{t-1}^4) + 4\phi E(b_t^3) E(y_{t-1}^4) \]

\[+ 6\phi^2 E(\varepsilon_t^2) E(y_{t-1}^2) + 6 E(\varepsilon_t^2) E(b_t^2) E(y_{t-1}^2) + 12\phi E(\varepsilon_t) E(b_t) E(y_{t-1}^2) \]
\[ E(y_t^4) = \frac{3\sigma^4 \epsilon E(\epsilon^4) + 6\sigma^2 z E(b_t^2) E(\epsilon^2) E(y_{t-1}^2) + 6\sigma^2 \phi^2 E(\epsilon^2) E(y_{t-1}^2)}{1 - \phi^4 - E(b_t^4) - 6\phi^2 E(b_t^2) - 4\phi E(b_t^3)} \]

\[ E(y_t^4) = \frac{3\sigma^4 \epsilon E(\epsilon^4) + 6\sigma^2 z E(\epsilon^2) E(b_t^2) + \phi^2 E(y_{t-1}^2)}{1 - \phi^4 - E(b_t^4) - 6\phi^2 E(b_t^2) - 4\phi E(b_t^3)} \]

Using the second order stationarity condition and by putting values of \( E(\epsilon^2) \), \( E(\epsilon^2) \) and \( E(b_t^2) \), we get the required value of \( E(y_t^4) \) as given in the statement of the theorem and

\[
E(\epsilon^2) = \frac{\omega^2 (1 + a_1)}{(1 - a_1)(1 - a_2)}
\]

\[
E(\epsilon^4) = \frac{\omega^4 (1 + 3a_1 + 5a_2 + 3a_1a_2 + 3a_3 + 5a_1a_3 + 3a_2a_3 + a_1a_2a_3)}{(1 - a_1)(1 - a_2)(1 - a_3)}
\]

\[
a_1 = \frac{1}{\sqrt{2\pi}} (\alpha_{1,+} + \alpha_{1,-}) + \beta_1, a_2 = \frac{1}{2} (\alpha_{1,+}^2 + \alpha_{1,-}^2) + \frac{2}{\sqrt{2\pi}} \beta_1 (\alpha_{1,+} + \alpha_{1,-}) + \beta_1^2
\]

\[
a_3 = \sqrt{\frac{2}{\pi}} (\alpha_{1,+}^3 + \alpha_{1,-}^3) + \frac{3}{2} \beta_1 (\alpha_{1,+}^2 + \alpha_{1,-}^2) + \frac{3}{\sqrt{2\pi}} \beta_1^2 (\alpha_{1,+} + \alpha_{1,-}) + \beta_1^3
\]

3. We can get \( K(y) = \frac{E(y_t^4)}{[E(y_t^4)]^2} \). The Kurtosis of the process follows from its definition of \( K(y) \). \( \square \)

**Theorem 3.2.** Consider the following doubly stochastic time series with given conditions:

\[
y_t = (\phi + b_t)y_{t-1} + \epsilon_t + \theta \epsilon_{t-1}
\]

\[
b_{t+1} = ab_t + (1 + b_t) \nu_{t+1}
\]

a) \( (\epsilon_t, \nu_t) \sim N\left( \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{cc} \sigma^2 \ & 0 \\ 0 & \sigma^2 \end{array} \right) \right) \)

b) \( 1 - a^2 - \sigma^2_v < 1 \)

c) \( 1 - a^2 - 2\sigma^2_v - \phi^2 + \phi^2 a^2 + \phi^2 \sigma^2_v < 1 \)

where \( \epsilon_t \) is an identically distributed independent sequence of variables with mean zero and variance \( \sigma^2_\epsilon \). Then we have results for \( E(y_t^2) \), \( E(y_t^4) \) and Kurtosis as follows:

1. \( E(y_t^2) = \frac{\omega^2 \sigma^2_\epsilon (1 + \theta^2)(1 - a^2 - \sigma^2_v)}{(1 - a^2 - 2\sigma^2_v - \phi^2 + \phi^2 a^2 + \phi^2 \sigma^2_v)} \)

2. \( E(y_t^4) = \frac{3\sigma^4_\epsilon (1 + 6\theta^2 + \theta^4) + 6\sigma^2_\epsilon (1 + \theta^2) [(\phi^2 + E(b_t^2)] E(y_{t-1}^2)}{1 - \phi^4 - E(b_t^4) - 6\phi^2 E(b_t^2) - 4\phi E(b_t^3)} \)
3. The Kurtosis of the process follows the definition \( K(y) = \frac{E(y_t^4)}{[E(y_t^2)]^2} \).

**Proof.** 1. First we calculate the value of \( E(y_t^2) \):

\[
E(y_t^2) = E[(\phi + b_t)y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}]^2
\]

\[
E(y_t^2) = \phi^2 E(y_{t-1}^2) + E(b_t^2) E(y_{t-1}^2) + E(\varepsilon_t^2) + \theta^2 E(\varepsilon_{t-1}^2)
+ 2E(\phi + b_t)E(y_{t-1})E(\varepsilon_t) + 2\theta E(\varepsilon_t)E(\varepsilon_{t-1})
+ 2\theta E(\phi + b_t)E(y_{t-1})E(\varepsilon_{t-1})
\]

\[
E(y_t^2) - \phi^2 E(y_{t-1}^2) - E(b_t^2) E(y_{t-1}^2) = E(\varepsilon_t^2) + \theta^2 E(\varepsilon_{t-1}^2)
\]

Using the second order stationarity condition and by replacing the values of \( E(\sigma_t^2) \) and \( E(b_t^3) \) we get:

\[
E(y_t^2) = \frac{\sigma_\varepsilon^2(1 + \theta^2)}{(1 - \phi^2 - E(b_t^2))} = \frac{\sigma_\varepsilon^2(1 + \theta^2)}{(1 - \phi^2 - \frac{\sigma_b^2}{(1 - a^2 - \sigma_\varepsilon^2)})}
\]

2. Now we calculate \( E(y_t^4) \):

\[
E(y_t^4) = E[(\phi + b_t)y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}]^4
\]

Because the odd moments of \( y_t \) are zero, we can write the above expression by expanding the above equation and applying \( E \) we get:

\[
E(y_t^4) = E[ (\phi + b_t)^4 y_{t-1}^4 + 6(\phi + b_t)^2 y_{t-1}^2 (\varepsilon_t + \theta \varepsilon_{t-1})^2 + (\varepsilon_t + \theta \varepsilon_{t-1})^4 ]
\]

Using the stationarity condition and \( E(\varepsilon_t) = E(\varepsilon_{t-1}) = 0 \)

\[
E(y_t^4) = E(\phi + b_t)^4 E(y_{t-1}^4) + E(\varepsilon_t^4) + \theta^4 E(\varepsilon_t^4) + 6\theta^2 E(\varepsilon_t^4)
+ 6E(\phi + b_t)^2 E(\varepsilon_t^2) E(y_{t-1}^2) + 6\theta^2 E(\phi + b_t^2) E(\varepsilon_t^2) E(y_{t-1}^2)
E(y_t^4) = \frac{3\sigma_\varepsilon^4 (1 + \theta^2 + \theta^4) + 6\sigma_\varepsilon^2 (1 + \theta^2) [(\phi^2 + E(b_t^2)] E(y_{t-1}^2)}{1 - \phi^4 - E(b_t^4) - 6\phi^2 E(b_t^2) - 4\phi E(b_t^3)}
\]

3. The Kurtosis of the process follows from its definition

\[
K(y) = \frac{E(y_t^4)}{[E(y_t^2)]^2}
\]

\[
K(y) = \frac{3\sigma_\varepsilon^4 (1 + \theta^2 + \theta^4) + 6\sigma_\varepsilon^2 (1 + \theta^2) [(\phi^2 + E(b_t^2)] E(y_{t-1}^2)}{1 - \phi^4 - E(b_t^4) - 6\phi^2 E(b_t^2) - 4\phi E(b_t^3)} = \frac{\sigma_\varepsilon^2(1 + \theta^2)(1 - a^2 - \sigma_\varepsilon^2)}{(1 - a^2 - 2\sigma_\varepsilon^2 - \phi^2 + \phi^2 a^2 + \phi^2 \sigma_\varepsilon^2)}.
\]

\( \square \)
Theorem 3.3. Consider a doubly stochastic model of the following form with TGARCH–(1, 1)

\[ y_t = (\phi + b_t)y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1} \]
\[ b_{t+1} = ab_t + (1 + b_t)v_{t+1} \]
\[ \varepsilon_t = \sigma_t z_t \]
\[ \sigma_t = \omega + \alpha_1, + \varepsilon_{t-1}^\dagger - \alpha_1, - \varepsilon_{t-1}^\dagger + \beta_1 \sigma_{t-j} \]

Then we have the following results:

1. \( E(y_t^2) = \frac{\omega^2 \sigma_z^2 (1 + \theta^2)(1 + a_1)(1 - a^2 - \sigma_v^2)}{(1 - a_1)(1 - a_2)(1 - a^2 - 2\sigma_v^2 - \phi^2 + \phi^2 a^2 + \phi^2 \sigma_v^2)} (1 + 6\theta^2 + \theta^4) \]

2. \( E(y_t^4) = \frac{f_1}{f_2} \), where

\[ f_1 = \sigma_z^2 \omega^4 \frac{(1 + 3a_1 + 5a_2 + 3a_1 a_2 + 3a_3 + 5a_1 a_3 + 3a_2 a_3 + a_1 a_2 a_3)}{(1 - a_1)(1 - a_2)(1 - a_3)} (1 + 6\theta^2 + \theta^4) \]
\[ + \frac{6\omega^4 \sigma_z^4 (1 + \theta^2)^2 (1 + a_1)^2 [\phi^2 (1 - a^2) + \sigma_v^2 (1 - \phi^2)]}{(1 - a_1)^2 (1 - a_2)^2 (1 - a^2 - 2\sigma_v^2 - \phi^2 + \phi^2 a^2 + \phi^2 \sigma_v^2)} \]
\[ f_2 = 1 - \phi^4 - 3\sigma_v^4 \frac{(-1 - a^2 - 5\sigma_v^2 + a^3 + a^5 - 16a^3 \sigma_v^2 + 3a\sigma_v^2 - 9a\sigma_v^4)}{(-1 + a^3 + 3a\sigma_v^2)(-1 + a^2 + \sigma_v^2)(-1 + 6a^2 \sigma_v^2 + a^4 + 3\sigma_v^4)} \]
\[ -6\phi^2 \frac{\sigma_v^2}{(1 - a^2 - \sigma_v^2)} - 4\phi 6a \frac{\sigma_v^4}{(1 - a^3 - 3a\sigma_v^2)(1 - a^2 - \sigma_v^2)} \]

3. The Kurtosis of the process follows from its definition \( K(y) = \frac{E(y_t^4)}{[E(y_t^2)]^2} \)

Proof. 1. First we calculate the value of \( E(y_t^2) \)

\[ E(y_t^2) = E[(\phi + b_t)y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}]^2 \]
\[ E(y_t^2) = \phi^2 E(y_{t-1}^2) + E(b_t^2) E(y_{t-1}^2) + E(\varepsilon_t^2) + \theta^2 E(\varepsilon_{t-1}^2) + 2E(\phi + b_t) E(y_{t-1}^2) E(\varepsilon_t) \]
\[ + 2\theta E(\varepsilon_t) E(\varepsilon_{t-1}) + 2\theta E(\phi + b_t) E(y_{t-1}) E(\varepsilon_{t-1}) \]

Using the second order stationarity condition and by replacing the values of \( E(\sigma_t^2) \) and \( E(b_t^2) \) we get:

\[ E(y_t^2) = \frac{\sigma_z^2 (1 + \theta^2) E(\sigma_t^2)}{(1 - \phi^2 - E(b_t^2))} \]
\[ E(y_t^2) = \frac{\omega^2 \sigma_z^2 (1 + \theta^2)(1 + a_1)(1 - a^2 - \sigma_v^2)}{(1 - a_1)(1 - a_2)(1 - a^2 - 2\sigma_v^2 - \phi^2 + \phi^2 a^2 + \phi^2 \sigma_v^2)} \]
2. Now we calculate $E(y_t^4)$

$$E(y_t^4) = E[(\phi + b_t)y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}]^4$$

Taking expectation $E$ we get:

$$E(y_t^4) = E(\phi + b_t)^4E(y_{t-1}^4) + E(\varepsilon_t^4) + \theta^4E(\varepsilon_t^4) + 6\theta^2E(\varepsilon_t^4)$$

$$+ 6E(\phi + b_t)^2E(\varepsilon_t^2)E(y_{t-1}^2) + 6\theta^2E(\phi + b_t)^2E(\varepsilon_t^2)E(y_{t-1}^2)$$

Using the second order stationarity condition and by replacing the values of $E(\varepsilon_t^4)$ and $E(\varepsilon_t^2)$

$$E(y_t^4) = \frac{3\sigma_z^4E(\sigma_t^4)(1 + 6\theta^2 + \theta^4) + 6\sigma_z^2E(\sigma_t^2)(1 + \theta^2)[(\phi^2 + E(b_t^2)]E(y_t^2)}{1 - \phi^4 - E(b_t^4) - 6\theta^2E(b_t^2) - 4\phi E(b_t^2)}$$

Now by replacing the values of $E(\sigma_t^4), E(\sigma_t^2)$ and $E(b_t^2), E(b_t^3), E(b_t^4)$ we obtain the required value of $E(y_t^4)$ as:

$$E(y_t^4) = \frac{f_1}{f_2},$$

where $f_1$ and $f_2$ are given in the statement of the theorem. \qed

**Theorem 3.4.** Consider a doubly stochastic model of the following form with TARCH($1,1$) errors as:

$$y_t = (\phi + b_t + \Phi s_t)y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$$

$$b_{t+1} = ab_t + (1 + b_t)v_{t+1}$$

$$\varepsilon_t = \sigma_t z_t$$

$$\sigma_t = \omega + \alpha_1, \varepsilon_{t-1}^+ - \alpha_1, \varepsilon_{t-1}^- + \beta_1 \sigma_{t-j}$$

Then we have the following results:

1. $$E(y_t^2) = \frac{\omega^2\sigma_z^2(1 + \theta^2)(1 + a_1)(1 - a^2 - \sigma_v^2)}{(1-a_1)(1-a_2)(1-a^2-2\sigma_v^2-\Phi^2+\Phi^2a^2+\Phi^2\sigma_v^2-\phi^2+\phi^2a^2+\phi^2\sigma_v^2)}$$

2. $$E(y_t^4) = \frac{f_1}{f_2},$$

where

$$f_1 = 3\sigma_z^4\omega^4(1 + \theta^4 + 6\theta^2)(1 + 3a_1 + 5a_2 + 3a_1a_2 + 3a_3 + 5a_1a_3 + 3a_2a_3 + a_1a_2a_3)\frac{(1-a_1)(1-a_2)(1-a_3)}{(1-a_1)(1-a_2)(1-a_3)}$$

$$+ \frac{6\sigma_z^4\omega^4(1 + \theta^2)^2(1 + a_1)^2 [(\phi^2 + \Phi^2)(1 - a^2 - \sigma_v^2) + \sigma_v^2]}{(1-a_1)^2(1-a_2)^2(1-a^2-2\sigma_v^2-\Phi^2+\Phi^2a^2+\Phi^2\sigma_v^2-\phi^2+\phi^2a^2+\phi^2\sigma_v^2)}$$

$$f_2 = 1 - \phi^4 - 3\sigma_v^4\frac{(-1 - a^2 - 5\sigma_v^2 + a^3 + a^5 - 16a^3\sigma_v^2 + 3a\sigma_v^2 - 9a\sigma_v^4)}{(-1 + a^3 + 3a\sigma_v^2)(-1 + a^2 + \sigma_v^2)(-1 + 6a^2\sigma_v^2 + a^4 + 3\sigma_v^4)}$$

$$- 6(\phi^2 + \Phi^2)[\frac{\sigma_v^2}{(1-a^2-2\sigma_v^2)} - \Phi^4 - 6\phi^2\Phi^2]$$
3. The Kurtosis of the process follows from its definition $K(y) = \frac{E(y_t^4)}{[E(y_t^2)]^2}$

Proof. 1. First we calculate the value of $E(y_t^2)$

$$E(y_t^2) = E[\left(\phi + b_t + \Phi s_t\right)y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}]^2$$

$$E(y_t^2) = \phi^2 E(y_{t-1}^2) + \Phi^2 E(y_{t-1}^2) + E(b_t^2)E(y_{t-1}^2) + E(\varepsilon_t^2) + \theta^2 E(\varepsilon_{t-1}^2) + 2E(\phi + b_t + \Phi s_t)E(y_{t-1})E(\varepsilon_t) + 2\theta E(\varepsilon_t)E(\varepsilon_{t-1}) + 2\theta E(\phi + b_t + \Phi s_t)E(y_{t-1})E(\varepsilon_{t-1})$$

Using the second order stationarity condition and by replacing the values of $E(\sigma_t^2)$ and $E(b_t^2)$ we get:

$$E(y_t^2) = \frac{\sigma_z^2(1 + \theta^2)E(\sigma_t^2)}{-(\phi^2 - \Phi^2 - E(b_t^2))}$$

$$E(y_t^2) = \frac{\omega^2 \sigma_z^2(1 + \theta^2)(1 + a_1)(1 - a^2 - \sigma_v^2)}{(1 - a_1)(1 - a_2)(1 - a^2 - 2\sigma_v^2 - \Phi^2 + \Phi^2 a^2 + \Phi^2 \sigma_v^2 - \phi^2 + \phi^2 a^2 + \phi^2 \sigma_v^2)}$$

2. Now we calculate $E(y_t^4)$

$$E(y_t^4) = E[\left(\phi + b_t + \Phi s_t\right)y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}]^4$$

By applying expectation $E$, the simplification of the above step is the same as the one in the previous theorem and we can write:

$$E(y_t^4) = \frac{3\varepsilon_t^4(1 + \theta^2 + \phi^2) + 6\varepsilon_t^2(1 + \theta^2)(\phi^2 + \Phi^2 + E(b_t^2))E(y_{t-1}^2)}{1 - \phi^4 - 3E(b_t^2) - 6(\phi^2 + \Phi^2)E(b_t^2) - \Phi^4 - 6\phi^2 \Phi^2}$$

By replacing the values of $\varepsilon_t^4, \varepsilon_t^2, E(b_t^2), E(b_t^4)$ and then using the second order stationarity condition and the values of $E(\sigma_t^4), E(\sigma_t^2)$ and $E(\varepsilon_t^2)$ we obtain the final expression of $E(y_t^4)$ in terms of $f_1/f_2$ where $f_1$ and $f_2$ are given in the statement of the theorem.

3. The Kurtosis of the process follows from its definition $K(y) = \frac{E(y_t^4)}{[E(y_t^2)]^2}$. □

4. CONCLUSIONS

In this paper, Kurtosis of doubly stochastic models with TGARCH innovations are derived. Statistical inferences to these doubly stochastic models with other asymmetric GARCH errors and state space modeling can be viewed as a special case for nonlinear time series.
REFERENCES


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