SOME PROPERTIES OF SYMBOL ALGEBRAS
OF DEGREE THREE

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In this paper, we study some properties of the matrix representations of the symbol algebras of degree three. Moreover, we study some equations with coefficients in these algebras, we define the Fibonacci symbol elements and we obtain some of their properties.

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0. PRELIMINARIES

Let $n$ be an arbitrary positive integer, let $K$ be a field whose $\text{char}(K)$ does not divide $n$ and contains $\omega$, a primitive $n$-th root of the unity. Let $K^* = K \setminus \{0\}$, $a, b \in K^*$ and let $S$ be the algebra over $K$ generated by elements $x$ and $y$ where

$$x^n = a, y^n = b, yx = \omega xy.$$ 

This algebra is called a symbol algebra (also known as a power norm residue algebra) and it is denoted by $\left(\frac{a, b}{K, \omega}\right)$. J. Milnor, in [13], calls it ”symbol algebra” because of its connection with the $K$-theory and with the Steinberg symbol. For $n = 2$, we obtain the quaternion algebra. For details about Steinberg symbol, the reader is referred to [11] and for other details about quaternions, the reader is referred to [14] and [16].

In the paper [2], using the associated trace form for a symbol algebra, the author studied some properties of such objects and gave some conditions for a symbol algebra to be with division or not. Starting from these results, we intend to find some examples of division symbol algebras. Since such an example is not easy to provide, we try to find first sets of invertible elements in a symbol algebra and study their algebraic properties and structure (see, for example, Proposition 4.4).

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In the special cases of quaternion algebras, octonion algebras and symbol algebras, the definition of a division algebra is equivalent to the fact that all its nonzero elements are invertible (since these algebras admit a type of norm \( n \) that permits composition, \( i.e. \ n(ab) = n(a)n(b) \), for all elements \( a, b \) in such an algebra). From this idea, for a symbol algebra with \( a = b = 1 \) (Theorem 4.7), we proved that all Fibonacci symbol elements are invertible. This set is included in the set of the elements denoted by \( M \) from Proposition 4.4, in which we proved that \( M \) is a \( Z \)-module. We intend to complete this set with other invertible elements such that the obtained set to be an algebra not only a \( Z \)-module and we intend to study some properties of this module (if there exist some properties similar with the properties of the modules from [18]).

The study of symbol algebras in general, and of degree three in particular, involves very complicated calculus and, usually, can be hard to find examples for some notions.

Hence, the present paper is rather technical, which is however unavoidable given the subject.

The paper is structured in four sections. In the first section, some definitions and general properties of these algebras are introduced. Since the theory of symbol algebras has many applications in different areas of mathematics and other applied sciences, in sections two, we studied matrix representations of symbol algebras of degree three and, in section three, we used some of these properties to solve some equations with coefficients in these algebras.

Since symbol algebras generalize the quaternion algebras, starting from some results given in the paper [9], in which the author defined and studied Fibonacci quaternions, we define in a similar manner the Fibonacci symbol elements and we study their properties. We computed the formula for the reduced norm of a Fibonacci symbol element (Proposition 4.5) and, using this expression, we find an infinite set of invertible elements (Theorem 4.7).

1. INTRODUCTION

In the following, we assume that \( K \) is a commutative field with \( char K \neq 2, 3 \) and \( A \) is a finite dimensional algebra over the field \( K \). The center \( C(A) \) of an algebra \( A \) is the set of all elements \( c \in A \) which commute and associate with all elements \( x \in A \). An algebra \( A \) is a simple algebra if \( A \) is not a zero algebra and \( \{0\} \) and \( A \) are the only ideals of \( A \). The algebra \( A \) is called central simple if the algebra \( A_F = F \otimes_K A \) is simple for every field extension \( F \) of \( K \). Equivalently, a central simple algebra is a simple algebra with \( C(A) = K \). We remark that each simple algebra is central simple over its center. If \( A \) is a central simple algebra, then \( \text{dim } A = n = m^2 \), with \( m \in \mathbb{N} \). The degree of the central simple algebra \( A \), denoted by \( \text{Deg }A \), is \( \text{Deg }A = m \).
If $A$ is an algebra over the field $K$, a subfield of the algebra $A$ is a subalgebra $L$ of $A$ such that $L$ is a field. The subfield $L$ is called a maximal subfield of the algebra $A$ if there is not a subfield $F$ of $A$ such that $L \subset F$. If the algebra $A$ is a central simple algebra, the subfield $L$ of the algebra $A$ is called a strictly maximally subfield of $A$ if $[L : K] = m$, where $[L : K]$ is the degree of the extension $K \subset L$.

Let $L \subset M$ be a field extension. This extension is called a cyclic extension if it is a Galois extension and the Galois group $G(M/L)$ is a cyclic group. A central simple algebra $A$ is called a cyclic algebra if there is $L$, a strictly maximally subfield of the algebra $A$, such that $L/K$ is a cyclic extension.

**Proposition 1.1** ([14], Proposition a, p. 277). Let $K \subset L$ be a cyclic extension with the Galois cyclic group $G = G(L/K)$ of order $n$ and generated by the element $\sigma$. If $A$ is a cyclic algebra and contains $L$ as a strictly maximally subfield, then there is an element $x \in A - \{0\}$ such that:

1. $A = \bigoplus_{0 \leq j \leq n-1} x^j L$;
2. $x^{-1} \gamma x = \sigma(\gamma)$, for all $\gamma \in L$;
3. $x^n = a \in K^*$.

We will denote a cyclic algebra $A$ with $(L, \sigma, a)$.

We remark that a symbol algebra is a central simple cyclic algebra of degree $n$. For details about central simple algebras and cyclic algebras, the reader is referred to [14].

**Definition 1.2** ([14]). Let $A$ be an algebra over the field $K$. If $K \subset L$ is a finite field extension and $n$ a natural number, then a $K$-algebra morphism $\varphi : A \to \mathcal{M}_n(L)$ is called a representation of the algebra $A$. The $\varphi$-characteristic polynomial of the element $a \in A$ is $P_\varphi(X, a) = \det(XI_n - \varphi(a))$, the $\varphi$-norm of the element $a \in A$ is $\eta_\varphi(a) = \det \varphi(a)$ and the $\varphi$-trace of the element $a \in A$ is $\tau_\varphi(a) = tr(\varphi(y))$. If $A$ is a $K-$central simple algebra such that $n = \text{Deg } A$, then the representation $\varphi$ is called a splitting representation of the algebra $A$.

**Remark 1.3.** i) If $X, Y \in \mathcal{M}_n(K)$, $K$ an arbitrary field, then we know that $tr(X^t) = tr(X)$ and $tr(X^tY) = tr(XY^t)$. It results that $tr(XYX^{-1}) = tr(Y)$.

ii) ([14], p. 296) If $\varphi_1 : A \to \mathcal{M}_n(L_1)$, $\varphi_2 : A \to \mathcal{M}_n(L_2)$ are two splitting representations of the $K-$algebra $A$, then $P_{\varphi_1}(X, a) = P_{\varphi_2}(X, a)$. It results that the $\varphi_1$-characteristic polynomial is the same with the $\varphi_2$-characteristic polynomial, $\varphi_1$-norm is the same with $\varphi_2$-norm, the $\varphi_1$-trace is the same with $\varphi_2$-trace and we will denote them by $P(X, a)$ instead of $P_{\varphi}(X, a)$, $\eta_{A/K}$ instead of $\eta_{\varphi_1}$ or simply $\eta$ and $\tau$ instead of $\tau_{\varphi}$, when is no confusion in
notation. In this case, the polynomial \( P(X, a) \) is called the *characteristic polynomial*, the norm \( \eta \) is called the *reduced norm* of the element \( a \in A \) and \( \tau \) is called the *trace* of the element \( a \in A \).

**Proposition 1.4 ([14], Corollary a, p. 296).** If \( A \) is a central simple algebra over the field \( K \) of degree \( m \), \( \phi : A \to M_r(L) \) is a matrix representation of \( A \), then \( m | r \), \( \eta_\phi = \eta^{r/m} \), \( \tau_\phi = (r/m) \tau \) and \( \eta_\phi(a), \tau_\phi(a) \in K \), for all \( a \in A \).

2. **Matrix Representations for the Symbol Algebras of Degree Three**

Let \( \omega \) be a cubic root of unity, \( K \) be a field such that \( \omega \in K \) and \( S = (\frac{a, b}{K, \omega}) \) be a symbol algebra over the field \( K \) generated by elements \( x \) and \( y \) where

\[
(2.1) \quad x^3 = a, y^3 = b, xy = \omega yx, a, b \in K^*.
\]

In [17], the author gave many properties of the left and right matrix representations for the real quaternion algebra. Since symbol algebras generalize the quaternion algebras, using some ideas from this paper, in the following, we will study the left and the right matrix representations for the symbol algebras of degree 3.

A basis in the algebra \( S \) is

\[
(2.2) \quad B = \{1, x, x^2, y, y^2, xy, x^2y, x^2y^2, xy^2\},
\]

(see [2, 4, 11, 14]).

Let \( z \in S \),

\[
(2.3) \quad z = c_0 + c_1 x + c_2 x^2 + c_3 y + c_4 y^2 + c_5 xy + c_6 x^2y + c_7 x^2 y^2 + c_8 xy^2
\]

and \( \Lambda(z) \in \mathcal{M}_9(K) \) be the matrix with the coefficients in \( K \) which its columns are the coordinates of the elements \( \{z \cdot 1, zx, zx^2, zy, zy^2, zxy, zx^2y, zx^2y^2, zxy^2\} \) in the basis \( B \):

\[
\Lambda(z) = \begin{pmatrix}
c_0 & ac_2 & ac_1 & bc_4 & bc_3 & ab\omega^2c_6 & ab\omega^2c_5 & ab\omega c_8 & ab\omega c_7 \\
c_1 & c_0 & ac_2 & bc_8 & bc_5 & ab\omega^2c_4 & ab\omega^2c_7 & ab\omega c_6 & b\omega c_3 \\
c_2 & c_1 & c_0 & bc_6 & bc_7 & b\omega^2c_8 & aw^2c_3 & b\omega c_4 & b\omega c_5 \\
c_3 & a\omega c_7 & a\omega^2c_5 & c_0 & bc_4 & ac_2 & ab\omega c_8 & ac_1 & a\omega c_2 \\
c_4 & a\omega^2c_6 & a\omega c_8 & c_3 & c_0 & a\omega c_7 & ac_1 & a\omega^2c_5 & ac_2 \\
c_5 & \omega c_3 & a\omega^2c_7 & c_1 & bc_8 & c_0 & ab\omega c_6 & ac_2 & b\omega^2c_4 \\
c_6 & \omega^2c_8 & \omega c_4 & c_7 & c_2 & \omega c_5 & c_0 & \omega^2c_3 & c_1 \\
c_7 & \omega c_5 & \omega^2c_3 & c_2 & bc_6 & c_1 & b\omega c_4 & c_0 & b\omega^2c_8 \\
c_8 & \omega^2c_4 & a\omega c_6 & c_5 & c_1 & \omega c_3 & ac_2 & a\omega^2c_7 & c_0
\end{pmatrix}.
\]
Let $\alpha_{ij} \in M_3(K)$ be the matrix with 1 in position $(i,j)$ and zero in the rest and
\[
\gamma_1 = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
\beta_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}, \quad \beta_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \omega & 0 & 0 \end{pmatrix}.
\]

**Proposition 2.1.** The map $\Lambda : S \to M_9(K)$, $z \mapsto \Lambda(z)$ is a $K$-algebra morphism.

**Proof.** With the above notations, let $\Lambda(x) = X = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \alpha_{31} & a\beta_2 \\ 0 & \beta_1 & \alpha_{13} \end{pmatrix} \in \mathcal{M}_9(K)$ and $\Lambda(y) = Y = \begin{pmatrix} 0 & b\alpha_{12} & \omega b\beta_4 \\ \beta_3 & \alpha_{21} & 0 \\ \omega^2 \alpha_{23} & \omega \alpha_{33} & \omega^2 \alpha_{12} \end{pmatrix}$.

By straightforward calculations, we obtain:
\[
\Lambda(x^2) = X^2, \quad \Lambda(y^2) = Y^2, \quad \Lambda(xy) = XY,
\]
\[
\Lambda(x^2y) = \Lambda(x^2)\Lambda(y) = \Lambda(x)\Lambda(xy) = X^2Y,
\]
\[
\Lambda(xy^2) = \Lambda(x)\Lambda(y^2) = \Lambda(xy)\Lambda(y) = XY^2,
\]
\[
\Lambda(x^2y^2) = \Lambda(x^2)\Lambda(y^2) = \Lambda(x)\Lambda(xy^2) = \Lambda(x^2y)\Lambda(y) = X^2Y^2.
\]

Therefore, we have $\Lambda(z_1z_2) = \Lambda(z_1)\Lambda(z_2).$ It results that $\Lambda$ is a $K$-algebra morphism. \qed

The morphism $\Lambda$ is called the left matrix representation for the algebra $S$.

**Definition 2.2.** For $Z \in S$, we denote by $\overrightarrow{Z} = (c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)^t \in \mathcal{M}_{9\times 1}(K)$ the vector representation of the element $Z$.

**Proposition 2.3.** Let $Z, A \in S$, then:

i) $\overrightarrow{Z} = \Lambda(Z)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $0 \in \mathcal{M}_{8\times 1}(K)$ is the zero matrix.

ii) $\overrightarrow{AZ} = \Lambda(A)\overrightarrow{Z}$.

**Proof.** ii) From i), we obtain that
\[
\overrightarrow{AZ} = \Lambda(AZ)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \Lambda(A)\Lambda(Z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \Lambda(A)\overrightarrow{Z}. \quad \square
\]

**Remark 2.4.** 1) We remark that an element $z \in S$ is an invertible element in $S$ if and only if $\det \Lambda(z) \neq 0$. 


2) The $\Lambda$-norm of the element $z \in S$ is $\eta_\Lambda (z) = \det \Lambda (z) = \eta^3 (z)$ and $\tau_\Lambda (z) = 9 \text{tr} \Lambda (z)$. Indeed, from Proposition 1.4, if $A = S, K = L, m = 3, r = 9, \varphi = \Lambda$, we obtain the above relation.

3) We have that $\tau_\Lambda (z) = \text{tr} \Lambda (z) = 9c_0$.

Let $z \in S, z = A + By + Cy^2$, where $A = c_0 + c_1x + c_2x^2, B = c_3 + c_5x + c_7x^2, C = c_4 + c_8x + c_6x^2$. We denote by $z_\omega = A + \omega By + \omega^2 Cy^2, z_{\omega^2} = A + \omega^2 By + \omega Cy^2$.

**Proposition 2.5.** Let $S = \begin{pmatrix} 1 & 1 \\ K & \omega \end{pmatrix}$. Then $\eta_\Lambda (z) = \eta_\Lambda (z_\omega) = \eta_\Lambda (z_{\omega^2})$.

*Proof.* The left matrix representation for the element $z \in S$ is $\Lambda (z) = \begin{pmatrix} c_0 & c_2 & c_1 & c_4 & c_3 & \omega^2 c_6 & \omega^2 c_5 & \omega c_8 & \omega c_7 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_8 & \omega^2 c_4 & \omega c_6 & c_5 & c_4 & \omega c_3 & \omega^2 c_2 & \omega \end{pmatrix}$

and for the element $z_\omega$ is

$$\Lambda (z_\omega) = \begin{pmatrix} c_0 & c_2 & c_1 & \omega^2 c_4 & \omega c_3 & \omega c_6 & c_5 & c_8 & \omega^2 c_7 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_8 & \omega c_4 & \omega c_6 & c_5 & c_4 & \omega c_3 & \omega^2 c_2 & \omega \end{pmatrix}.$$

Denoting by $D_{rs} = \left( d_{ij}^{rs} \right) \in M_9 (K)$ the matrix defined such that $d_{kk}^{rs} = 1$ for $k \notin \{ r, s \}, d_{rr}^{rs} = d_{ss}^{rs} = 0, d_{rs}^{rs} = d_{sr}^{sr} = 1$ and zero in the rest, we have that $\det D_{rs} = -1$. If we multiply a matrix $A$ to the left with $D_{rs}$, the new matrix is obtained from $A$ by changing the line $r$ with the line $s$ and if we multiply a matrix $A$ to the right with $D_{rs}$, the new matrix is obtained from $A$ by changing the column $r$ with the column $s$. By straightforward calculations, it results that $\Lambda (z_\omega) = D_{79}D_{48}D_{46}D_{57}D_{21}D_{23}\Lambda (z) D_{12}D_{23}D_{49}D_{79}D_{48}D_{59}$, therefore $\eta_\Lambda (z) = \det \Lambda (z) = \det \Lambda (z_\omega) = \eta_\Lambda (z_{\omega^2})$. In the same way, we get that $\eta_\Lambda (z) = \det \Lambda (z) = \det \Lambda (z_{\omega^2}) = \eta_\Lambda (z_{\omega^2})$. □
Similar to the matrix $\Lambda (z) , z \in S$, we define $\Gamma (z) \in M_9 (K)$ to be the matrix with the coefficients in $K$ of the basis $B$ for the elements $\{ z \cdot 1, xz, x^2z, yz, y^2z, xyz, x^2y^2z, x^2yz, xy^2z \}$ in their columns. This matrix is

$$\Gamma (z) = \begin{pmatrix}
  c_0 & ac_2 & ac_1 & bc_4 & bc_3 & ab\omega^2c_6 & ab\omega^2c_5 & ab\omega c_8 & ab\omega c_7 \\
  c_1 & c_0 & ac_2 & bw_3 & bw_2c_5 & bc_4 & ab\omega c_7 & ab\omega^2c_6 & bc_3 \\
  c_2 & c_1 & c_0 & bw^2c_6 & bw_2c_7 & bw_3c_8 & ab\omega c_3 & bc_4 & bw^2c_5 \\
  c_3 & ac_7 & ac_5 & c_0 & bc_4 & aw^2c_2 & ab\omega c_2 & awc_1 & ab\omega c_6 \\
  c_4 & ac_6 & ac_8 & c_3 & c_0 & aw^2c_7 & aw^2c_1 & awc_5 & awc_2 \\
  c_5 & c_3 & ac_7 & \omega c_1 & bw^2c_8 & c_0 & ab\omega c_6 & aw^2c_2 & bc_4 \\
  c_6 & c_8 & c_4 & \omega^2c_7 & \omega c_2 & \omega c_5 & c_0 & c_3 & \omega^2c_1 \\
  c_7 & c_5 & c_3 & \omega^2c_2 & bw_3 & \omega c_1 & bc_4 & c_0 & bw^2c_8 \\
  c_8 & c_4 & ac_6 & \omega c_5 & \omega^2c_1 & c_3 & awc_2 & aw^2c_7 & c_0
\end{pmatrix}$$

Let $\alpha_{ij} \in M_3 (K)$ be the matrix with 1 in position $(i, j)$ and zero in the rest and

$$\beta_5 = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & \omega & 0
\end{pmatrix}, \quad \beta_6 = \begin{pmatrix}
  0 & 1 & 0 \\
  \omega & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix},$$

$$\beta_7 = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 1 & 0
\end{pmatrix}, \quad \beta_8 = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0
\end{pmatrix}.$$

**Proposition 2.6.** With the above notations, we have:

i) $\Gamma (d_1z_1 + d_2z_2) = d_1 \Gamma (z_1) + d_2 \Gamma (z_2)$, for all $z_1, z_2 \in S$ and $d_1, d_2 \in K$.

ii) $\Gamma (z_1z_2) = \Gamma (z_2) \Gamma (z_1)$, for all $z_1, z_2 \in S$.

**Proof.** Let $\Gamma (x) = U = \begin{pmatrix}
  \gamma_1 & 0 & 0 \\
  0 & \omega \alpha_{31} & aw\beta_6 \\
  0 & \omega \beta_5 & \omega^2\alpha_{13}
\end{pmatrix} \in M_9 (K)$ and

$$\Gamma (y) = V = \begin{pmatrix}
  0 & b\alpha_{12} & \omega b\beta_8 \\
  \beta_7 & \alpha_{21} & 0 \\
  \alpha_{23} & \alpha_{33} & \alpha_{12}
\end{pmatrix}.$$ 

By straightforward calculations, we obtain:

$\Gamma (x^2) = U^2 , \Gamma (y^2) = V^2 , \Gamma (yx) = UV ,
\Gamma (x^2y) = \Gamma (y) \Gamma (x^2) = \Gamma (y) \Gamma (x) = VU^2 ,
\Gamma (xy^2) = \Gamma (y^2) \Gamma (x) = \Gamma (y) \Gamma (xy) = V^2U ,
\Gamma (x^2y^2) = \Gamma (y^2) \Gamma (x^2) = \Gamma (y) \Gamma (x^2y) = V^2U^2.$

Therefore, we have that $\Gamma (z_1z_2) = \Gamma (z_2) \Gamma (z_1)$ and $\Gamma$ is a $K$–algebra morphism. □
Proposition 2.7. Let $Z, A \in S$, then:

i) $\overrightarrow{Z} = \Gamma (Z) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $0 \in M_{8 \times 1} (K)$ is the zero matrix.

ii) $\overrightarrow{ZA} = \Gamma (A) \overrightarrow{Z}$.

iii) $\Lambda (A) \Gamma (B) = \Gamma (B) \Lambda (A)$.

Proposition 2.8. Let $S = \left( \begin{pmatrix} 1 \\ K, \omega \end{pmatrix} \right)$, therefore $\eta (\Gamma (z)) = \eta (\Gamma (z \omega)) = \eta (\Gamma (z \omega^2))$.

Theorem 2.9. Let $S = \left( \begin{pmatrix} a, b \\ K, \omega \end{pmatrix} \right)$ be a symbol algebra of degree three and

$M_9 = (1, \frac{1}{a} x, \frac{1}{a} x^2, \frac{1}{b} y, \frac{1}{b} y^2, \frac{1}{ab} xy, \frac{1}{ab} x^2 y, \frac{1}{ab} x y^2, \frac{1}{ab} x^2 y^2)$,
$N_9 = (1, x^2, x, y^2, y, x^2 y^2, xy, xy^2, x^2 y)^t$,
$M_{10} = (1, \frac{1}{a} x^2, \frac{1}{a} x, \frac{1}{b} y^2, \frac{1}{b} y, \frac{1}{ab} x^2 y^2, \frac{1}{ab} xy, \frac{1}{ab} x y^2, \frac{1}{ab} x^2 y)$,
$N_{10} = (1, x, x^2, y, y^2, xy, x^2 y, x y^2, x^2 y)^t$. The following relation is true

$M_9 \Lambda (z) N_9 = M_{10} \Gamma (z) N_{10} = 3 z, z \in S$.

3. SOME EQUATIONS WITH COEFFICIENTS IN A SYMBOL ALGEBRA OF DEGREE THREE

Using some properties of left and right matrix representations found in the above section, we solve some equations with coefficients in a symbol algebra of degree three.

Let $S$ be an associative algebra of degree three. For $z \in S$, let $P (X, z)$ be the characteristic polynomial for the element $a$

$$P (X, z) = X^3 - \tau (z) X^2 + \pi (z) X - \eta (z) \cdot 1,$$

where $\tau$ is a linear form, $\pi$ is a quadratic form and $\eta$ a cubic form.

Proposition 3.1 ([1]). With the above notations, denoting by $z^* = z^2 - \tau (z) z + \pi (z) \cdot 1$, for an associative algebra of degree three, we have:

i) $\pi (z) = \tau (z^*)$.

ii) $2 \pi (z) = \tau (z^2) - \tau (z^2)$.

iii) $\tau (zw) = \tau (wz)$.

iv) $z^{**} = \eta (z) z$.

v) $(zw)^* = w^* z^*$.

vi) $\pi (zw) = \pi (wz)$.

In the following, we will solve some equations with coefficients in the symbol algebra of degree three $S = \left( \begin{pmatrix} a, b \\ K, \omega \end{pmatrix} \right)$. For each element $Z \in S$ relation (3.1) holds. First, we remark that if the element $Z \in S$ has $\eta (Z) \neq 0$, then $Z$ is an invertible element. Indeed, from (3.1.), we have that $ZZ^* = \eta (Z)$, therefore $Z^{-1} = \frac{Z}{\eta (Z)}$. 

We consider the following equations:

(3.2) \[ AZ =ZA \]
(3.3) \[ AZ = ZB \]
(3.4) \[ AZ -ZA = C \]
(3.5) \[ AZ - ZB = C , \]
with \( A, B, C \in S . \)

**Proposition 3.2.**

i) Equation (3.2) has non-zero solutions in the algebra \( S . \)

ii) If equation (3.3) has nonzero solutions \( Z \) in the algebra \( S \) such that \( \eta (Z) \neq 0 , \) then \( \tau (A) = \tau (B) \) and \( \eta (A) = \eta (B) \).

iii) If equation (3.4) has solution, then this solution is not unique.

iv) If \( \Lambda (A) - \Gamma (B) \) is an invertible matrix, then equation (3.5) has a unique solution.

**Proof.** i) Using vector representation, we have that \( \overrightarrow{AZ} = \overrightarrow{ZA} \). It results that \( \Lambda (A) \overrightarrow{Z} = \Gamma (A) \overrightarrow{Z} \), therefore \( (\Lambda (A) - \Gamma (A)) \overrightarrow{Z} = 0 . \) The matrix \( \Lambda (A) - \Gamma (A) \) has the determinant equal with zero (first column is zero), then equation (3.2) has non-zero solutions.

ii) We obtain \( \eta (AZ) = \eta (BZ) \) and \( \eta (A) = \eta (B) \) since \( Z \) is an invertible element in \( S . \) Using representation \( \Lambda , \) we get \( \Lambda (A) \Lambda (Z) = \Lambda (Z) \Lambda (B) \).

From Remark 1.3 i), it results that \( \Lambda (A) = \Lambda (Z) \Lambda (B) (\Lambda (Z))^{-1} . \) Therefore \( \tau (A) = tr (\Lambda (A)) = tr (\Lambda (Z) \Lambda (B) (\Lambda (Z))^{-1}) = tr (\Lambda (B)) = \tau (B) \in K . \)

iii) Using the vector representation, we have \( (\Lambda (A) - \Gamma (A)) \overrightarrow{Z} = C \) and, since the matrix \( \Lambda (A) - \Gamma (A) \) has the determinant equal with zero (first column is zero), if equation (3.4) has solution, then this solution is not unique.

iv) Using vector representation, we have \( (\Lambda (A) - \Gamma (B)) \overrightarrow{Z} = C . \) \( \square \)

**Proposition 3.3.** Let 
\[ A = a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 y^2 + a_5 xy + a_6 x^2 y + a_7 x^2 y^2 + a_8 xy^2 \in S , \]
\[ B = b_0 + b_1 x + b_2 x^2 + b_3 y + b_4 y^2 + b_5 xy + b_6 x^2 y + b_7 x^2 y^2 + b_8 xy^2 \in S , \]
\( A_0 = A - a_0 \neq 0 \) and \( B_0 = B - b_0 \neq 0 . \) If \( a_0 = b_0 , A_0 \neq -B_0 , \eta (A_0) = \eta (B_0) = 0 \) and \( \pi (A_0) = \pi (B_0) \neq 0 , \) then all solutions of equation (3.3) are in the \( K \)-algebra \( \mathcal{A} (A,B) , \) the subalgebra of \( S \) generated by the elements \( A \) and \( B , \) and have the form \( \lambda_1 X_1 + \lambda_2 X_2 , \) where \( \lambda_1, \lambda_2 \in K , \) \( X_1 = A_0 + B_0 \) and \( X_2 = \pi (A_0) - A_0 B_0 . \)

**Proof.** First, we verify that \( X_1 \) and \( X_2 \) are solutions of the equation (3.3).

Now, we prove that \( X_1 \) and \( X_2 \) are linearly independent elements. If \( \alpha_1 X_1 + \alpha_2 X_2 = 0 , \) it results that \( \alpha_2 \pi (A_0) = 0 , \) therefore \( \alpha_2 = 0 . \) We obtain
that $\alpha_1 = 0$. Obviously, each element of the form $\lambda_1X_1 + \lambda_2X_2$, where $\lambda_1, \lambda_2 \in K$ is a solution of the equation (3.3) and since $\pi(A_0) \neq 0$ we have that $A(A, B) = A(X_1, X_2)$. Since each solution of the equation (3.3) belongs to algebra $A(A, B)$, it results that all solutions have the form $\lambda_1X_1 + \lambda_2X_2$, where $\lambda_1, \lambda_2 \in K$. □

4. FIBONACCI SYMBOL ELEMENTS

In this section, we will introduce the Fibonacci symbol elements and we will compute the reduced norm of such an element. This relation helps us to find an infinite set of invertible elements. First of all, we recall [3], [7] and give some properties of Fibonacci numbers, properties which will be used in our proofs.

Fibonacci numbers are the following sequence of numbers

$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots,$

with the $n$th term given by the formula:

$$f_n = f_{n-1} + f_{n-2}, \ n \geq 2,$$

where $f_0 = 0, f_1 = 1$. The expression for the $n$th term is

$$f_n = \frac{1}{\sqrt{5}} [\alpha^n - \beta^n],$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Remark 4.1. Let $(f_n)_{n \geq 0}$ be the Fibonacci sequence $f_0 = 0, f_1 = 1, f_{n+2} = f_{n+1} + f_n, (\forall) \ n \in \mathbb{N}$. Then

$$f_n + f_{n+3} = 2f_{n+2}, (\forall)n \in \mathbb{N}.$$  

$$f_n + f_{n+4} = 3f_{n+2}, (\forall)n \in \mathbb{N}.$$  

Remark 4.2. Let $(f_n)_{n \geq 0}$ be the Fibonacci sequence $f_0 = 0, f_1 = 1, f_{n+2} = f_{n+1} + f_n, (\forall) \ n \in \mathbb{N}$. Let $\omega$ be a primitive root of unity of order 3, let $K = \mathbb{Q}(\omega)$ be the cyclotomic field and let $S = \left( \frac{a, b}{K, \omega} \right)$ be the symbol algebra of degree 3. Thus, $S$ has a $K$- basis $\{x^iy^j|0 \leq i, j < 3\}$ such that $x^3 = a \in K^*$, $y^3 = b \in K^*$, $yx = \omega xy$.

Let $z \in S, z = \sum_{i,j=1}^{n} x^iy^jc_{ij}$. The reduced norm of $z$ is

$$\eta(z) = a^2 \cdot (c_{20}^3 + bc_{21}^3 + b^2c_{22}^3 - 3bc_{20}c_{21}c_{22}) + a \cdot (c_{10}^3 + bc_{11}^3 + b^2c_{12}^3 - 3bc_{10}c_{11}c_{12}) -$$

$$-3a \cdot (c_{00}c_{10}c_{20} + bc_{01}c_{11}c_{21} + b^2c_{02}c_{12}c_{22}) -$$

$$-3ab\omega (c_{00}c_{12}c_{21} + c_{01}c_{10}c_{22} + c_{02}c_{11}c_{20}) -$$
(4.2) \[-3ab\omega^2 (c_{00}c_{11}c_{22}+c_{02}c_{10}c_{21}+c_{01}c_{12}c_{20}) + c_{00}^3 + bc_{01}^3 + b^2c_{02}^3 - 3bc_{00}c_{01}c_{02}.\]

(See [14], p. 299).

We define the \textit{n th Fibonacci symbol element} to be the element

\[F_n = f_n \cdot 1 + f_{n+1} \cdot x + f_{n+2} \cdot x^2 + f_{n+3} \cdot y+\]

(4.3) \[+f_{n+4} \cdot xy + f_{n+5} \cdot x^2y + f_{n+6} \cdot y^2 + f_{n+7} \cdot xy^2 + f_{n+8} \cdot x^2y^2.\]

In [8] A.F. Horadam generalized the Fibonacci numbers, giving by:

\[h_n = h_{n-1} + h_{n-2}, \quad n \in \mathbb{N}, n \geq 2,\]

\[h_0 = p, h_1 = q, \text{ where } p, q \text{ are arbitrary integers. From [8], relation 7, we have that these numbers satisfy the equality } h_{n+1} = pf_n + qf_{n+1}, \forall n \in \mathbb{N}.\]

In the following will be easy to use instead of \(h_{n+1}\) the notation \(h_{n+1}^{p,q}\). It is obvious that

\[h_{n+1}^{p,q} = h_{n+1}^{p',q'} = h_{n}^{p+p',q+q'},\]

for \(n\) a positive integers number and \(p, q, p', q'\) integers numbers. (See [4]).

\textbf{Remark 4.3.} Let \((f_n)_{n \geq 0}\) be the Fibonacci sequence \(f_0 = 0, f_1 = 1, f_{n+2} = f_{n+1} + f_n, \forall n \in \mathbb{N}\). Then

i) \(f_{n}^2 + f_{n-1}^2 = f_{2n-1}, \forall n \in \mathbb{N}^*;\)

ii) \(f_{n+1}^2 - f_n^2 = f_{2n}, \forall n \in \mathbb{N}^*;\)

iii) \(f_{n+3}^2 = 2f_{n+2}^2 + 2f_{n+1}^2 - f_n^2, \forall n \in \mathbb{N};\)

iv) \(f_{n}^2 - f_{n-1}f_{n+1} = (-1)^n - 1, \forall n \in \mathbb{N}^*;\)

v) \(f_{2n} = f_{n}^2 + 2f_{n}f_{n-1}\)

We define the \textit{n th generalized Fibonacci symbol element} to be the element

\[H_{n}^{p,q} = h_{n}^{p,q} \cdot 1 + h_{n+1}^{p,q} \cdot x + h_{n+2}^{p,q} \cdot x^2 + h_{n+3}^{p,q} \cdot y+\]

(4.4) \[+h_{n+4}^{p,q} \cdot xy + h_{n+5}^{p,q} \cdot x^2y + h_{n+6}^{p,q} \cdot y^2 + h_{n+7}^{p,q} \cdot xy^2 + h_{n+8}^{p,q} \cdot x^2y^2.\]

What algebraic structure has the set of Fibonacci symbol elements or the set of generalized Fibonacci symbol? The answer will be found in the following proposition.

\textbf{Proposition 4.4.} Let \(M\) the set

\[M = \left\{ \sum_{i=1}^{n} H_{n_i}^{p_i,q_i} | n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Z}, \forall i = 1/n \right\} \cup \{0\}\]

is a \(\mathbb{Z}\)- module.
Proof. The proof is immediate if we remark that for \( n_1, n_2 \in \mathbb{N}, p_1, p_2, \alpha_1, \alpha_2 \in \mathbb{Z} \), we have:

\[
\alpha_1 h_{n_1}^{p_1, q_1} + \alpha_2 h_{n_2}^{p_2, q_2} = h_{n_1}^{\alpha_1 p_1, \alpha_1 q_1} + h_{n_2}^{\alpha_2 p_2, \alpha_2 q_2}.
\]

Therefore, we obtain that \( M \) is a \( \mathbb{Z} \)- submodule of the symbol algebra \( S \). \( \square \)

In the following, we will compute the reduced norm for the \( n \)th Fibonacci symbol element.

**Proposition 4.5.** Let \( F_n \) be the \( n \)th Fibonacci symbol element. Let \( \omega \) be a primitive root of order 3 of unity and let \( K = \mathbb{Q}(\omega) \) be the cyclotomic field. Then the norm of \( F_n \) is

\[
\eta(F_n) = 4a^2 h_{n+3}^{211, 14} \cdot \left(h_{2n}^{84, 135} - 2f_n^2 \right) + 8h_{n+3}^{8, 3} \cdot \left(h_{2n}^{12, 20} - f_n^2 \right) +
+a[f_{n+2} \left(\omega h_{2n}^{30766, 27923} + h_{2n}^{22358, 20533} \right) + f_{n+3} \cdot \left(\omega h_{2n}^{4368, 1453} + h_{2n}^{14128, 12163} \right) -
\left(-1\right)^n \left(\omega h_{n+3}^{1472, 26448} + h_{n+3}^{12982, 24138} \right)].
\]

(4.5)

Proof. In this proof, we denote with \( E(x, y, z) = x^3 + y^3 + z^3 - 3xyz \). We obtain:

\[
\eta(F_n) = \eta(F_n) + 3 \cdot (1 + \omega + \omega^2) \cdot (f_n^3 + f_{n+1}^3 + f_{n+2}^3 + \ldots + f_{n+8}^3)
= a^2 \cdot (f_{n+2}^3 + f_{n+5}^3 + f_{n+8}^3 f_{n+2} f_{n+5} f_{n+8})
+ a \cdot (f_{n+1}^3 + f_{n+4}^3 + f_{n+7}^3 f_{n+1} f_{n+4} f_{n+7})
+ a \cdot (f_n^3 + f_{n+1}^3 + f_{n+2}^3 f_n f_{n+1} f_{n+2})
+ a \cdot (f_{n+3}^3 + f_{n+4}^3 + f_{n+5}^3 f_{n+3} f_{n+4} f_{n+5})
+ a \cdot (f_{n+6}^3 + f_{n+7}^3 + f_{n+8}^3 f_{n+6} f_{n+7} f_{n+8})
+ a \cdot (\omega^3 f_n^3 + f_{n+5}^3 + f_{n+7}^3 f_n f_{n+5} f_{n+7})
+ a \cdot (f_{n+1}^3 + f_{n+3}^3 + \omega^3 f_{n+8}^3 f_{n+1} f_{n+3} f_{n+8})
+ a \cdot (f_{n+2}^3 + f_{n+4}^3 + \omega^3 f_{n+6}^3 f_{n+2} f_{n+4} f_{n+6})
+ a \cdot (f_n^3 + f_{n+4}^3 + \omega^6 f_{n+8}^3 f_n f_{n+4} f_{n+8})
+ a \cdot (\omega^6 f_{n+1}^3 + f_{n+5}^3 + f_{n+6}^3 f_{n+1} f_{n+5} f_{n+6})
+ a \cdot (f_{n+2}^3 + f_{n+3}^3 + \omega^6 f_{n+7}^3 - 3\omega^2 f_{n+2} f_{n+3} f_{n+7})
+ (f_n^3 + f_{n+3}^3 + f_{n+6}^3 f_n f_{n+3} f_{n+6})
= a^2 \cdot E(f_{n+2}, f_{n+5}, f_{n+8}) + a \cdot E(f_{n+1}, f_{n+4}, f_{n+7})
+ a \cdot E(f_n, f_{n+1}, f_{n+2}) + a \cdot E(f_{n+3}, f_{n+4}, f_{n+5})
+ a \cdot E(f_{n+6}, f_{n+7}, f_{n+8}) + a \cdot E(\omega f_n, f_{n+5}, f_{n+7})
+ a \cdot E(f_{n+1}, f_{n+3}, \omega f_{n+8}) + a \cdot E(f_{n+2}, f_{n+4}, \omega f_{n+6})
+a \cdot E(f_n, f_{n+4}, \omega^2 f_{n+8}) + a \cdot E(\omega^2 f_{n+1}, f_{n+5}, f_{n+6}) \\
\quad + a \cdot E(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}) + E(f_n, f_{n+3}, f_{n+6}) \cdot \\

Now, we compute $E(f_{n+2}, f_{n+5}, f_{n+8})$.

$$E(f_{n+2}, f_{n+5}, f_{n+8}) =$$

$$\frac{1}{2} (f_{n+2} + f_{n+5} + f_{n+8}) \left[ (f_{n+5} - f_{n+2})^2 + (f_{n+8} - f_{n+5})^2 + (f_{n+8} - f_{n+2})^2 \right].$$

Using Remark 4.1, Remark 4.2, Remark 4.3 (iii) and the recurrence of the Fibonacci sequence, we obtain:

$$E(f_{n+2}, f_{n+5}, f_{n+8}) =$$

$$\frac{1}{2} (2f_{n+4} + f_{n+8}) \left[ (f_{n+4} + f_{n+1})^2 + (f_{n+7} + f_{n+4})^2 + (f_{n+1} + 2f_{n+4} + f_{n+7})^2 \right] =$$

$$= \frac{1}{2} (f_{n+4} + 3f_{n+6}) \cdot \left[ (2f_{n+3})^2 + (2f_{n+6})^2 + (2f_{n+3} + 2f_{n+6})^2 \right] =$$

$$= 4(11f_{n+2} + 14f_{n+3}) \cdot (135f_{n+1}^2 + 82f_n^2 - 51f_{n-1}^2).$$

Then, we have:

$$E(f_{n+2}, f_{n+5}, f_{n+8}) = 4(11f_{n+2} + 14f_{n+3}) \cdot (135f_{n+1}^2 + 82f_n^2 - 51f_{n-1}^2).$$

Using Remark 4.3 (i, ii) and the definition of the generalized Fibonacci sequence it is easy to compute that:

$$E(f_{n+2}, f_{n+5}, f_{n+8}) = 4(11f_{n+2} + 14f_{n+3}) \cdot (135f_{2n+84f_{2n-1}} - 2f_n^2) =$$

$$= 4(11f_{n+2} + 14f_{n+3}) \cdot \left( h_{2n}^{84,135} - 2f_n^2 \right) = 4h_{n+3}^{11,14} \cdot \left( h_{2n}^{84,135} - 2f_n^2 \right).$$

Therefore, we obtain

$$E(f_{n+2}, f_{n+5}, f_{n+8}) = 4h_{n+3}^{11,14} \cdot \left( h_{2n}^{84,135} - 2f_n^2 \right).$$

Replacing $n \rightarrow n - 1$ in relation (4.6) and using Remark 4.3, (iii) and the recurrence of the Fibonacci sequence, we obtain:

$$E(f_{n+1}, f_{n+4}, f_{n+7}) = 4(3f_{n+2} + 11f_{n+3}) \cdot (51f_{n+1}^2 + 33f_n^2 - 20f_{n-1}^2).$$

Now, we calculate $E(f_n, f_{n+1}, f_{n+2})$.

$$E(f_n, f_{n+1}, f_{n+2}) = \frac{1}{2} (f_n + f_{n+1} + f_{n+2}) \cdot \left[ (f_{n+1} - f_n)^2 + (f_{n+2} - f_{n+1})^2 + (f_{n+2} - f_n)^2 \right] =$$

$$= f_{n+2} \cdot (f_{n-1}^2 + f_n^2 + f_{n+1}^2).$$

So, we obtain

$$E(f_n, f_{n+1}, f_{n+2}) = f_{n+2} \cdot (f_{n+1}^2 + f_n^2 + f_{n-1}^2).$$
Replacing \( n \to n + 3 \) in relation (4.9.), using Remark 4.3 (iii) and the recurrence of the Fibonacci sequence we obtain:

\[
E(f_{n+3}, f_{n+4}, f_{n+5}) = (f_{n+2} + 2f_{n+3}) \cdot (23f_{n+1}^2 + 15f_n^2 - 9f_{n-1}^2).
\]

Replacing \( n \to n + 3 \) in relation (4.10), using Remark 4.3 (iii) and the recurrence of the Fibonacci sequence we obtain:

\[
E(f_{n+6}, f_{n+7}, f_{n+8}) = (3f_{n+2} + 4f_{n+3}) \cdot (635f_{n+1}^2 + 387f_n^2 - 239f_{n-1}^2).
\]

Adding equalities (4.8)–(4.11), after straightforward calculation, we have:

\[
E(f_{n+1}, f_{n+4}, f_{n+7}) + E(f_{n}, f_{n+1}, f_{n+2}) + E(f_{n+3}, f_{n+4}, f_{n+5})
+ E(f_{n+6}, f_{n+7}, f_{n+8}) = f_{n+2} \cdot (2511f_{n+1}^2 + 1573f_n^2 - 965f_{n-1}^2) + f_{n+3} \cdot (4790f_{n+1}^2 + 3030f_n^2 - 1854f_{n-1}^2).
\]

Using Remark 4.3 (i, ii), it is easy to compute that:

\[
E(f_{n+1}, f_{n+4}, f_{n+7}) + E(f_{n}, f_{n+1}, f_{n+2}) + E(f_{n+3}, f_{n+4}, f_{n+5})
+ E(f_{n+6}, f_{n+7}, f_{n+8}) = f_{n+2} \cdot h_{2n+1}^{965,1546} + f_{n+3} \cdot h_{2n+1}^{1854,2936} + 27f_n^2 \cdot h_{n+4}^{67,1}.
\]

Then, we obtained that:

\[
E(f_{n+1}, f_{n+4}, f_{n+7}) + E(f_{n}, f_{n+1}, f_{n+2}) + E(f_{n+3}, f_{n+4}, f_{n+5})
+ E(f_{n+6}, f_{n+7}, f_{n+8}) = f_{n+2} \cdot h_{2n+1}^{965,1546} + f_{n+3} \cdot h_{2n+1}^{1854,2936} + 27f_n^2 \cdot h_{n+4}^{67,1}.
\]

Replacing \( n \) by \( n - 1 \) in relation (4.8), using Remark 4.3 (iii) and the recurrence of the Fibonacci sequence we obtain:

\[
E(f_n, f_{n+3}, f_{n+6}) = 8(8f_{n+2} + 3f_{n+3}) \cdot (20f_{n+1}^2 + 11f_n^2 - 8f_{n-1}^2).
\]

Using Remark 4.3 (i,ii), the definition of the generalized Fibonacci sequence, we have:

\[
E(f_n, f_{n+3}, f_{n+6}) = 8(8f_{n+2} + 3f_{n+3}) \cdot [20(f_{n+1}^2 - f_{n-1}^2) + 12(f_{n+1}^2 + f_{n}^2) - f_{n}^2] = 8(8f_{n+2} + 3f_{n+3}) \cdot (20f_{2n+12}f_{2n-1} - f_{n}^2) = 8h_{n+3}^{8,3} \cdot (h_{2n}^{12,20} - f_{n}^2).
\]

Therefore,

\[
E(f_n, f_{n+3}, f_{n+6}) = 8h_{n+3}^{8,3} \cdot (h_{2n}^{12,20} - f_{n}^2).
\]

Now, we compute \( E(\omega f_n, f_{n+5}, f_{n+7}) \).

\[
E(\omega f_n, f_{n+5}, f_{n+7}) = \frac{1}{2}(\omega f_n + f_{n+5} + f_{n+7}) \cdot [(f_{n+7} - f_{n+5})^2 + (f_{n+5} - \omega f_n)^2]
+ (f_{n+7} - \omega f_n)^2.
\]
By repeatedly using of Fibonacci sequence recurrence and Remark 4.3 (iii), we obtain:

\[ E(\omega f_n, f_{n+5}, f_{n+7}) = \frac{1}{2} \left[ 2 (2+\omega) f_{n+2} + (7 - \omega) f_{n+3} \right] \cdot \]

\[ \cdot \left[ 104 f_{n+1}^2 + 65 f_n^2 - 40 f_{n-1}^2 + ((8 - \omega)f_{n+1} + (\omega - 3)f_{n-1})^2 \right. \]

\[ + ((11 - \omega)f_{n+1} + (\omega - 8)f_{n-1})^2 \right] . \]

In the same way, we have:

\[ E(\omega f_n, f_{n+5}, f_{n+7}) = \frac{1}{2} \left[ 2 (2+\omega) f_{n+2} + (7 - \omega) f_{n+3} \right] \cdot \left[ (-40\omega+285) f_{n+1}^2 + (-24\omega+31) f_{n+1}^2 - (64\omega+285) f_n^2 - 4 (16\omega-55) \cdot (-1)^n \right] . \]

(4.16)

Now, we calculate \( E(f_{n+1}, f_{n+3}, \omega f_{n+3}) \).

\[ E(f_{n+1}, f_{n+3}, \omega f_{n+3}) = \frac{1}{2} (f_{n+1} + f_{n+3} + \omega f_{n+3}) \cdot \]

\[ \left[ (f_{n+3} - f_{n+1})^2 + (\omega f_{n+3} - f_{n+3})^2 + (\omega f_{n+3} - f_{n+1})^2 \right] . \]

By repeatedly using of Fibonacci sequence recurrence and Remark 4.3 (iii), it is easy to compute that:

\[ E(f_{n+1}, f_{n+3}, \omega f_{n+3}) = [(5\omega - 1) f_{n+2} + 2 (4\omega - 1) f_{n+3}] \cdot [-2 (696\omega + 578) f_{n+1}^2 - (182\omega + 169) f_{n+1}^2 + 2 (485\omega + 441) f_n^2 + (970\omega + 881) \cdot (-1)^n] . \]

(4.17)

Now, we calculate \( E(f_{n+2}, f_{n+4}, \omega f_{n+4}) \).

\[ E(f_{n+2}, f_{n+4}, \omega f_{n+4}) = \]

\[ = \frac{1}{2} (f_{n+2} + f_{n+4} + \omega f_{n+4}) \cdot \left[ (f_{n+4} - f_{n+2})^2 + (\omega f_{n+4} - f_{n+4})^2 + (\omega f_{n+4} - f_{n+2})^2 \right] . \]

In the same way, we obtain:

\[ E(f_{n+2}, f_{n+4}, \omega f_{n+4}) = \frac{-1}{2} \left[ 2 (\omega+1) f_{n+2} + (3\omega+1) f_{n+3} \right] \cdot [(520\omega + 309) f_{n+1}^2 + (80\omega+47) f_{n+1}^2 + (112\omega - 21) f_n^2 + 8 (14\omega+3) \cdot (-1)^n] . \]

(4.18)

Adding relations (4.16), (4.17), (4.18) and making some calculus, it results:

\[ E(\omega f_n, f_{n+5}, f_{n+7}) + E(f_{n+1}, f_{n+3}, \omega f_{n+3}) + E(f_{n+2}, f_{n+4}, \omega f_{n+4}) = \]

\[ = f_{n+2} [(10138\omega+6127) f_{n+1}^2 - (1210\omega+5231) f_n^2 + (301\omega+1132) f_{n-1}^2 - (-1)^n (1235\omega+5315)] + f_{n+3} [(4103\omega+13544) f_{n+1}^2 - (3148\omega+8566) f_n^2 + \]

\[ + (307\omega+1771) f_{n-1}^2 - (-1)^n (11396\omega+16279)] . \]

(4.19)
Now, we compute \( E(f_n, f_{n+4}, \omega^2 f_{n+8}) \).

\[
E(f_n, f_{n+4}, \omega^2 f_{n+8}) = \frac{1}{2} (f_n + f_{n+4} + \omega^2 f_{n+8}) \cdot [(f_{n+4} - f_n)^2 + (\omega^2 f_{n+8} - f_{n+4})^2 + (\omega^2 f_{n+8} - f_n)^2].
\]

Using Remark 4.2, the recurrence of the Fibonacci sequence we obtain:

\[
f_n + f_{n+4} = 3f_{n+2}; \quad f_{n+4} - f_n = 3f_{n+1} + f_n; \quad f_{n+8} = 5f_{n+2} + 8f_{n+3}.
\]

Using Remark 4.3. (iv) it is easy to compute that:

\[
E(f_n, f_{n+4}, \omega^2 f_{n+8}) = [(-5\omega + 8) f_{n+2} + 8 (-\omega + 1) f_{n+3}] \cdot [(1460\omega + 325) f_{n+1}^2 + (208\omega + 42) f_{n-1}^2 - (1030\omega + 127) f_n^2 - (1030\omega + 127) \cdot (-1)^n]. \tag{4.20}
\]

Similarly, it is easy to compute that:

\[
E(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}) = - [(3\omega + 2) f_{n+2} + (5\omega + 4) f_{n+3}] \cdot [(546\omega + 112) f_{n+1}^2 + (80\omega + 17) f_{n-1}^2 - (418\omega + 87) f_n^2 - (418\omega + 87) \cdot (-1)^n].
\]

Now, we compute \( E(f_{n+5}, f_{n+6}, \omega^2 f_{n+1}) \).

\[
E(f_{n+5}, f_{n+6}, \omega^2 f_{n+1}) = \frac{1}{2} (f_{n+5} + f_{n+6} + \omega^2 f_{n+1}) \cdot [(f_{n+6} - f_{n+5})^2 + (f_{n+5} - \omega^2 f_{n+1})^2 + (f_{n+6} - \omega^2 f_{n+1})^2].
\]

Using the recurrence of the Fibonacci sequence, we have:

\[
f_{n+5} + f_{n+6} + \omega^2 f_{n+1} = (\omega + 4) f_{n+2} + (-\omega + 4) f_{n+3}; \quad f_{n+6} - f_{n+5} = 3f_{n+1} + 2f_n;
\]

\[
f_{n+5} - \omega^2 f_{n+1} = (\omega + 6) f_{n+1} + 3f_n; \quad f_{n+6} - \omega^2 f_{n+1} = (\omega + 9) f_{n+1} + 4f_n.
\]

Using Remark 4.3 (iv) it is easy to compute that:

\[
E(f_{n+5}, f_{n+6}, \omega^2 f_{n+1}) = [(\omega + 4) f_{n+2} + (-\omega + 4) f_{n+3}] \cdot [(23\omega + 151) f_{n+1}^2 + 19f_{n-1}^2 - (6\omega + 107) f_n^2 - (6\omega + 107) \cdot (-1)^n]. \tag{4.22}
\]

Adding equalities (4.20)–(4.22), we have:

\[
E(f_n, f_{n+4}, \omega^2 f_{n+8}) + E(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}) + E(f_{n+5}, f_{n+6}, \omega^2 f_{n+1}) = f_{n+2} \cdot [(17785\omega + 11895) f_{n+1}^2 - (13037\omega + 7667) f_n^2 + (2542\omega + 1658) f_{n-1}^2 - (13037\omega + 7667) \cdot (-1)^n] + f_{n+3} \cdot [(2608\omega + 2048) f_{n-1}^2 - (15052\omega + 7859) f_n^2 - (2650\omega - 6171) f_{n+1}^2 - (15052\omega + 7859) \cdot (-1)^n]. \tag{4.23}
\]

Adding (4.12) (4.19), (4.23), it results:
Some properties of symbol algebras of degree three

(4.24) \[ E(f_{n+1}, f_{n+4}, f_{n+7}) + E(f_n, f_{n+1}, f_{n+2}) + E(f_{n+3}, f_{n+4}, f_{n+5}) + E(f_{n+6}, f_{n+7}, f_{n+8}) + E(\omega f_n, f_{n+5}, f_{n+7}) + E(f_{n+1}, f_{n+3}, \omega f_{n+8}) + E(f_{n+2}, f_{n+4}, \omega f_{n+6}) + E(f_n, f_{n+4}, \omega^2 f_{n+8}) + E(\omega^2 f_{n+1}, f_{n+5}, f_{n+6}) + E(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}) = f_{n+2} \cdot (2511 f_{n+1}^2 - 965 f_{n-1}^2) + f_{n+3} \cdot (4790 f_{n+1}^2 + 3030 f_n^2 - 1854 f_{n-1}^2) + f_{n+2} \cdot [(29723 \omega + 18022) f_{n+1}^2 - (14247 \omega + 12989) f_n^2 + (2843 \omega + 2790) f_{n-1}^2 - (14272 \omega + 12928) \cdot (-1)^n] + f_{n+3} \cdot [(1453 \omega + 7373) f_{n+1}^2 - (18200 \omega + 16425) f_n^2 + (2915 \omega + 3819) f_{n-1}^2 - (26448 \omega + 24138) \cdot (-1)^n].

Using Remark 4.3 (i, ii) and the definition of the generalized Fibonacci numbers, it is easy to compute that:

\[ E(f_{n+1}, f_{n+4}, f_{n+7}) + E(f_n, f_{n+1}, f_{n+2}) = f_{n+2} \cdot \left( \omega h_{2n}^{30766,27923} + h_{2n}^{22358,20533} \right) + f_{n+3} \cdot \left( \omega h_{2n}^{4368,1453} + h_{2n}^{14128,12163} \right) - f_n \cdot \left( \omega h_{n+3}^{45013,22563} + h_{n+3}^{3683,27523} \right) + (-1)^{n+1} \cdot \left( \omega h_{n+3}^{1472,26448} + h_{n+3}^{12982,24138} \right). \]

Therefore, we obtain:

(4.25) \[ E(f_{n+1}, f_{n+4}, f_{n+7}) + E(f_n, f_{n+1}, f_{n+2}) + E(f_{n+3}, f_{n+4}, f_{n+5}) + E(f_{n+6}, f_{n+7}, f_{n+8}) + E(\omega f_n, f_{n+5}, f_{n+7}) + E(f_{n+1}, f_{n+3}, \omega f_{n+8}) + E(f_{n+2}, f_{n+4}, \omega f_{n+6}) + E(f_n, f_{n+4}, \omega^2 f_{n+8}) + E(\omega^2 f_{n+1}, f_{n+5}, f_{n+6}) + E(f_{n+2}, f_{n+3}, \omega^2 f_{n+7}) = f_{n+2} \cdot \left( \omega h_{2n}^{30766,27923} + h_{2n}^{22358,20533} \right) + f_{n+3} \cdot \left( \omega h_{2n}^{4368,1453} + h_{2n}^{14128,12163} \right) - f_n \cdot \left( \omega h_{n+3}^{45013,22563} + h_{n+3}^{3683,27523} \right) + (-1)^{n+1} \cdot \left( \omega h_{n+3}^{1472,26448} + h_{n+3}^{12982,24138} \right). \]

Adding equalities (4.7), (4.15), (4.25), we have:

\[ \eta(F_n) = 4a^2 h_{n+3}^{211,14} \cdot \left( h_{2n}^{84,135} - 2f_n^2 \right) + 8h_{n+3}^{8,3} \cdot \left( h_{2n}^{12,20} - f_n^2 \right) + a[f_{n+2} \left( \omega h_{2n}^{30766,27923} + h_{2n}^{22358,20533} \right) + \ldots] \]
\[ f_{n+3} \cdot \left( \omega h_{2n}^{4368,1453} + h_{2n}^{14128,12163} \right) - f_n^2 \left( \omega h_{n+3}^{45013,22563} + h_{n+3}^{33683,27523} \right) \]
\[ - (-1)^n \left( \omega h_{n+3}^{1472,26448} + h_{n+3}^{12982,24138} \right). \]

**Corollary 4.6.** Let \( F_n \) be the \( n \)th Fibonacci symbol element. Let \( \omega \) be a primitive root of order 3 of unity and let \( K = \mathbb{Q}(\omega) \) be the cyclotomic field. Then, the norm of \( F_n \) is
\[
\eta(F_n) = f_{n+2} \left( \omega h_{2n}^{30766,27923} + h_{2n}^{26822,27753} \right) + f_{n+3} \cdot \left( \omega h_{2n}^{4368,1453} + h_{2n}^{19120,20203} \right) - f_n^2 \cdot \left( \omega h_{n+3}^{45013,22563} + h_{n+3}^{33835,27659} \right) + (-1)^{n+1} \left( \omega h_{n+3}^{1472,26448} + h_{n+3}^{12982,24138} \right).
\]

**Proof.** From the relation (4.7), we know that
\[
E(f_{n+2}, f_{n+5}, f_{n+8}) = 4 \left( 11f_{n+2} + 14f_{n+3} \right) \cdot \left( h_{2n}^{84,135} - 2f_n^2 \right).
\]
Therefore, we obtain
\[
(4.26) \quad E(f_{n+2}, f_{n+5}, f_{n+8}) = f_{n+2} \left( h_{2n}^{3696,5940} - 88f_n^2 \right) + f_{n+3} \left( h_{2n}^{4704,7560} - 112f_n^2 \right).
\]
From the relation (4.15), we know that
\[
E(f_n, f_{n+3}, f_{n+6}) = 8 \left( 8f_{n+2} + 3f_{n+3} \right) \cdot \left( h_{2n}^{12,20} - f_n^2 \right).
\]
Then, we obtain
\[
(4.27) \quad E(f_n, f_{n+3}, f_{n+6}) = f_{n+2} \left( h_{2n}^{768,1280} - 64f_n^2 \right) + f_{n+3} \left( h_{2n}^{288,480} - 24f_n^2 \right).
\]
Adding equalities (4.26) and (4.27) it is easy to compute that:
\[
E(f_{n+2}, f_{n+5}, f_{n+8}) + E(f_n, f_{n+3}, f_{n+6}) =
\begin{align*}
&= f_{n+2} \left( h_{2n}^{3696,5940} + h_{2n}^{768,1280} \right) + f_{n+3} \left( h_{2n}^{4704,7560} + h_{2n}^{288,480} \right) - f_n^2 \left( 152f_{n+2} + 136f_{n+3} \right) \\
&= f_{n+2} h_{2n}^{4464,7220} + f_{n+3} h_{2n}^{4992,8040} - f_n^2 h_{n+3}^{152,136}.
\end{align*}
\]
It results
\[
\eta(F_n) = f_{n+2} h_{2n}^{4464,7220} + f_{n+3} h_{2n}^{4992,8040} - f_n^2 h_{n+3}^{152,136}
\begin{align*}
&+ f_{n+2} \left( \omega h_{2n}^{30766,27923} + h_{2n}^{22358,20533} \right) + f_{n+3} \cdot \\
&\cdot \left( \omega h_{2n}^{4368,1453} + h_{2n}^{14128,12163} \right) - f_n^2 \left( \omega h_{n+3}^{45013,22563} + h_{n+3}^{33683,27523} \right) \\
&- (-1)^n \left( \omega h_{n+3}^{1472,26448} + h_{n+3}^{12982,24138} \right).
\end{align*}
\]
Therefore
\[
\eta(F_n) = f_{n+2} \left( \omega h_{2n}^{30766,27923} + h_{2n}^{26822,27753} \right) + f_{n+3} \cdot \left( \omega h_{2n}^{4368,1453} + h_{2n}^{19120,20203} \right)
\begin{align*}
&- f_n^2 \cdot \left( \omega h_{n+3}^{45013,22563} + h_{n+3}^{33835,27659} \right) + (-1)^{n+1} \left( \omega h_{n+3}^{1472,26448} + h_{n+3}^{12982,24138} \right).
\end{align*}
\]
In conclusion, even indices of the top of generalized Fibonacci numbers are very large, the expressions of the norm $\eta(F_n)$ from Proposition 4.5 and from Corollary 4.6 are much shorter than the formula founded in [14] and, in addition, the powers of Fibonacci numbers in these expressions are 1 or 2. □

Let $S = \left( \frac{1,1}{Q,\omega} \right)$ be a symbol algebra of degree 3. Using the norm form given in the Corollary 4.6, we obtain:

$$\eta(F_n) = \omega \left( f_{n+2}h_{2n}^{30766,27923} + f_{n+3}h_{2n}^{4368,1453} - f_n^2h_{n+3}^{45013,22563} + (-1)^{n+1}h_{n+3}^{1472,26448} \right) + \left( f_{n+2}h_{2n}^{26822,27753} + f_{n+3}h_{2n}^{19120,20203} - f_n^2h_{n+3}^{33835,27659} + (-1)^{n+1}h_{n+3}^{12982,24138} \right).$$

If $\eta(F_n) \neq 0$, we know that the element $F_n$ is invertible. From relation $\eta(F_n) \neq 0$, it results

$$\left( 4.28 \right) f_{n+2}h_{2n}^{30766,27923} + f_{n+3}h_{2n}^{4368,1453} - f_n^2h_{n+3}^{45013,22563} + (-1)^{n+1}h_{n+3}^{1472,26448} \neq 0$$
or

$$\left( 4.29 \right) f_{n+2}h_{2n}^{26822,27753} + f_{n+3}h_{2n}^{19120,20203} - f_n^2h_{n+3}^{33835,27659} + (-1)^{n+1}h_{n+3}^{12982,24138} \neq 0.$$

Using relation (4.1) and Remark 4.3, i) and v), we obtain that

$$f_{n+2}h_{2n}^{26822,27753} + f_{n+3}h_{2n}^{19120,20203} - f_n^2h_{n+3}^{33835,27659} + (-1)^{n+1}h_{n+3}^{12982,24138} = 26822f_{n+2}f_{2n-1} + 27753f_{n+2}f_{2n} + 19120f_{n+3}f_{2n-1} + 20203f_{n+3}f_{2n} - 33835f_n^2f_{n+2} - 27659f_n^2f_{n+3} + (-1)^{n+1}12982f_{n+2} + (-1)^{n+1}24138f_{n+3}$$

$$= 26822f_{n+2}f_{2n-1} + 26822f_{n+2}f_{2n} + 27753f_{n+2}f_{2n} + 2 \cdot 27753f_{n-1}f_nf_{n+2} + 19120f_{n+3}f_n^2 + 19120f_{n+3}f_{n-1} + 20203f_{n+3}f_{2n} - 33835f_n^2f_{n+2} - 27659f_n^2f_{n+3} + (-1)^{n+1}12982f_{n+2} + (-1)^{n+1}24138f_{n+3} = 26822f_{n+2}f_{2n-1}$$

$$+ (-1)^{n+1}12982f_{n+2} + 26822f_{n+2}f_{2n} + 27753f_{n+2}f_{2n} - 27659f_n^2f_{n+3} + 2 \cdot 27753f_{n-1}f_nf_{n+2} + (-1)^{n+1}24138f_{n+3} + 19120f_{n+3}f_n^2 + 20203f_{n+3}f_{2n} - 33835f_n^2f_{n+2} + 19120f_{n+3}f_{n-1}f_{n+2}.$$

We remark that $26822f_{n+2}f_{2n-1} + (-1)^{n+1}12982f_{n+2} > 0$.

We have $26822f_{n+2}f_{2n}^2 + 27753f_{n+2}f_{2n}^2 - 27659f_n^2f_{n+3} = 54575f_{n+2}f_{n}^2 - 27659f_n^2f_{n+3} \geq 0$, since $54575f_{n+2} > 27659f_n^2f_{n+3}$ is equivalently with $\frac{f_{n+2}}{f_{n+3}} > \frac{27659}{54575} \approx 0.506$, which is true for all $n \geq 1$. For $n = 0$, we have $54575f_{n+2}f_{2n}^2 - 27659f_n^2f_{n+3} = 0$.

Obviously, $2 \cdot 27753f_{n-1}f_nf_{n+2} + (-1)^{n+1}24138f_{n+3} = 55506f_{n-1}f_nf_{n+2} + (-1)^{n+1}24138f_{n+3} > 0$, since $\frac{f_{n+3}}{f_{n+2}} < \frac{55506}{24138} \approx 2, 29.$
And, finally, $19120 f_{n+3} f_n^2 + 20203 f_{n+3} f_{2n} - 33835 f_n^2 f_{n+2} > 0$, since $f_{2n} > f_n^2$ and $19120 + 20203 = 39323 > 33835$.

From the above, we have that the term $f_{n+2} h_{2n}^{26822,27753} + f_{n+3} h_{2n}^{19120,20203} - f_n^2 h_{n+3}^{33835,27659} + (-1)^{n+1} h_{n+3}^{12982,24138} > 0$ for all $n \in \mathbb{N}$.

We just proved the following result:

**Theorem 4.7.** Let $S = \left( \frac{1}{1, \omega} \right)$ be a symbol algebra of degree 3. All symbol Fibonacci elements are invertible elements.

## 5. CONCLUSIONS

In this paper, we studied some properties of the matrix representation of symbol algebras of degree 3. Using some properties of left matrix representations, we solved equations with coefficients in these algebras. The study of symbol algebras of degree three involves very complicated calculus and, usually, can be hard to find examples for some notions. We introduced the Fibonacci symbol elements, we gave an easier expression of reduced norm of Fibonacci symbol elements, and, from this formula, we found examples of many invertible elements, namely all Fibonacci symbol elements are invertible. Using some ideas from this paper, we expect to obtain some interesting new results in a further research and we hope that this kind of sets of invertible elements will help us to provide, in the future, examples of symbol division algebra of degree three or of degree greater than three. This idea will be developed in our next researches.

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## REFERENCES


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