VECTOR COMPARISON OPERATORS IN CONE METRIC SPACES

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Communicated by the former editorial board

The aim of this paper is to show that some nonlinear contractive conditions on TVS-cone metric spaces can be reduced to nonlinear contractive conditions on usual metric spaces, extending the results of W.S. Du [4].

AMS 2010 Subject Classification: 47H10, 54H25.

Key words: cone metric space, TVS-cone metric spaces, comparison function, fixed point.

1. INTRODUCTION AND PRELIMINARIES

The Banach-Piccard-Caccioppoli contraction principle plays an important role in several branches of mathematics and applied mathematics. For this reason, it has been extended in many directions (see *e.g.* [10] and references therein). One of such directions is to consider the cone metric spaces instead of metric spaces (see *e.g.* [1, 2, 6] for more details in this topic).

Recently, W.S. Du used in [4] the scalarization function ξ_e and investigated the equivalence of vectorial versions of fixed point theorems in cone metric spaces and scalar versions of fixed point theorems in metric spaces. He showed that if (X, ρ) is a $TVS-$ cone metric space then $d_\rho = \xi_e \circ \rho$ is a metric on X. Thus, many of the fixed point results in cone metric spaces for maps satisfying contractive linear conditions can be considered as the corollaries of corresponding theorems in metric spaces. In this paper, we shall prove that some contractive conditions of nonlinear type on cone metric spaces can be reduced to nonlinear contractive conditions on metric spaces.

For the convenience of the reader we recall some definitions and facts (see [4]). Let Y be a topological vector space (for short TVS) with its zero vector θ.

Definition 1.1 ([4, 6]). A subset K of Y is called a cone whenever the following three assertions hold:

(i) K is closed, nonempty and $K \neq \{\theta\}$;
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in K$ imply $ax + by \in K$;
(iii) $K \cap -K = \{\theta\}$.

If, further, $\text{int}K \neq \emptyset$ we say that K is solid cone.

Let $K \subset Y$ be a cone. We define the partial ordering $\preceq_K$ with respect to K by $x \preceq_K y$ or, more simple, $x \preceq y$, if and only if $y - x \in K$. We shall write $x \ll y$ provided that $y - x \in \text{int}K$ (interior of K). So, $x \not\ll y$ means $y - x \notin \text{int}K$.

Remark 1.1. (1) If $b \not\ll a$ and $b \preceq c$ then $c \not\ll a$.

(2) If $b \ll a$ and $b \not\ll c$, then $a \not\ll c$.

Proof. (1) If we suppose $c \ll a$, then $a - c \in \text{int}K$. It follows

$$a - b = c - b + a - c \subset K + \text{int}K \subset \text{int}K$$

which contradicts $b \not\ll a$.

(2) In the same way, if $a \ll c$, then $c - a \in \text{int}K$. So,

$$c - b = c - a + a - b \in \text{int}K + \text{int}K \subset \text{int}K,$$

contradicting the hypothesis $b \not\ll c$. $\square$

In the sequel, we suppose that Y is a locally convex Hausdorff space with null vector θ and $\mathcal{S}$ its family of seminorms. We consider a solid cone K in Y and $e \in \text{int}K$.

In [4], W.S. Du defines the TVS-cone metric spaces and, based upon the idea of Huang and Zhang [6], the convergence and completeness in the respective spaces as follows:

Definition 1.2. Let X be a nonempty set. Suppose that a map $\rho : X \times X \to Y$ satisfies:

(cm$_1$) $\theta \preceq \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = \theta$ if and only if $x = y$;

(cm$_2$) $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$;

(cm$_3$) $\rho(x, y) \preceq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a TVS-cone metric on X and (X, ρ) is called a TVS-cone metric space.

Definition 1.3. Let (X, ρ) be a TVS-cone metric space, $x \in X$ and $(x_n)_n$ a sequence in X. We say that:

(a) $(x_n)_n$ TVS-cone converges to x whenever, for every $c \in Y$ with $\theta \ll c$, there is a natural number N such that $\rho(x_n, x) \ll c$, for all $n \geq N$;

(b) $(x_n)_n$ is a TVS-cone Cauchy sequence whenever, for every $c \in Y$ with $\theta \ll c$, there is a natural number N such that $\rho(x_m, x_n) \ll c$, for all $m, n \geq N$;

(c) (X, ρ) is TVS-cone complete if every TVS-cone Cauchy sequence in X is TVS-cone convergent.

The following nonlinear scalarization function is of fundamental importance for our survey. The original version is due to Gerstewitz[1] (1983). In

[1]Gerstewitz, Chr. (Tammer), *Nichtkonvexe dualitat in der vektaroptimierung*, Wissenschaftliche Zeitschrift Leuna-merseburg **25** (1983), 357–364

order to define such a function we need first the following lemmas:

Lemma 1.1 ([5], L. 2.1). *One has* $Y = \bigcup_{r>0} (r \cdot e - \mathrm{int}K)$.

Lemma 1.2 ([3], Pr. 1.41). *For every* $y \in Y$, *the set* $\{r \in \mathbb{R},\ r \cdot e - y \in K\}$ *is bounded from below and a closed subset in* $\mathbb{R}$.

Let us define a *scalarization function* $\xi_e : Y \to \mathbb{R}$ given by

$$\xi_e(y) = \inf\{r \in \mathbb{R};\ r \cdot e - y \in K\}. \tag{1.1}$$

The previous lemmas assure that ξ is well defined and, further, we can write

$$\xi_e(y) = \min\{r \in \mathbb{R};\ r \cdot e - y \in K\}.$$

The main properties of ξ_e which are often used in the sequel are revealed in the following lemma.

Lemma 1.3 ([3]). *For each* $r \in \mathbb{R}$ *and* $y \in Y$, *the following statements hold:*

(i) $\xi_e(y) \geq r$ *if and only if* $y \notin r \cdot e - \mathrm{int}K$;
(ii) $\xi_e(y) < r$ *if and only if* $y \in r \cdot e - \mathrm{int}K$;
(iii) $\xi_e(\cdot)$ *is positively homogeneous and continuous on* Y;
(iv) *if* $y_2 \in y_1 + K$ *then* $\xi_e(y_1) \leq \xi_e(y_2)$;
(v) [2] *if* $y_2 \in y_1 + \mathrm{int}K$ *then* $\xi_e(y_1) < \xi_e(y_2)$ *that is* ξ_e *is strictly monotone;*
(vi) $\xi_e(r \cdot e) = r$ *for all* $r \in \mathbb{R}$, *particularly* $\xi_e(\theta) = 0$;
(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$, *for all* $y_1, y_2 \in Y$;
(viii) [3] *if* $\theta \preceq u \ll c$ *for each* $c \in \mathrm{int}K$, *then* $u = \theta$.
(ix) [4] $\xi_e(y) = r \Leftrightarrow y \in r \cdot e - \partial K$, *where* ∂K *means the frontier of the set* K.

The following lemma is crucial for our purpose.

Lemma 1.4 ([4], Th. 2.1). *Let us consider a TVS-metric space* (X, ρ). *Then the map* $\mathrm{d}_\rho : X \times X \to [0, \infty)$ *defined by* $\mathrm{d}_\rho := \xi_e \circ \rho$ *is a metric on* X.

Proof. We proceed to verify the axioms of metric. Thus, by Definition 1.2(cm_1), (cm_2) and Lemma 1.3, one has

$$\mathrm{d}_\rho(x, y) = \xi_e\big(\rho(x, y)\big) \geq 0 \text{ and } \mathrm{d}_\rho(x, y) = \mathrm{d}_\rho(y, x),$$

for all $x, y \in X$.

If $x = y$, then, by (cm_1), $\mathrm{d}_\rho(x, y) = \xi_e(\theta) = 0$. Conversely, $\mathrm{d}_\rho(x, y) = 0$ implies, by taking $r = 0$ in Lemma 1.3(ix), $\rho(x, y) \in K \cap -K = \{\theta\}$, so $x = y$.

[2] see [3] Proposition 1.55
[3] see [9] Remark 1.3(4)
[4] see [3] Proposition 1.43(iii)

The triangle inequality follows using again Lemma 1.3. We have

$$\mathrm{d}_\rho(x,y) = \xi_e\big(\rho(x,y)\big) \leq \xi_e\big(\rho(x,z)+\rho(z,y)\big)$$
$$\leq \xi_e\big(\rho(x,z)\big) + \xi_e\big(\rho(z,y)\big) = \mathrm{d}_\rho(x,z) + \mathrm{d}_\rho(z,y),\ \forall\, x,y,z \in X. \quad \square$$

THEOREM 1.1 ([4], Th. 2.2). *The metric space (X, d_ρ) is complete provided that (X,ρ) is TVS-cone complete.*

2. MAIN RESULTS

We work within the context of the previous section.

Definition 2.1. A map $\varphi : K \to K$ is called a vector comparison operator if the following assertions are satisfied:

(c_1) $k_1 \preceq k_2$ implies $\varphi(k_1) \preceq \varphi(k_2)$;
(c_2) $\varphi(r\cdot e) \ll r\cdot e$ for each $r > 0$;
(c_3) for every $t_0 > 0$ and any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\varphi(t\cdot e) - \varphi(t_0\cdot e) \ll \varepsilon\cdot e,\ \forall\, t \in (t_0, t_0+\delta).$$

Let us consider the following map $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\psi(t) = \xi_e(\varphi(t\cdot e)), \tag{2.1}$$

where $\varphi : K \to K$. Notice that, since $\theta \preceq \varphi(u)$, for any $u \in K$, by using Lemma 1.3 $(iv),(vi)$, it follows that $\psi(t) = \xi_e\big(\varphi(t\cdot e)\big) \geq \xi_e(\theta) = 0$, consequently ψ is well defined.

PROPOSITION 2.1. *Whenever φ satisfies (c_2) from Definition 2.1, the mapping ψ defined in (2.1) satisfies the following properties:*

(i) $\psi(t) < t$ *for all* $t > 0$;
(ii) *is right continuous at* $t = 0$.

Proof. (i) Let be $t > 0$. Since ξ_e is strictly monotone and $t\cdot e \in \mathrm{int}K$ for each $t > 0$, we deduce, using (c_2)

$$\psi(t) = \xi_e\big(\varphi(t\cdot e)\big) < \xi_e(t\cdot e) = t.$$

(ii) We need to show that $\lim\limits_{\substack{t\to 0\\ t>0}} \psi(t) = 0$. We suppose by reductio ad absurdum that there exist $\varepsilon_0 > 0$ and a sequence of real numbers $t_n \searrow 0$ such that $\psi(t_n) = \xi_e(\varphi(t_n\cdot e)) \geq \varepsilon_0$. So, by Lemma 1.3$(i)$, one has $\varphi(t_n\cdot e) \not\ll \varepsilon_0\cdot e$. On the other hand $\varphi(t_n\cdot e) \ll t_n\cdot e$, for any n. Therefore, via Remark 1.1, we get $t_n\cdot e \not\ll \varepsilon_0\cdot e$. However, $t_n \to 0$ implies $\varepsilon_0 - t_n > 0$ for any n greatest than some $n_0 \geq 1$. So, $(\varepsilon_0 - t_n)\cdot e \in \mathrm{int}K$, meaning that $t_n\cdot e \ll \varepsilon_0\cdot e$. This is a contradiction. $\square$

Proposition 2.2. *Assume that φ satisfies $(c_1),(c_2)$ from Definition 2.1. Then $\varphi(\theta)=\theta$.*

Proof. Firstly, we observe that, for some arbitrary $c \in \operatorname{int}K$, there are $\delta_1 > 0$ and a seminorm $s_0 \in \mathcal{S}$ such that $c + \{x : \ s_0(x) < \delta_1\} \subset \operatorname{int}K$, hence $c - x \in \operatorname{int}K$, for any x with $s_0(x) < \delta_1$. Thus, one has

$$x \in Y,\ s_0(x) < \delta_1 \ \Rightarrow\ x \ll c. \tag{2.2}$$

Since $\lim\limits_{\varepsilon\to 0} s_0(\varepsilon\cdot e) = 0$, one can find $\eta > 0$ such that $s_0(\varepsilon \cdot e) < \delta_1$, so, by (2.2),

$$\varepsilon \cdot e \ll c, \quad \forall\, \varepsilon \in (0,\eta). \tag{2.3}$$

Next, for some $\varepsilon_0 \in (0,\eta)$, from the right continuity of ψ (Proposition 2.1), it follows that there is $\delta_2 > 0$, such that $\psi(t) < \varepsilon_0$, for all $t \in (0,\delta_2)$. Hence, $\xi_e\big(\varphi(t\cdot e)\big) < \varepsilon_0$, that is, according to Lemma 1.3 (ii),

$$\varphi(t\cdot e) \ll \varepsilon_0 \cdot e, \quad \forall\, t \in (0,\delta_2). \tag{2.4}$$

Consequently, according to (c_1), (2.3), (2.4)

$$\theta \preceq \varphi(\theta) \preceq \varphi(t\cdot e) \ll \varepsilon_0 \cdot e \ll c$$

thence,

$$\theta \preceq \varphi(\theta) \ll c.$$

Finally, by using now Lemma 1.3 $(viii)$, we obtain $\varphi(\theta)=\theta$, completing the proof. $\square$

Lemma 2.1 ([9], Rem. 1.4). *If $c \in \operatorname{int}K$, $\theta \preceq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.*

Notice that the converse assertion from the previous lemma is not generally true as follows by considering in the Banach space $Y := \mathcal{C}^1_{\mathbb{R}}[0,1]$ endowed with $\|x\| = \|x\|_\infty + \|x'\|_\infty$, the cone $K = \big\{x \in \mathcal{C}^1_{\mathbb{R}}[0,1];\ x(t) \geq 0,\ \forall\, t \in [0,1]\big\}$ and the sequence $x_n(t) := \frac{t^n}{n} \in K$.

Proposition 2.3. *Let us assume that $\varphi : K \to K$ verifies (c_1) from Definition 2.1. If anyone of the following assertions occurs*

(α) φ satisfies a right continuity condition, i.e.

$$\lim_{t \searrow t_0} \varphi(t\cdot e) = \varphi(t_0 \cdot e); \tag{2.5}$$

or

(β) φ satisfies, in addition, (c_2) and the following subadditivity condition

$$\varphi(t_1\cdot e + t_2 \cdot e) \preceq \varphi(t_1\cdot e) + \varphi(t_2\cdot e),\ \forall\, t_1,t_2 > 0, \tag{2.6}$$

then φ satisfies (c_3).

Proof. (α) We proceed by reductio ad absurdum. Suppose that there are $t_0 > 0$ and $\varepsilon_0 > 0$ such that for each $n = 1, 2, \ldots,$ one can find $t_n \searrow t_0$ with $\varphi(t_n \cdot e) - \varphi(t_0 \cdot e) \not\ll \varepsilon_0 \cdot e$. At the same time, by (2.5) and the preceding lemma, putting $c = \frac{1}{2}\,\varepsilon_0 \cdot e \in \operatorname{int}K$, there is a natural number N such that

$$\varphi(t_n \cdot e) - \varphi(t_0 \cdot e) \ll \frac{1}{2} \cdot \varepsilon_0 \cdot e.$$

From Remark 1.1(2), we deduce $\frac{1}{2}\,\varepsilon_0 \cdot e \not\ll \varepsilon_0 \cdot e$. This is a contradiction.

(β) Let $t_0 > 0$ and $\varepsilon > 0$ be arbitrary chosen. By the same argument as in the proof of the preceding proposition, one can find $\delta > 0$ such that $\varphi(t \cdot e) \ll \varepsilon \cdot e$, for all $t \in (0, \delta)$. By hypothesis, one has

$$\varphi(t \cdot e) - \varphi(t_0 \cdot e) \preceq \varphi(t \cdot e - t_0 \cdot e) + \varphi(t_0 \cdot e) - \varphi(t_0 \cdot e) =$$

$$\varphi\big((t - t_0) \cdot e\big) \ll \varepsilon \cdot e, \quad \forall\, t \in (t_0, t_0 + \delta). \quad \square$$

Corollary 2.1. *If φ satisfies* (2.6)*, then then map ψ defined in (2.1) is subadditive, that is $\psi(t_1 + t_2) \leq \psi(t_1) + \psi(t_2)$, for all $t_1, t_2 > 0$.*

Proof. Let be $t_1, t_2 > 0$. Then, using Lemma 1.3 $(iv), (vii)$, we get

$$\psi(t_1 + t_2) = \xi_e[\varphi(t_1 \cdot e + t_2 \cdot e)] \leq$$

$$\leq \xi_e[\varphi(t_1 \cdot e) + \varphi(t_2 \cdot e)] \leq \xi_e(\varphi(t_1 \cdot e)) + \xi_e(\varphi(t_2 \cdot e)) = \psi(t_1) + \psi(t_2). \quad \square$$

Definition 2.2. A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a (scalar) comparison function provided that the following two assertions hold
(s_1) $0 \leq t_1 \leq t_2$ implies $\psi(t_1) \leq \psi(t_2)$;
(s_2) $\psi^n(t) \to 0$ for all $t > 0$.

The most important (scalar) comparison function is $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(t) = ct$, where $0 < c < 1$.

The following theorem gives a crucial result for our work.

Theorem 2.1 ([8]). *Let (X, d) a complete metric space and $f : X \to X$ be a $\psi-$contraction, i.e. ψ is a (scalar) comparison function and*

$$\mathrm{d}\big(f(x), f(y)\big) \leq \psi\big(\mathrm{d}(x, y)\big), \quad \textit{for all } x, y \in X.$$

Then f has a unique fixed point x^*. Further, for each $x \in X$, $f^n(x) \underset{n}{\longrightarrow} x^*$.

Theorem 2.2. *If $\varphi : K \to K$ is a vector comparison operator, then ψ defined via (2.1) is a scalar comparison function.*

Proof. The assertion (s_1) comes obviously from (c_1) and Lemma 1.3 (iv).

We divide the proof of (s_2) into two steps.

First step: we establish that ψ is right continuous at any $t_0 > 0$. For this purpose, take some $t_0 > 0$ and let be $\varepsilon > 0$. By (c_3), there is $\delta > 0$ such that

$$\psi(t) - \psi(t_0) = \xi_e\big(\varphi(t \cdot e)\big) - \xi_e\big(\varphi(t_0 \cdot e)\big) \le$$

$$\xi_e\big(\varphi(t \cdot e) - \varphi(t_0 \cdot e)\big) < \xi_e(\varepsilon \cdot e) = \varepsilon, \ \ \forall\, t \in (t_0, t_0 + \delta),$$

where, in the first inequality we have used Lemma 1.3 (vii) and in the second one, the axiom (c_3).

Second step: let be $t > 0$. Using Proposition 2.1(i) and (s_1), we deduce that $0 \le \psi^{n+1}(t) \le \psi^n(t) \le t$, for any $n \ge 1$. So there is $x = \lim\limits_n \psi^n(t) \ge 0$. If $x > 0$ then $\psi(x) < x$. On the other hand, from the right continuity of ψ at x, one obtain $x \le \psi(\psi^n(t)) \to \psi(x)$. Thus, $x \le \psi(x)$ which is a contradiction. Hence, $x = 0$.

Consequently, ψ is a scalar comparison function. $\square$

THEOREM 2.3. *Let us consider a TVS-cone metric space (X, ρ) and an operator $T : X \to X$. Suppose that $\varphi : K \to K$ is a vector comparison operator such that*

$$\rho\big(T(x), T(y)\big) \preceq \varphi\big(\rho(x, y)\big), \ \ \textit{for all} \ \ x, y \in X.$$

Then

$$\mathrm{d}_\rho\big(T(x), T(y)\big) \le \psi\big(\mathrm{d}_\rho(x, y)\big), \ \ \textit{for all} \ \ x, y \in X,$$

ψ being defined in (2.1).

Further, whenever (X, ρ) is TVS-cone complete, T has a unique fixed point x^*. Moreover, for each $x \in X$, the iterative sequence $T^n(x)$ TVS-cone converges to x^*.

Proof. Notice that, given $z \in Y$, by (1.1), there is a sequence of real numbers $(r_n)_n$ which converges to $\xi_e(z)$ such that $z \in r_n \cdot e - K$. K being closed, one deduce $\xi_e(z) \cdot e - z \in K$, that is $z \preceq \xi_e(z) \cdot e$.

Let $x, y \in X$. Taking $z = \rho(x, y)$, we obtain $\rho(x, y) \preceq \xi_e\big(\rho(x, y)\big) \cdot e$.

Thereby,

$$\mathrm{d}_\rho\big(T(x), T(y)\big) = \xi_e\big(\rho(T(x), T(y))\big) \le \xi_e\big(\varphi(\rho(x, y))\big)$$

$$\le \xi_e\Big(\varphi\big(\xi_e(\rho(x, y)) \cdot e\big)\Big) = \xi_e \circ \varphi\big(\mathrm{d}_\rho(x, y) \cdot e\big) = \psi\big(\mathrm{d}_\rho(x, y)\big).$$

This means that T is a scalar ψ-contraction in the metric space (X, d_ρ).

Let suppose now that (X, ρ) is a TVS-cone complete metric space. Then, in view of Theorem 1.1, the metric space (X, d_ρ) is also complete, hence, by Matkowski's Theorem (Theorem 2.1), there is a unique $x^* \in X$ such that $T(x^*) = x^*$ and, more, if $x \in X$, that $T^n(x) \underset{n}{\longrightarrow} x^*$ (with respect to d_ρ).

It remains to prove that $\big(T^n(x)\big)_n$ TVS-cone converges to x^*. For this purpose, choose $c \in \mathrm{int}K$. We consider (2.3) from the first part of the proof of

Proposition 2.2. Thus, there exists $\eta > 0$ such that $\varepsilon \cdot e \ll c$ for any $\varepsilon \in (0, \eta)$. For such an ε, there is a natural number N such that

$$\mathrm{d}_\rho(T^n(x), x^*) = \xi_e\big(\rho(T^n(x), x^*)\big) < \varepsilon, \ \forall\, n \geq N.$$

Lemma 1.3 (ii) implies $\rho(T^n(x), x^*) \ll \varepsilon \cdot e \ll c$, for all $n \geq N$, as required. $\square$

Remark 2.1. If $\varphi(k) = \lambda \cdot k$, $\lambda \in [0, 1)$, then we obtain Theorem 2.3 of W.S. Du [4].

Remark 2.2. Let (X, p) a cone metric space. When $\varphi(k) = \lambda \cdot k$, $\lambda \in [0, 1)$, then one obtain the results of L.G. Huang and Zhang Xian [6].

In what follows, we give some examples of vector comparison operators.

Example 2.1. If K is an arbitrary cone in a Banach space E and $\lambda \in (0, 1)$, then $\varphi : K \to K$ defined by $\varphi(k) = \lambda k$ is a vector comparison operator.

Example 2.2. We consider the Euclidian space $E = \mathbb{R}^2$, $K = \big\{(x, y) \mid x, y \geq 0\big\}$ and $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ two arbitrary scalar continuous comparison mappings, *e.g.* ψ_1, ψ_2 may have one of the forms $t \mapsto \dfrac{t}{t+1}$, $t \mapsto \dfrac{\alpha t}{\alpha t + \alpha + 1}$, $\alpha > 0$, $t \mapsto \ln(t + 1)$ and so on. Then $\varphi : K \to K$, defined by $\varphi(x, y) = \big(\psi_1(x), \psi_2(y)\big)$ is a vector comparison operator.

Proof. The conditions $(c_1), (c_2)$ are simple to check and (c_3) follows immediately from Proposition 2.3 (α). $\square$

Example 2.3. Let $Y := \mathcal{C}_{\mathbb{R}}[0, 1]$ be the Banach space of continuous real valued functions on the unit interval $[0, 1]$ endowed with the uniform metric and set $K := \big\{x \in \mathcal{C}_{\mathbb{R}}[0, 1];\ x \geq 0\big\}$. Then K is solid cone in Y and $\varphi : K \to K$ defined by $\varphi(x) = \ln(x + 1)$ is a vector comparison operator, for any choice of $e \in \mathrm{int}K$.

Proof. Clearly K is solid cone. Next, it is also easy to check that φ obeys the axioms $(c_1), (c_2)$ from Definition 2.1. To establish (c_3) we show that φ satisfies the condition (2.6) and next we apply Proposition 2.3 (β).

So, choose $e \in \mathrm{int}K$ and $t_1, t_2 > 0$. Firstly, we observe that, for any $\tau \in [0, 1]$, one has

$$t_1 \cdot e(\tau) + t_2 \cdot e(\tau) + 1 \leq \big(t_1 \cdot e(\tau) + 1\big)\big(t_2 \cdot e(\tau) + 1\big).$$

Thereby,

$$\varphi(t_1 \cdot e + t_2 \cdot e) \preceq \varphi(t_1 \cdot e) + \varphi(t_2 \cdot e). \quad \square$$

APPLICATION

In the sequel we investigate the (countable) iterated function systems on a cone metric space consisting of φ-contractions, where φ is a vector comparison operator. We shall show that such a systems of functions has an attractor.

Let us consider a solid cone K in a locally convex Hausdorff space and (X, ρ) a TVS-cone metric space. For some fixed $e \in \mathrm{int}K$, we also consider the metric space (X, d_ρ) defined above.

For each $c \in \mathrm{int}K$ and $x \in X$ let us define the ρ-open ball with center at x and radius c as follows

$$\mathrm{B}_\rho(x, c) = \{y \in X: \ \rho(x, y) \ll c\}.$$

We further denote, for some $\varepsilon > 0$, by $\mathrm{B}_{\mathrm{d}_\rho}(x, \varepsilon)$ the open ball centered at x with radius ε in metric space (X, d_ρ).

The topology on X induced by the cone metric ρ is given by[5]

$$\tau_\rho = \big\{\emptyset\big\} \cup \big\{D \subset X: \ \forall\, x \in D, \ \exists\, c \in K \text{ such that } \mathrm{B}_\rho(x, c) \subset D\big\},$$

while the metric topology generated by d_ρ will be denoted by τ_{d_ρ}.

Proposition 2.4. *One has $\tau_\rho = \tau_{\mathrm{d}_\rho}$.*

Proof. To establish the equality from the statement it is enough to prove that, for every $x \in X$, any ρ-open ball centered at x include a d_ρ-open ball with center at x and conversely.

To this end, let us take firstly $c \in \mathrm{int}K$ and $x \in X$. We prove that there is $\varepsilon_0 > 0$ such that $\mathrm{B}_{\mathrm{d}_\rho}(x, \varepsilon_0) \subset \mathrm{B}_\rho(x, c)$. Arguing by contradiction we assume that, for any $\varepsilon > 0$, one can find $y_\varepsilon \in \mathrm{B}_{\mathrm{d}_\rho}(x, \varepsilon)$ such that $y_\varepsilon \notin \mathrm{B}_\rho(x, c)$. So, for each $n = 1, 2, \ldots$, there is $y_n \in \mathrm{B}_{\mathrm{d}_\rho}(x, \frac{1}{n})$,

$$y_n \notin \mathrm{B}_\rho(x, c). \tag{2.7}$$

Since $\mathrm{d}_\rho(x, y_n) < \frac{1}{n}$, according to Lemma 1.3 (ii), one obtains

$$\rho(x, y_n) \ll \frac{1}{n} \cdot e, \quad \text{for all} \ \ n \geq 1.$$

By using the same argument as in the proof of Proposition 2.2 (2.3), we deduce that there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n} \cdot e \ll c$, for any $n \geq n_0$. Thus, $\rho(x, y_n) \ll c$, for every $n \geq n_0$, contradicting (2.7).

In order to continue we shall show that, for a given $x \in X$ and $\varepsilon > 0$, there exists $c \in \mathrm{int}K$ such that $\mathrm{B}_\rho(x, c) \subset \mathrm{B}_{\mathrm{d}_\rho}(x, \varepsilon)$. Thus, we put $c = \varepsilon \cdot e$ and choose $y \in \mathrm{B}_\rho(x, c)$, that is $\rho(x, y) \ll \varepsilon \cdot e$. Lemma 1.3 (v) implies that

$$\mathrm{d}_\rho(x, y) = \xi_e\big(\rho(x, y)\big) < \xi_e(\varepsilon \cdot e) = \varepsilon.$$

[5] [9, 2.2.]

Hence, $y \in \mathrm{B}_{\mathrm{d}_\rho}(x, \varepsilon)$.

Accordingly, the two topologies coincide. $\square$

In the sequel we briefly recall some basic facts[6] concerning the theory of Iterated Function Systems (abbreviated IFS) and of Countable Iterated Function Systems (CIFS).

Let us denote by $\mathcal{K}(X)$ the collection of all nonempty compact sets of a complete metric space (X, d). We equip $\mathcal{K}(X)$ with the Hausdorff-Pompeiu metric d_H. This new metric space is complete (resp. compact) provided that (X, d) is complete (compact). An *iterated function system* (IFS) consists of a finite set of contraction mappings $\omega_n : X \to X$, $n = 1, 2, \ldots, N$. We define the *Hutchinson operator* $\mathcal{S} : \mathcal{K}(X) \to \mathcal{K}(X)$, $\mathcal{S}(B) = \bigcup\limits_{1 \le n \le N} \omega_n(B)$. It is known that $\mathcal{S}$ is a contraction map. By Banach's contraction principle, it follows that there exists a unique $A \in \mathcal{K}(X)$ such that $\mathcal{S}(A) = A$, named the *attractor* of IFS $(\omega_n)_{n=1}^N$.

Analogously, if (X, d) is compact, then a sequence $(\omega_n)_{n \ge 1}$ of contraction maps on X into itself having the supremum of its ratios less than 1 is called a *countable iterated function System* (CIFS). The Hutchinson operator will be in this case $\mathcal{S}(B) = \overline{\bigcup\limits_{n \ge 1} \omega_n(B)}$ and its unique set "fixed point" is called the *attractor* of the considered CIFS.

In both IFS and CIFS cases, the attractor A is approximated in $\big(\mathcal{K}(X), \mathrm{d}_H\big)$ by $\big(\mathcal{S}^p(B)\big)_p$, for any $B \in \mathcal{K}(X)$, where $\mathcal{S}^p$ means p-time composition of $\mathcal{S}$.

The following theorems are extensions of a classical results proved in the case when ω_n are φ_n-contractions instead of contractions (φ_n being a (scalar) comparison function).

THEOREM 2.4 ([11], Th. 4.1). *Let us suppose that (X, d) is a complete metric space and, for $n = 1, 2, \ldots, N$, ω_n is a φ_n-contraction. Then IFS $(\omega_n)_{n=1}^N$ has an attractor $A \in \mathcal{K}(X)$. Furthermore, $\mathcal{S}^p(B) \underset{p}{\to} A$, for all $B \in \mathcal{K}(X)$.*

THEOREM 2.5 ([11], Th. 4.6.). *Assume that (X, d) is a compact metric space and, for every $n \ge 1$, ω_n is a φ_n-contraction, where $(\varphi_n)_{n \ge 1}$ are comparison functions. If, besides, $\sup\limits_{n \ge 1} \varphi_n(t) < t$ for all $t > 0$, then the CIFS $(\omega_n)_n$ has an attractor $A \in \mathcal{K}(X)$ which is successively approximated by $\big(\mathcal{S}^p(B)\big)_p$, for any $B \in \mathcal{K}(X)$.*

[6] More details about this topics can be found in: J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747 and, respectively, N.A. Secelean, *Countable Iterated Function Systems*, Far East J. Dyn. Syst. **3**(2) (2001), 149–167.

As an application of our main results, we now give a generalization of IFS considering the mappings ω_n on some TVS-cone metric space.

THEOREM 2.6. *For $n = 1, 2, \dots, N$ let $\varphi_n : K \to K$ be some vector comparison operators and $\omega_n : X \to X$ be such that*

$$\rho\big(\omega_n(x), \omega_n(y)\big) \preceq \varphi_n\big(\rho(x, y)\big), \quad \textit{for all} \ \ x, y \in X.$$

If (X, ρ) is a complete TVS-cone metric space, then there is a unique nonempty compact set $A \subset X$ such that $A = \bigcup_{1 \le n \le N} \omega_n(A)$. Furthermore, for every compact $\emptyset \ne B \subset X$, $\mathcal{S}^p(B) \underset{p}{\to} A$ the convergence being considered with respect to the Hausdorff-Pompeiu metric defined by means of a certain metric on X.

Proof. Proposition 2.4 assures that a set is compact with respect to τ_ρ if and only if it is compact in the topology τ_{d_ρ}. According to Theorem 1.1, we infer that the metric space (X, d_ρ) is complete.

Next, from Theorem 2.3 one deduces that each ω_n is a ψ_n-contraction in the complete metric space (X, d_ρ), where ψ_n is defined in (2.1) and represents a scalar comparison mapping as it follows from Theorem 2.2. The conclusion now comes by applying Theorem 2.4. $\square$

THEOREM 2.7. *We assume that (X, ρ) is a compact TVS-cone metric space and, for each $n \ge 1$, $\omega_n : X \to X$ is such that*

$$\rho\big(\omega_n(x), \omega_n(y)\big) \preceq \varphi\big(\rho(x, y)\big), \quad \textit{for all} \ \ x, y \in X.$$

Then there exists a unique nonempty compact set $A \subset X$ such that $A = \overline{\bigcup_{n \ge 1} \omega_n(A)}$. Furthermore, for every compact $\emptyset \ne B \subset X$, $\mathcal{S}^p(B) \underset{p}{\to} A$ with respect to the Hausdorff-Pompeiu metric defined by means of a certain metric on X.

Proof. We use the same argument as in the proof of the previous theorem. Thus, any compact set of (X, ρ) is also compact in (X, d_ρ) and conversely. So, the metric space (X, d_ρ) is compact too.

Next, all mappings ω_n $(n \ge 1)$ defined in (2.1), are ψ-contractions in (X, d_ρ). The conclusion now follows from Theorem 2.5. $\square$

REFERENCES

[1] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces.* J. Math. Anal. Appl. **341** (2008), 416–420.

[2] I. Arandjelović, Z. Kadelburg and S. Radenović, *Boyd-Wong-type common fixed point results in cone metric spaces.* Appl. Math. Comput. **217** (2011), 7167–7171.

[3] G.Y. Chen, X.X. Huang and X.Q. Yang, *Vector Optimization.* Springer-Verlag, Berlin, Heidelberg, Germany, 2005.

[4] Wei-Shih Du, *A note on cone metric fixed point theory and its equivalence.* Nonlinear Anal. **72** (2010), 2259–2261.

[5] C. Gerth and P. Weidner, *Nonconvex separation theorems and some applications in vector optimization.* J. Optim. Theory Appl. **67** (1990), *2*, 297–320.

[6] L.G. Huang and X. Zhang, *Cone metric spaces and fixed point theorems of contractive mappings.* J. Math. Anal. Appl. **332** (2007), 1468–1476.

[7] G. Jungck, *Common fixed points for noncontinuous nonself maps on non-metric spaces*, Far East J. Math. Sci. **4** (1996), 199–215.

[8] J. Matkowski, *Integrable solutions of functional equations.* Dissertationes Math. **127** (1975), 1–68.

[9] S. Radenović and Z. Kadelburg, *Quasi-Contractions on symetric and cone symetric spaces.* Banach J. Math. Anal. **1** (2011), 38–50.

[10] I.A. Rus, A. Petruşel and G. Petruşel, *Fixed Point Theory.* Cluj University Press, 2008.

[11] N.A. Secelean, *The existence of the attractor of countable iterated function systems.* Mediterr. J. Math. **9** (2012), *1*, 65–84.

Received 5 November 2012

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