QUASIRECOGNITION BY PRIME GRAPH OF $L_n(2^\alpha)$ FOR SOME n AND α

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Let G be a finite group. The prime graph of G is denoted by $\Gamma(G)$. In this paper as the main result, we show that if G is a finite group such that $\Gamma(G) = \Gamma(L_n(2^\alpha))$, where $n = 4m+3$, $3 \nmid \alpha$ and α is odd, then G has a unique nonabelian composition factor isomorphic to $L_n(2^\alpha)$. We also show that if G is a finite group satisfying $|G| = |L_n(2^\alpha)|$ and $\Gamma(G) = \Gamma(L_n(2^\alpha))$, then $G \cong L_n(2^\alpha)$. As a consequence of our result we give a new proof for a conjecture of W. J. Shi and J. X. Bi for $L_n(2^\alpha)$. Application of this result to the problem of recognition of finite simple groups by the set of element orders are also considered. Specially it is proved that by the above conditions, $L_n(2^\alpha)$ is quasirecognizable by spectrum.

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1. INTRODUCTION

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. The *spectrum* of a finite group G which is denoted by $\omega(G)$ is the set of its element orders. We construct the *prime graph* of G which is denoted by $\Gamma(G)$ as follows: the vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (we write $p \sim q$) if and only if G contains an element of order pq. Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_i(G)$, $i = 1, ..., s(G)$, be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$ we always suppose that $2 \in \pi_1(G)$. In graph theory a subset of vertices of a graph is called an independent set if its vertices are pairwise non-adjacent. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise non-adjacent in $\Gamma(G)$. In other words, if $\rho(G)$ is some independent set with the maximal number of vertices in $\Gamma(G)$, then $t(G) = |\rho(G)|$. Similarly if $p \in \pi(G)$, then let $\rho(p, G)$ be some independent set with the maximal number of vertices in $\Gamma(G)$ containing p and $t(p, G) = |\rho(p, G)|$.

A finite group G is called *recognizable by prime graph* if $\Gamma(H) = \Gamma(G)$ implies that $H \cong G$. A nonabelian simple group P is called *quasirecognizable by prime graph* if every finite group whose prime graph is $\Gamma(P)$ has a unique nonabelian composition factor isomorphic to P (see [11]). Obviously recognition (quasirecognition) by prime graph implies recognition (quasirecognition) by spectrum, but the converse is not true in general. Also some method of recognition by spectrum cannot be used for recognition by prime graph.

Hagie in [8], determined finite groups G satisfying $\Gamma(G) = \Gamma(S)$, where S is a sporadic simple group. It is proved that if $q = 3^{2n+1}$ $(n > 0)$, then the simple group ${}^2G_2(q)$ is uniquely determined by its prime graph [11, 31]. A group G is called a CIT group if G is of even order and the centralizer in G of any involution is a 2-group. In [14], finite groups with the same prime graph as a CIT simple group are determined. It is proved that the simple group $F_4(q)$, where $q = 2^n > 2$, (see [12]), and ${}^2F_4(q)$, (see [1]), are quasirecognizable by prime graph. Also in [10], it is proved that if p is a prime number which is not a Mersenne or Fermat prime and $p \neq 11, 13, 19$ and $\Gamma(G) = \Gamma(\mathrm{PGL}(2, p))$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2, p)$ and if $p = 13$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2, 13)$ or $\mathrm{PSL}(2, 27)$. Then it is proved that if p and $k > 1$ are odd and $q = p^k$ is a prime power, then $\mathrm{PGL}(2, q)$ is uniquely determined by its prime graph [2].

In [3], it is proved that if $p = 2^n + 1 \geq 5$ is a prime number, then ${}^2D_p(3)$ is quasirecognizable by prime graph. Then in [5], the authors proved that ${}^2D_{2^m+1}(3)$ is recognizable by prime graph.

In [13, 19], finite groups with the same prime graph as $\mathrm{PSL}(2, q)$, where q is not prime, are determined. Also in [15], it is proved that if $p > 11$ is a prime number and $p \not\equiv 1 \pmod{12}$, then PSL(2, p) is uniquely determined by its prime graph. In [16–18, 20] finite groups with the same prime graph as $L_n(2)$ are obtained. Let $G = L_n(2^k)$. In [7, 22, 28], it is proved that the recognizablity problem by spectrum is solved for $n = 4$, $n = 16$ and for $n = 2^m \geq 32$. Also recognizability by spectrum for simple groups $PSL(2, q)$, was proved in [5].

In this paper as the main result, we show that if G is a finite group such that $\Gamma(G) = \Gamma(L_n(2^\alpha))$, where $n = 4m + 3 \geq 27$, $3 \nmid \alpha$ and α is odd, then G has a unique nonabelian composition factor isomorphic to $L_n(2^\alpha)$, *i.e.* the simple group $L_n(2^\alpha)$ is quasirecognizable by prime graph. As a consequence of our result it is proved that $L_n(2^\alpha)$ is quasirecognizable by spectrum.

In this paper, all groups are finite and by simple groups we mean nonabelian simple groups. All further unexplained notations are standard and refer to [6]. Throughout the proof we use the classification of finite simple groups. In ([25], Tables 2–9), independent sets also independent numbers for all simple groups are listed and we use these results in the proof of the main theorem of this paper.

2. PRELIMINARY RESULTS

LEMMA 2.1 ([27], Theorem 1]). *Let G be a finite group with $t(G) \geq 3$ and $t(2, G) \geq 2$. Then the following hold:*

1. *there exists a finite nonabelian simple group S such that $S \leq \bar{G} = G/K \leq \mathrm{Aut(S)}$ for the maximal normal soluble subgroup K of G.*
2. *for every independent subset ρ of $\pi(G)$ with $|\rho| \geq 3$ at most one prime in ρ divides the product $|K||\bar{G}/S|$. In particular, $t(S) \geq t(G) - 1$.*
3. *one of the following holds:*
   (a) *every prime $r \in \pi(G)$ non-adjacent to 2 in $\Gamma(G)$ dose not divide the product $|K||\bar{G}/S|$; in particular, $t(2, S) \geq t(2, G)$.*
   (b) *there exists a prime $r \in \pi(K)$ non-adjacent to 2 in $\Gamma(G)$; in which case $t(G) = 3$, $t(2, G) = 2$, and $S \cong \mathrm{Alt}_7$ or $L_2(q)$ for some odd q.*

Remark 2.2. In Lemma 2.1, for every odd prime $p \in \pi(S)$, we have $t(p, S) \geq t(p, G) - 1$.

LEMMA 2.3 (Zsigmondy Theorem, [32]). *Let p be a prime and let n be a positive integer. Then one of the following holds:*

(i) *there is a primitive prime p' for $p^n - 1$, that is , $p' \mid (p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leq m < n$, (usually p' is denoted by r_n)*

(ii) *$p = 2$, $n = 1$ or 6,*

(iii) *p is a Mersenne prime and $n = 2$.*

Remark 2.4 ([23]). Let p be a prime number and $(q, p) = 1$. Let $k \geq 1$ be the smallest positive integer such that $q^k \equiv 1 \pmod{p}$. Then k is called *the order of q with respect to p* and we denote it by $\mathrm{ord}_p(q)$. Obviously by the Fermat's little theorem it follows that $\mathrm{ord}_p(q)|(p-1)$. Also if $q^n \equiv 1 \pmod{p}$, then $\mathrm{ord}_p(q)|n$. Similarly if $m > 1$ is an integer and $(q, m) = 1$, we can define $\mathrm{ord}_m(q)$. If a is odd, then $\mathrm{ord}_a(q)$ is denoted by $e(a, q)$, too. If q is odd, let $e(2, q) = 1$ if $q \equiv 1 \pmod{4}$ and $e(2, q) = 2$ if $q \equiv -1 \pmod{4}$

LEMMA 2.5 ([21]). *Let N be a normal subgroup of G. Assume that G/N is a Frobenius group with Frobenius kernel F and cyclic Frobenius complement C. If $(|N|, |F|) = 1$, and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$, where p is a prime divisor of $|N|$.*

LEMMA 2.6 ([9]). *Let G be a finite simple group $A_{n-1}(q)$.*

1. *If there exists a primitive prime divisor r of $q^n - 1$, then G contains a Frobenius subgroup with kernel of order r and cyclic complement of order n.*
2. *G contains a Frobenius subgroup with kernel of order q^{n-1} and cyclic complement of order $(q^{n-1} - 1)/(n, q - 1)$.*

Lemma 2.7 ([9]). *Let G be a finite simple group.*

1. *If $G = C_n(q)$, then G contains a Frobenius subgroup with kernel of order q^n and cyclic complement of order $(q^n - 1)/(2, q-1)$.*
2. *If $G = {}^2D_n(q)$, and there exists a primitive prime divisor r of $q^{2n-2} - 1$, then G contains a Frobenius subgroup with kernel of order q^{2n-2} and cyclic complement of order r.*
3. *If $G = B_n(q)$ or $D_n(q)$, and there exists a primitive prime divisor r_m of $q^m - 1$ where $m = n$ or $n-1$ such that m is odd, then G contains a Frobenius subgroup with kernel of order $q^{m(m-1)/2}$ and cyclic complement of order r_m.*

Lemma 2.8 ([25], Proposition 2.1]). *Let $G = A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic p. Let r and s be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Denote $k = e(r,q)$ and $l = e(s,q)$ and suppose that $2 \leq k \leq l$. Then r and s are non-adjacent if and only if $k + l > n$, and k does not divide l.*

3. MAIN RESULTS

We prove a refinement of Lemma 2.5, which is used in the proof of the main theorem.

Lemma 3.1. *Let G be a group satisfying the conditions of Lemma 2.1, and let the groups K and S be as in the claim of Lemma 2.1. Let there exist $p \in \pi(K)$ and $p' \in \pi(S)$ such that $p \nsim p'$ in $\Gamma(G)$, and S contains a Frobenius subgroup with kernel F and cyclic complement C such that $(|F|, |K|) = 1$. Then $p|C| \in \omega(G)$.*

Proof. We claim that $F \nleq KC_G(K)/K$. Since $KC_G(K)/K \trianglelefteq G/K$, so $S \cap KC_G(K)/K \trianglelefteq S$. Let $S \cap KC_G(K)/K = S$. Then $S \leq KC_G(K)/K$. So for every $t' \in \pi(S)$ and $t \in \pi(K)$ we have $t' \sim t$, which is a contradiction. Consequently $S \cap KC_G(K)/K = 1$, since S is a simple group. So $F \nleq KC_G(K)/K$, since $F \leq S$. Therefore $p|C| \in \omega(G)$, by Lemma 2.5. □

Remark 3.2. In the sequel of this paper, let $M = L_n(q)$, where $n = 4m + 3 \geq 27$, $q = 2^\alpha$, $3 \nmid \alpha$, and α is odd. Also we denote a primitive prime divisor of $q^i - 1$ by r_i. We note that $\pi(M) = \pi(q^{n(n-1)/2} \prod_{i=2}^{n}(q^i - 1))$. Also we know that $\rho(2, M) = \{2, r_{n-1}, r_n\}$, $t(M) = [(n+1)/2] \geq 14$ and $\rho(M) = \{r_i \mid [n/2] < i \leq n\}$.

Lemma 3.3. *By the above assumptions, we have $\rho(5, M) = \{5, r_{n-2}, r_{n-1}, r_n\}$ also we have $t(13, M) = 12$.*

Proof. We know that $e(5, 2^\alpha) = 4$, since α is odd. If $r_i \in \pi(M)$, then $r_i \nsim 5$ in $\Gamma(M)$, if and only if $i + 4 > n$ and $4 \nmid i$, by Lemma 2.8. Therefore 5 is not adjacent to r_n, r_{n-1} and r_{n-2}. Also by $\rho(M)$ in Remark 3.2, r_n, r_{n-1} and r_{n-2} are pairwise nonadjacent in $\Gamma(M)$, so $\rho(5, M) = \{5, r_{n-2}, r_{n-1}, r_n\}$.

On the other hand, we know that $e(13, 2^\alpha) = 12$, since $(12, \alpha) = 1$. If $r_i \in \pi(M)$, then $r_i \nsim 13$ in $\Gamma(M)$, if and only if $i + 12 > n$ and $12 \nmid i$, by Lemma 2.8. Therefore $t(13, M) = 12$ and the proof is completed. □

Lemma 3.4. *Let $G = D_n^\varepsilon(q')$, where $4 \mid (n-1)$ and $e(5, q') = 4$.*

1. *If $\varepsilon = +$, then $\rho(5, S) = \{5, r'_n, r'_{n-2}, r'_{2(n-1)}\}$.*
2. *If $\varepsilon = -$, then $\rho(5, S) = \{5, r'_{2n}, r'_{2(n-1)}, r'_{2(n-2)}\}$.*

Proof. If $r'_i \in \pi(G)$, such that $r'_i \nsim 5$ in $\Gamma(G)$, then by ([26], Proposition 2.5), we have:

$$2\eta(i) + 4 > 2n - (1 - \varepsilon(-1)^{i+4}) \qquad \text{and} \qquad \text{i/4} \quad \text{is not odd.}$$

Let $\varepsilon = +$. Therefore 5 is not adjacent to r'_n, r'_{n-2} and $r'_{2(n-1)}$. Also by [25, Table 8], r'_n, r'_{n-2} and $r'_{2(n-1)}$ are pairwise nonadjacent in $\Gamma(G)$, so $\rho(5, G) = \{5, r'_n, r'_{n-2}, r'_{2(n-1)}\}$.

Similarly, if $\varepsilon = -$, then $\rho(5, S) = \{5, r'_{2n}, r'_{2(n-1)}, r'_{2(n-2)}\}$. □

Lemma 3.5. *In the prime graph of M, either $r_n \nsim 7 \nsim r_{n-1}$ or $7 \nsim r_{n-2}$.*

Proof. we know that $e(7, 2^\alpha) = 3$, since $3 \nmid \alpha$. If $r_i \in \pi(M)$, then $r_i \nsim 7$ in $\Gamma(M)$, if and only if $i + 3 > n$ and $3 \nmid i$, by Lemma 2.8. Therefore if $3 \mid (n-2)$, then $r_n \nsim 7 \nsim r_{n-1}$, otherwise $7 \nsim r_{n-2}$ in $\Gamma(M)$. □

Theorem 3.6. *The simple group $M = L_n(q)$, where $n = 4m + 3 \geq 27$, $q = 2^\alpha$, $3 \nmid \alpha$, and α is odd, is quasirecognizable by prime graph, i.e. if G is a finite group such that $\Gamma(G) = \Gamma(M)$, then G has a unique nonabelian composition factor isomorphic to M.*

Proof. We know that $t(M) \geq 14$, and $t(2, M) = 3$. By Lemma 2.1, there exists a nonabelian simple group S, such that $S \leq \bar{G} = G/K \leq \mathrm{Aut}(S)$, where K is the maximal normal soluble subgroup of G.

We know that $\{r_n, r_{n-1}\} \subseteq \rho(2, G)$. Therefore $\{r_n, r_{n-1}\} \subseteq \pi(S)$ and $r_n \nsim 2 \nsim r_{n-1}$ in $\Gamma(S)$. By Lemma 2.1, we know that $t(S) \geq 13$ and $t(2, S) \geq 3$. In the sequel we consider each possibility for S by ([29], Tables 1a–1c):

Step 1. The simple group S cannot be isomorphic to an alternating group $A_{n'}$, where $n' \geq 5$.

Since $t(S) \geq 13$, it follows that $n' \geq 17$. If $x \in \pi(A_{n'})$ such that $x \nsim 13$, then $n' - 13 < x \leq n'$, by [25, Proposition 1.1]. On the other hand, there exist at least $[13/2] + [13/3] - [13/6] = 8$ elements of $[n' - 12, n']$, which are divisible by 2 or 3. Therefore at most 5 elements of $[n'-12, n']$ are prime numbers. Hence, $t(13, S) \leq 6$. Therefore by Remark 2.2, $t(13, G) \leq 7$, which is a contradiction, by Lemma 3.3.

Step 2. If S is a classical Lie type group, then we will prove that $S \cong M$. We prove the result in the following cases.

We denote by $D^{+}_{n'}(q')$ the simple group $D_{n'}(q')$, and by $D^{-}_{n'}(q')$ the simple group ${}^2D_{n'}(q')$.

Case 1. Let $S \cong D^{\varepsilon}_{n'}(q')$, where $q' = p^f$.

(a) Let $p \neq 2$.

We know that $t(2, S) \geq 3$, which implies that $n' \equiv 1 \pmod 2$ and if $\varepsilon = +$, then $q' \equiv 5 \pmod 8$ and if $\varepsilon = -$, then $q' \equiv 3 \pmod 8$, by ([25], Table 6). Therefore n' is odd. In the rest of this case we consider the simple groups $D_{n'}(q')$ and ${}^2D_{n'}(q')$ simultaneously.

- If $S \cong D_{n'}(q')$, then each r'_i, where $i \notin \{n', 2(n'-1)\}$, is adjacent to 2 in $\Gamma(S)$ and $\rho(2, S) = \{2, r'_{n'}, r'_{2(n'-1)}\}$, by [25, Proposition 4.4]. We know that $r_n \nsim 2 \nsim r_{n-1}$ in $\Gamma(S)$. Therefore r_n and r_{n-1} are some primitive prime divisors of $q'^{n'} - 1$ and $q'^{2(n'-1)} - 1$, say $r'_{n'}$ and $r'_{2(n'-1)}$. In the sequel of this paper for simplicity we use the following notation to illustrate these relations:

$$\{r_n, r_{n-1}\} \approx \{r'_{n'}, r'_{2(n'-1)}\}. \tag{1}$$

On the other hand, using Lemma 3.3 we have $r_n \nsim 5 \nsim r_{n-1}$ in $\Gamma(G)$, and so $r'_{n'} \nsim 5 \nsim r'_{2(n'-1)}$ in $\Gamma(S)$.

- ∘ If $S \cong {}^2D_{n'}(q')$, then $\rho(2, S) = \{2, r'_{2n'}, r'_{2(n'-1)}\}$, by ([25], Proposition 4.4). Similarly to the above we have

$$\{r_n, r_{n-1}\} \approx \{r'_{2n'}, r'_{2(n'-1)}\}, \tag{2}$$

and $r'_{2n'} \nsim 5 \nsim r'_{2(n'-1)}$ in $\Gamma(S)$.

now we consider two following cases:

▶ First suppose that $p = 13$, and so $q' = 13^f$. We know that $e(5, q') \mid 4$, by Fermat's little theorem.

Step 1. In this step, we determine $\rho(5, S)$.

- If $S \cong D_{n'}(q')$, then we claim that $e(5, q') = 4$. If $e(5, q') = 1$, then $5 \sim r'_{n'}$ in $\Gamma(S)$, by ([26], Proposition 2.5), since n' is odd, which is a contradiction. If $e(5, q') = 2$, then $5 \sim r'_{2(n'-1)}$, by [26, Proposition 2.5], which is a contradiction. Therefore $e(5, q') = 4$. As we mentioned above $5 \nsim r'_{2(n'-1)}$ in $\Gamma(S)$. Therefore by ([26], Proposition 2.5), we have $2(n'-1) + 4 > 2n' - (1 - (-1)^{4+2(n'-1)})$ and $2(n'-1)/4$ is not an odd integer.

Since n' is odd we conclude that $4 \mid (n'-1)$. Now using Lemma 3.4, we have $\rho(5,S)=\{5, r'_{n'}, r'_{n'-2}, r'_{2(n'-1)}\}$.

- ∘ If $S \cong {}^2D_{n'}(q')$, then we claim that $e(5,q')=4$. If $e(5,q')=1$, then $5 \sim r'_{2(n'-1)}$ in $\Gamma(S)$, by ([26], Proposition 2.5), which is a contradiction. If $e(5,q')=2$, then $5 \sim r'_{2n'}$, by ([26], Proposition 2.5), which is a contradiction. Therefore $e(5,q')=4$. We know that $5 \nsim r'_{2(n'-1)}$ in $\Gamma(S)$ and similarly to the previous case, by using ([26], Proposition 2.5), we have $4 \mid (n'-1)$ and $\rho(5,S)=\{5, r'_{2n'}, r'_{2(n'-1)}, r'_{2(n'-2)}\}$, by Lemma 3.4.

Step 2. Now we prove that $r_{n-2} \in \pi(S)$.

Otherwise $r_{n-2} \in \pi(K) \cup \pi(\bar{G}/S)$. If $r_{n-2} \in \pi(\bar{G}/S)$, then $r_{n-2} \in \pi(\mathrm{Out}(S)) = \{2\} \cup \pi(f)$. We know that $r_{n-2} \neq 2$ and if $r_{n-2} \mid f$, then r_{n-2} is the order of a field automorphism of S. We know that a field automorphism centralizes the elements of $D^{\varepsilon}_{n'}(13)$ and $5 \in \pi(D^{\varepsilon}_{n'}(13))$. So 5 is adjacent to r_{n-2}, which is a contradiction, by Lemma 3.3. Therefore $r_{n-2} \in \pi(K)$.

- • If $S \cong D_{n'}(q')$, then by Lemma 2.7, $D_{n'}(q')$ contains a Frobenius subgroup of the form $q'^{n'(n'-1)/2} : r'_{n'}$. Since $r_n \in \pi(S)$, $r_n \nsim r_{n-2}$ in $\Gamma(G)$ and $(r_{n-2}, q')=1$, then by Lemma 3.1, $r_{n-2} \sim r'_{n'}$. Therefore using (1), $r_{n-2} \sim r_n$ or $r_{n-2} \sim r_{n-1}$, which is a contradiction, by Lemma 3.3.
- ∘ If $S \cong {}^2D_{n'}(q')$, then by Lemma 2.7, ${}^2D_{n'}(q')$ contains a Frobenius subgroup of the form $q'^{2(n'-1)} : r'_{2(n'-1)}$. Since $r_n \in \pi(S)$, $r_n \nsim r_{n-2}$ in $\Gamma(G)$ and $(r_{n-2}, q')=1$, then by Lemma 3.1, $r_{n-2} \sim r'_{2(n'-1)}$, which is a contradiction, by (2) and Lemma 3.3.

Thus, $r_{n-2} \in \pi(S)$. On the other hand, we note that $e(7,13)=2$, and so $e(7,q') \mid 2$.

Step 3. Finally we prove that $r_{n-2} \nsim 7$ in $\Gamma(G)$ and we get a contradiction.

- • Let $S \cong D_{n'}(q')$, so $r_{n-2} = r'_{n'-2}$, by Lemma 3.4. If $e(7,q')=1$, then $r'_{n'} \sim 7$ in $\Gamma(S)$, by ([26], Proposition 2.5), and if $e(7,q')=2$, then $r'_{2(n'-1)} \sim 7$ in $\Gamma(S)$, by ([26], Proposition 2.5). Therefore by (1) and Lemma 3.5, $r_{n-2} \nsim 7$ in $\Gamma(G)$ and so $r'_{n'-2} \nsim 7$ in $\Gamma(S)$. Using ([26], Proposition 2.5), we see that $r'_{n'-2} \sim 7$ in $\Gamma(S)$ either $e(7,q')=1$ or 2, which is a contradiction.
- ∘ Let $S \cong {}^2D_{n'}(q')$, so $r_{n-2} = r'_{2(n'-2)}$, by Lemma 3.4. Similarly to the above case, we have $r'_{2(n'-2)} \nsim 7$ in $\Gamma(S)$, which is a contradiction, by ([26], Proposition 2.5).

► Therefore $p \neq 13$. We know that $t(13,G)=12$, by Lemma 3.3, so $t(13,S) \geq 11$. On the other hand, $e(13,q') \mid 12$. Let $e(13,q')=t$. If $r'_i \in \pi(S)$, such that $r'_i \nsim 13$ in $\Gamma(S)$, then by ([26], Proposition 2.5), we have $12 + 2\eta(i) \geq 2\eta(t) + 2\eta(i) > 2n' - (1-\varepsilon(-1)^{t+i})$.

- Let $S \cong D_{n'}(q')$. We consider two following cases:
  (i) Let t be an even number. If i is an even number, then $i \in \{2(n'-5), 2(n'-4), 2(n'-3), 2(n'-2), 2(n'-1)\}$, and if i is an odd number, then $i \in \{n'-6, n'-4, n'-2, n'\}$. Therefore $t(13, S) \leq 10$, which is a contradiction.
  (ii) Let t be an odd number. If i is an even number, then $i \in \{2(n'-6), 2(n'-5), 2(n'-4), 2(n'-3), 2(n'-2), 2(n'-1)\}$, and if i is an odd number, then $i \in \{n'-4, n'-2, n'\}$. Therefore $t(13, S) \leq 10$, which is a contradiction, again.
- Let $S \cong {}^2D_{n'}(q')$. We consider two following cases:
  (i) Let t be an even number. If i is an even number, then $i \in \{2(n'-6), 2(n'-5), 2(n'-4), 2(n'-3), 2(n'-2), 2(n'-1), 2n'\}$, and if i is an odd number, then $i \in \{n'-4, n'-2\}$. Therefore $t(13, S) \leq 10$, which is a contradiction.
  (ii) Let t be an odd number. If i is an even number, then $i \in \{2(n'-5), 2(n'-4), 2(n'-3), 2(n'-2), 2(n'-1), 2n'\}$, and if i is an odd number, then $i \in \{n'-6, n'-4, n'-2\}$. Therefore $t(13, S) \leq 10$, which is a contradiction.

(b) By the above discussions $p = 2$ and so $q' = 2^f$.

If $n' \equiv 1 \pmod 2$, then since $\rho(2, S)$ is similar to the 2-independence set in (a), similarly to the proof of (a) we get a contradiction. So let $n' \equiv 0 \pmod 2$.

- If $S \cong D_{n'}(q')$, then similarly to the above $\{r_n, r_{n-1}\} \approx \{r'_{n'-1}, r'_{2(n'-1)}\}$. We know that $\pi(S) \subseteq \pi(G)$, so $2(n'-1)f \leq n\alpha$. Now we consider the following cases:
  ▶ Let $r_n = r'_{n'-1}$.
  Let p_0 be a primitive prime divisor of $2^{n\alpha}-1$. Then p_0 is a primitive prime divisor of $(2^\alpha)^n - 1$. Since $r_n = r'_{n'-1}$, it follows that $p_0 \mid (2^{(n'-1)f} - 1)$. Therefore $n\alpha \leq (n'-1)f$, which is a contradiction.
  ▶ Let $r_n = r'_{2(n'-1)}$.
  Let p_0 be a primitive prime divisor of $2^{n\alpha} - 1$. Similarly to the above $n\alpha \leq 2(n'-1)f$, since $r_n = r'_{2(n'-1)}$. Consequently $n\alpha = 2(n'-1)f$. By the assumptions, we know that n and α are odd, which is a contradiction.
- If $S \cong {}^2D_{n'}(q')$, similarly to the above $\{r_n, r_{n-1}\} \approx \{r'_{2n'}, r'_{2(n'-1)}, r'_{n'-1}\}$. We know that $\pi(S) \subseteq \pi(G)$, so $2n'f \leq n\alpha$. We consider the following cases:
  ▶ Let $r_n = r'_{n'-1}$.
  Let p_0 be a primitive prime divisor of $2^{n\alpha}-1$. Then p_0 is a primitive prime divisor of $(2^\alpha)^n - 1$. Since $r_n = r'_{n'-1}$, it follows that $p_0 \mid (2^{(n'-1)f} - 1)$.

Therefore $n\alpha \leq (n'-1)f$, which is a contradiction.

▶ Let $r_n = r'_{2(n'-1)}$. Similarly to the above we get a contradiction.

▶ Let $r_n = r'_{2n'}$.

Similarly to the above we have $n\alpha \leq 2n'f$. Therefore $n\alpha = 2n'f$. We know that n and α are odd, which is a contradiction.

Case 2. Let $S \cong C_{n'}(q')$ or $B_{n'}(q')$, where $q' = p^f$.

We know that $t(2, S) \geq 3$. Then $p = 2$ and $n' > 1$ is odd, by ([25], Table 4). In this case, we have $\rho(2, S) = \{2, r'_{n'}, r'_{2n'}\}$, by ([25], Proposition 4.3). Similarly to the above $\{r_n, r_{n-1}\} \approx \{r'_{n'}, r'_{2n'}\}$. On the other hand, we know that $\pi(S) \subseteq \pi(G)$, so $2n'f \leq n\alpha$. We consider the following cases:

▶ Let $r_n = r'_{n'}$.

Let p_0 be a primitive prime divisor of $2^{n\alpha} - 1$. Then p_0 is a primitive prime divisor of $(2^\alpha)^n - 1$. Since $r_n = r'_{n'}$, it follows that $p_0 \mid (2^{n'f} - 1)$. Therefore $n\alpha \leq n'f$, which is a contradiction.

▶ Let $r_n = r'_{2n'}$.

Similarly to the above we have $n\alpha \leq 2n'f$. Therefore $n\alpha = 2n'f$. We know that n and α are odd, which is a contradiction.

We denote by $A^+_{n'}(q')$ the simple group $A_{n'}(q')$, and by $A^-_{n'}(q')$ the simple group ${}^2A_{n'}(q')$.

Case 3. Let $S \cong A^{\varepsilon}_{n'-1}(q')$, where $q' = p^f$ and $\varepsilon \in \{+, -\}$.

We know that $t(S) \geq t(G)-1 \geq 13$. Therefore $[(n'+1)/2] \geq [(n+1)/2]-1$ which implies that $n' > n - 4$.

(a) Let $p \neq 2$.

We know that $t(2, S) \geq 3$, and so by ([25], Table 6), $2 < n'_2 = (q' - \varepsilon 1)_2$. In the sequel, for convenience we state the proof for $A_{n'-1}(q)$ and ${}^2A_{n'-1}(q)$, simultaneously.

- If $S \cong A_{n'-1}(q')$, then each r'_i, where $i \notin \{n'-1, n'\}$, is adjacent to 2 in $\Gamma(S)$ and $\rho(2, S) = \{2, r'_{n'-1}, r'_{n'}\}$, by ([25], Table 6) and ([25], Proposition 4.1). We know that $r_n \nsim 2 \nsim r_{n-1}$ in $\Gamma(S)$. Therefore similarly to the above

(3) $$\{r_n, r_{n-1}\} \approx \{r'_{n'}, r'_{n'-1}\},$$

On the other hand, $r_n \nsim 5 \nsim r_{n-1}$ in $\Gamma(G)$, by Lemma 3.3. Therefore $r'_{n'} \nsim 5 \nsim r'_{n'-1}$ in $\Gamma(S)$.

∘ If $S \cong {}^2A_{n'-1}(q')$, then each r'_i, where $i \notin \{n', 2(n'-1)\}$, is adjacent to 2 in $\Gamma(S)$ and $\rho(2, S) = \{2, r'_{n'}, r'_{2(n'-1)}\}$, by ([25], Table 6) and ([25], Proposition 4.1). Similarly to the above we have

(4) $$\{r_n, r_{n-1}\} \approx \{r'_{n'}, r'_{2(n'-1)}\}.$$

Also we know that $r_n \nsim 5 \nsim r_{n-1}$ in $\Gamma(G)$. Hence, $r'_{n'} \nsim 5 \nsim r'_{2(n'-1)}$ in $\Gamma(S)$.

Let $p \neq 5$. We know that $e(5, q') \mid 4$.

- If $S \cong A_{n'-1}(q')$ and $2 \mid e(5, q')$, then $5 \sim r'_{n'}$, by ([25], Proposition 2.1), since $n'_2 > 2$, which is a contradiction. Therefore $e(5, q') = 1$ and so $5 \mid (q' - 1)$.
- ◦ If $S \cong {}^2A_{n'-1}(q')$ and $e(5, q') = 4$ or $e(5, q') = 1$, then $5 \sim r'_{n'}$, by ([25], Proposition 2.2), which is a contradiction. Therefore $e(5, q') = 2$ and so $5 \mid (q' + 1)$.

Now we prove that $r_{n-2} \in \pi(S)$.

Otherwise, $r_{n-2} \in \pi(K) \cup \pi(\bar{G}/S)$. If $r_{n-2} \in \pi(\bar{G}/S)$, then $r_{n-2} \in \pi(\mathrm{Out}(S)) \subseteq \{2\} \cup \pi(f(n', q' - \varepsilon 1))$. We know that $r_{n-2} \neq 2$. If $r_{n-2} \mid (q' - \varepsilon 1)$, then $r_{n-2} \in \pi(S)$, since $(q' - \varepsilon 1) \mid |S|$, which is a contradiction. If $r_{n-2} \mid f$, then r_{n-2} is the order of a field automorphism of S. We know that a field automorphism centralizes the elements of $A^{\varepsilon}_{n'-1}(p)$ and $5 \in \pi(A^{\varepsilon}_{n'-1}(p))$. So 5 is adjacent to r_{n-2}, which is a contradiction, by Lemma 3.3. Therefore $r_{n-2} \in \pi(K)$.

- If $S \cong A_{n'-1}(q')$, then it is proved that $5 \mid (q' - 1)$. By Lemma 2.6, $A_{n'-1}(q')$ contains a Frobenius subgroup of the form $q'^{n'-1} : (q'^{n'-1} - 1)/(n', q' - 1)$. Since $r_n \in \pi(S)$, $r_n \nsim r_{n-2}$ in $\Gamma(G)$ and $(r_{n-2}, q') = 1$, then by Lemma 3.1, $r_{n-2} \sim \pi((q'^{n'-1} - 1)/(n', q' - 1))$. So $r_{n-2} \sim r'_{n'-1}$, which is a contradiction, by (3) and Lemma 3.3.
- ◦ If $S \cong {}^2A_{n'-1}(q')$, then $5 \mid (q' + 1)$. By [24], we have $C_{n'/2}(q') \leq {}^2A_{n'-1}(q')$. By Lemma 2.7, $C_{n'/2}(q')$ contains a Frobenius subgroup of the form $q'^{n'/2} : (q'^{n'/2} - 1)/2$. Since $r_n \in \pi(S)$, $r_n \nsim r_{n-2}$ in $\Gamma(G)$ and $(r_{n-2}, q') = 1$, then by Lemma 3.1, $r_{n-2} \sim \pi((q'^{n'/2} - 1)/2)$. Since $n'_2 > 2$, then $(q'^2 - 1) \mid (q'^{n'/2} - 1)$ so $r_{n-2} \sim 5$, which is a contradiction, by Lemma 3.3.

Thus, $r_{n-2} \in \pi(S)$ and so $t(5, S) \geq 4$. As we mentioned above $5 \mid (q - \varepsilon 1)$ and $5 \nsim r_{n-2}$, by Lemma 3.3.

- If $S \cong A_{n'-1}(q')$, then by [25, Proposition 4.1], each r'_i, where $i \notin \{n' - 1, n'\}$, is adjacent to 5 in $\Gamma(S)$. Therefore $t(5, S) \leq 3$ and we get a contradiction.
- ◦ If $S \cong {}^2A_{n'-1}(q')$, then by [25, Proposition 4.2], each r'_i, where $i \notin \{2(n' - 1), n'\}$, is adjacent to 5 in $\Gamma(S)$ and so $t(5, S) \leq 3$, which is a contradiction.

Therefore $p = 5$ and so $q' = 5^f$.

- Let $S \cong A_{n'-1}(q')$. We know that $e(19, 5^f) \mid 9$. Let $r'_i \in \pi(S)$, such that $r'_i \nsim 19$ in $\Gamma(S)$. If $e(19, 5^f) = 1$, then $i \in \{n' - 1, n'\}$, by ([25], Proposition 4.1), so $t(19, S) \leq 3$. Otherwise $i \in \{n', ..., n'-8\}$, by Lemma 2.8 so $t(19, S) \leq 9$. Therefore by Lemma 2.2, $t(19, G) \leq 10$. On the other hand, we know that $e(19, 2) = 18$. Then $e(19, 2^\alpha) = 18$, since $(18, \alpha) = 1$. If $r_i \in \pi(G)$, such that $r_i \nsim 19$ in $\Gamma(G)$, then by Lemma 2.8, $i + 18 > n$, $18 \nmid i$ and $i \nmid 18$. Therefore $t(19, G) \geq 12$, which is a contradiction.

- Let $S \cong {}^2A_{n'-1}(q')$. Since $q' = 5^f$, then $(q' + 1)_2 = 2$, which is a contradiction.

(b) Therefore $p = 2$ and so $q' = 2^f$.

- If $S \cong A_{n'-1}(q')$, then each r'_i, where $i \notin \{n' - 1, n'\}$, is adjacent to 2 in $\Gamma(S)$ and $\rho(2, S) = \{2, r'_{n'}, r'_{n'-1}\}$, by ([25], Proposition 3.1). Therefore similarly to the above $\{r_n, r_{n-1}\} \approx \{r'_{n'}, r'_{n'-1}\}$. We know that $\pi(S) \subseteq \pi(G)$, so $n'f \leq n\alpha$. Now we consider the following cases:
▶ Let $r_n = r'_{n'-1}$ and $r_{n-1} = r'_{n'}$.
Let p_0 be a primitive prime divisor of ${}^{n\alpha} - 1$. Then p_0 is a primitive prime divisor of $(2^\alpha)^n - 1$. Since $r_n = r'_{n'-1}$, it follows that $p_0 \mid (2^{(n'-1)f} - 1)$. Therefore $n\alpha \leq (n' - 1)f$, which is a contradiction.
▶ Let $r_n = r'_{n'}$ and $r_{n-1} = r'_{n'-1}$.
Similarly to the above $n\alpha \leq n'f$, since $r_n = r'_{n'}$. Therefore $n\alpha = n'f$. On the other hand, let p'_0 be a primitive prime divisor of $2^{(n-1)\alpha} - 1$, so $p'_0 \mid (2^{(n'-1)f} - 1)$. Consequently, $(n - 1)\alpha \mid (n' - 1)f$, and so $\alpha \mid f$. If $\alpha \neq f$, then $\alpha \leq f/2$ and so $n \geq 2n'$. Therefore $n \leq 8$, since $n - 4 < n'$, which is a contradiction. Consequently, $\alpha = f$ and $n = n'$. Hence, $S \cong M$.
- Let $S \cong {}^2A_{n'-1}(q')$. We consider the following cases:
(i) Let $n' \equiv 0 \pmod 4$. Then $\rho(2, S) = \{2, r'_{n'}, r'_{2(n'-1)}\}$, by ([25], Proposition 3.2). Therefore similarly to the above $\{r_n, r_{n-1}\} \approx \{r'_{n'}, r'_{2(n'-1)}\}$. We know that $\pi(S) \subseteq \pi(G)$, so $2(n' - 1)f \leq n\alpha$. Let $r_n = r'_{n'}$. If p_0 be a primitive prime divisor of $2^{n\alpha} - 1$, then p_0 is a primitive prime divisor of $(2^\alpha)^n - 1$. Therefore $p_0 \mid (2^{n'f} - 1)$, it follows that $n\alpha \leq n'f$, which is a contradiction. Therefore $r_n = r'_{2(n'-1)}$. Similarly, $n\alpha \leq 2(n' - 1)f$ and so $n\alpha = 2(n' - 1)f$, which is a contradiction, since n and α are odd.
(ii) Let $n' \equiv 1 \pmod 4$. Then $\rho(2, S) = \{2, r'_{2n'}, r'_{n'-1}\}$, by ([25], Proposition 3.2) and similarly to the above $\{r_n, r_{n-1}\} \approx \{r'_{2n'}, r'_{n'-1}\}$, and we get a contradiction.
(iii) Let $n' \equiv 2 \pmod 4$. Then $\rho(2, S) = \{2, r'_{n'/2}, r'_{2(n'-1)}\}$, by ([25], Proposition 3.2). Therefore similarly to the above we get a contradiction.
(iv) Let $n' \equiv 3 \pmod 4$. Then $\rho(2, S) = \{2, r'_{2n'}, r'_{(n'-1)/2}\}$, by ([25], Proposition 3.2) and similarly to the above we get a contradiction.

Therefore, $S \cong L_n(2^\alpha)$, and so the quasirecognition by prime graph is proved. $\square$

Theorem 3.7. *Let $q = 2^\alpha$, where α is odd and $3 \nmid \alpha$. If $\Gamma(G) = \Gamma(L_n(q))$, where $n = 4m+3 \geq 27$, then $L_n(q) \leq G/K \leq \mathrm{Aut}\ (L_n(q))$, where K is a p-group, where $p = 2$ or $p \mid (q^2-1)$.*

Proof. By Lemma 2.1, we know that $S \leq G/K \leq \mathrm{Aut}\ (S)$, where K is the maximal normal soluble subgroup of G. By Theorem 3.6, $S \cong L_n(2^\alpha)$. Let there exists p such that $p \mid |K|$. We claim that without loss of generality we can consider K as an elementary abelian p-group for $p \in \pi(G)$. Since K is soluble so there is $p \in \pi(G)$ such that $O^p(K) \neq K$. Then $K/O^p(K)$ is a nontrivial p-group. Let $\hat{K} = K/O^p(K)$ and $\hat{G} = G/O^p(K)$, since $O^p(K)$ is a characteristic subgroup of K and $K \lhd G$. If the Frattini subgroup of $\hat{K}$ is denoted by $\Phi(\hat{K})$, then $\hat{K}/\Phi(\hat{K})$ is an elementary abelian p-group and we have

$$\frac{G}{K} \cong \frac{\hat{G}}{\hat{K}} \cong \frac{\hat{G}/\Phi(\hat{G})}{\hat{K}/\Phi(\hat{K})}.$$

Therefore, without loss of generality we can assume that K is an elementary abelian p-group. Let $p \neq 2$.

We claim that if $t \in \pi(L_n(2^\alpha))$, then $t \nsim r_n$ or $t \nsim r_{n-1}$. Let $a = e(t, 2^\alpha)$. If $a = 1$, then $t \nsim r_n$ and $t \nsim r_{n-1}$, by ([25], Proposition 4.1). If $a > 1$, $t \sim r_n$ and $t \sim r_{n-1}$, then by Lemma 2.8, $a \mid n$ and $a \mid (n-1)$ or $(n-1)+a \leq n$, we get that $a = 1$, which is a contradiction.

By Lemma 2.7, $L_n(q)$ contains a Frobenius subgroup with kernel of order q^{n-1} and cyclic complement of order $(q^{n-1}-1)/(n, q-1)$. By assumption, $S \leq G/K$, and so G/K contains a Frobenius subgroup T/K of the form $q^{n-1} : (q^{n-1}-1)/(n, q-1)$. Since $p \neq 2$ and we know that $p \nsim r_n$ or $p \nsim r_{n-1}$, by Lemma 3.1, it follows that $p \sim r_{n-1}$. Also we know that $L_{n-1}(q) \hookrightarrow L_n(q)$, and so $L_{n-1}(q) \leq G/K$. Similarly G/K contains a Frobenius subgroup of the form $q^{n-2} : (q^{n-2}-1)/(n-1, q-1)$, by Lemma 2.7. Now $p \neq 2$ and $p \nsim r_n$ or $p \nsim r_{n-1}$. Therefore using Lemma 3.1, we get that $p \sim r_{n-2}$. Let $e(p, q) = l$. Since $p \sim r_{n-1}$, it follows that $(n-1)+l \leq n$ or $l \mid (n-1)$, by Lemma 2.8. Similarly since $p \sim r_{n-2}$ it follows that $(n-2)+l \leq n$ or $l \mid (n-2)$. Consequently, $l \leq 2$, so $p \mid (q^2-1)$. $\square$

Theorem 3.8. *Let G be a finite group satisfying $|G| = |L_n(2^\alpha)|$, where $n = 4m+3 \geq 27$, $3 \nmid \alpha$ and α is odd. If $\Gamma(G) = \Gamma(L_n(2^\alpha))$, then $G \cong L_n(2^\alpha)$.*

Proof. By assumption, $\Gamma(G) = \Gamma(L_n(2^\alpha))$. Now by Theorem 3.6, it follows that G has a normal series $S \leq G/K \leq \mathrm{Aut}\ (S)$ such that $S \cong L_n(2^\alpha)$. Also $|G| = |L_n(2^\alpha)|$ and so $K = 1$. Therefore, $G \cong L_n(2^\alpha)$. $\square$

Corollary 3.9. *Let G be a finite group satisfying $|G| = |L_n(2^\alpha)|$, where $n = 4m + 3 \geq 27$, $3 \nmid \alpha$ and α is odd. If $\omega(G) = \omega(L_n(2^\alpha))$, then $G \cong L_n(2^\alpha)$.*

We note that recently this theorem is proved for each finite simple group (see [30]).

Corollary 3.10. *Let $n = 4m + 1$, $3 \nmid \alpha$ and α be odd. Then $L_n(2^\alpha)$ is quasirecognizable by spectrum, i.e. if G is a finite group such that $\Gamma(G) = \Gamma(L_n(2^\alpha))$, then G has a unique nonabelian composition factor isomorphic to $L_n(2^\alpha)$.*

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