STABILITY AND REGULARIZATION FOR 2-D BACKWARD HEAT CONDUCTION EQUATION

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Consider a two-dimensional backward heat conduction problem for a simple domain such as a rectangle or a square. Based on the fundamental solution to the heat equation, we propose to solve this problem by a Fourier truncated method, through which we obtain a well-posed solution. The well-posedness of the proposed regularized problem and convergence property of the regularizing solution to the exact one are also be to proven. Some error estimates are given to show the efficiency of our method. The numerical example is presented to show the validity of the proposed methods.

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1. INTRODUCTION

Let $T > 0$ and $\Omega = (0, \pi)\times(0, \pi)$. We consider the problem of determining $u(x, y, 0)$ from the following system

$$(1)\qquad \begin{cases} u_t - u_{xx} - u_{yy} = f(x, y, t), \\ u(0, y, t) = u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) = 0, \\ u(x, y, T) = g(x, y), \end{cases}$$

where $(x, y, t) \in \Omega \times (0, T)$. The functions $f \in L^2(0, T; L^2(\Omega))$ and $g \in L^2(\Omega)$ may be given inexactly. The problem is called the backward heat problem (BHP) or the final value problem.

It is known in general that the backward problem is ill-posed, *i.e.*, a solution does not always exist, and in the case of existence, it does not depend continuously on the given datum. In fact, from a small noise contaminated physical measurement, the corresponding solutions may have a large error. This makes the numerical computation difficult. Hence, a regularization is needed. In the mathematical literatures, various methods have been proposed for solving backward Cauchy problems. We can notably mention the method of quasi-solution (QS method) of Tikhonov [25], the method of quasi-reversibility

(QR method) of Lattes and Lions [13, 17, 19], the method of logarithmic convexity [4, 3, 10, 15, 16, 20], the quasi boundary value method (Q.B.V. method) [6, 23, 27]. Physically, this problem arises from the requirement of recovering heat temperature at some earlier time using the knowledge of the final temperature. The problem is also involved in the situation of a particle moving in an environment with constant diffusion coefficient (see [7]) when one asks to determine the particle position history from its current place. The interest of backward heat equations also comes from financial mathematics, where the celebrated BlackScholes model [2] for call option can be transformed into a backward parabolic equation whose form is related closely to backward heat equations.

Although there are many papers on the linear homogeneous case of the backward problem, we only find a few papers on the nonhomogeneous case. Particularly, the two dimensional case of this one is rarely studied. For the nonhomogeneous and nonlinear cases, we refer the reader to some recent papers of Feng Xiao-Li [8], D.D. Trong and his group [26, 27, 29, 30, 31, 32]. Let us mention here some approaches of many earlier works and their technical difficulties. Physically, g can only be measured, there will be measurement errors, and we would actually have some data function g^ϵ, for which

$$\|g^\epsilon - g\|_{L_2} \leq \epsilon$$

where the constant $\epsilon > 0$ represents a bound on the measurement error. Let u and v^ϵ be the exact solution and the approximate solution of the given backward heat problem respectively. In [29, 30], the errors are of order $\frac{1}{1+\ln\frac{T}{\epsilon}}$. The error estimates in [27] are $\epsilon^{\frac{t}{T}}$ for $t > 0$ and $\left(\ln\frac{1}{\epsilon}\right)^{\frac{-1}{4}}$ for $t = 0$. Feng Xiao-Li and coauthors [8] gave the error estimates of order $\left(\frac{4T}{\ln\frac{1}{\epsilon}}\right)^{\frac{p}{2}}$ for $p > 0$. Very recently, in [30], Trong and Tuan improve the previous stability results with the order $\epsilon^{\frac{t}{T}}\left(\frac{T}{1+\ln(\frac{T}{\epsilon})}\right)^{1-\frac{t}{T}}$. From the discussed errors, we see that the error estimates of most regularization methods in the literatures are Hölder type, *i.e.*,

$$\|u(.,.,t) - v^\epsilon(.,.,t)\|_{L_2(\Omega)} \leq C\epsilon^q, \tag{2}$$

where C is the constant depending on u, $0 < q < 1$ is a real number which does not depend on t, u and ϵ is the noise level of final data $u(x, y, T)$. The major object of this paper is to provide a truncation regularization method to established the Hölder estimates such as (2). By Theorem 1, the solution of

Problem (1) is given by the form

(3)

$$u(x,y,t)=\sum_{m,n=1}^{\infty}\left(e^{(T-t)(m^2+n^2)}g_{mn}-\int_0^T e^{(s-t)(m^2+n^2)}f_{mn}(s)\mathrm{d}s\right)\sin(mx)sin(ny)$$

where

$$g(x,y)=\sum_{m,n=1}^{\infty}g_{mn}\sin(mx)\sin(ny),$$

$$f(x,y,t)=\sum_{n,m=1}^{\infty}f_{mn}(t)\sin(mx)\sin(ny)$$

are the expansions of g and f respectively.

Since $t<T$, we know from (3) that, when m^2+n^2 become large, $\exp\{(T-t)(m^2+n^2)\}$ increases very quickly. Therefore, the term $e^{(T-t)(m^2+n^2)}$ is the unstability's cause. So, we hope to recover the stability of Problem (3) by filtering the high frequencies with a suitable method. The essence of our regularization method is just to eliminate all high frequencies from the solution, and instead consider (3) only for $m^2+n^2\le M_\epsilon$, where M_ϵ is an appropriate positive constant satisfying $\lim\limits_{\epsilon\to 0}M_\epsilon=\infty$. The technique involved in the truncation method is applied for the heat problem in an arbitrary open, bounded domain Ω with smooth boundary in R^2. In fact, we state a few properties of the eigenvalues of the operator $-\Delta$ on an open, bounded domain with zero Dirichlet boundary condition. One can also refer to chapter 6.5 in [7].

Theorem on Eigenvalues of the Laplace operator.

1. *Each eigenvalues of $-\Delta$ is real. The family of eigenvalues $\{\lambda_p\}_{p=1}^{\infty}$ satisfy $0\le\lambda_1\le\lambda_2\le\lambda_3\le...$, and $\lambda_p\to\infty$ as $p\to\infty$.*

2. *There exists an orthonormal basis $\{X_p\}_{p=1}^{\infty}$ of $L^2(\Omega)$, where $X_p\in H_0^1(\Omega)$ is an eigenfunction corresponding to λ:*

$$\begin{cases}\Delta X_p(x)=-\lambda_pX_p(x),x\in\Omega\\ X_p(x)=0,x\in\partial\Omega,\end{cases}\tag{4}$$

for $p=1,2,...$ In practice, for an arbitrary domain Ω, the eigenvalues λ_p defined in (4) could be computed by some numerical methods and it may have the numerical errors. Hence, the regularized solution in this case is difficult to compute. In this paper, for simple theory and numerical computations, we choose the domain Ω is a square in R^2 in order to get the exactly eigenvalues, such as $\lambda_{mn}=m^2+n^2$.

The paper is organized as follows. In Section 2, we shall construct the regularized problem and show that it works even with a very weak condition on the exact solution. In Section 3, the error estimates are derived. Finally, in Section 4, we present a numerical example which show the validity of the proposed methods.

2. THE ILL-POSED BACKWARD HEAT PROBLEM

Let us first make clear what a weak solution of the Problem (1) is. We call a function $u \in C\left([0,T]; H^1(\Omega)\right) \cap C^1((0,T); L^2(\Omega))$ to be a weak solution for the Problem (1) if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle u(.,.,t), W\right\rangle_{L_2(\Omega)} - \left\langle u(.,.,t), \Delta W\right\rangle_{L_2(\Omega)} = \left\langle f(.,.,t), \Delta W\right\rangle_{L_2(\Omega)}, \tag{5}$$

and $\langle u(.,.,T), W\rangle_{L_2(\Omega)} = \langle g(.,.), W\rangle_{L_2(\Omega)}$for any function $W \in H^2(\Omega) \cap H_0^1(\Omega)$. In fact, it is enough to choose W in the orthogonal basis $\{\frac{2}{\pi}\sin(mx)\sin(ny)\}_{n,m\geq 1}$ and the formula (5) reduces to

$$u_{mn}(t) = e^{(T-t)(n^2+m^2)} g_{nm} - \int_t^T e^{(s-t)(n^2+m^2)} f_{nm}(s)\mathrm{d}s, \quad \forall m,n \geq 1 \tag{6}$$

where

$$\begin{aligned} u_{mn}(t) &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi u(x,y,t)\sin(mx)\sin(ny)\mathrm{d}x\mathrm{d}y \\ f_{mn}(s) &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y,s)\sin(mx)\sin(ny)\mathrm{d}x\mathrm{d}y \\ g_{mn} &= \frac{4}{\pi^2}\int_0^\pi\int_0^\pi g(x,y)\sin(mx)\sin(ny)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

The solution to the Problem (1) may also be written formally as

(7)

$$u(x,y,t)=\sum_{n,m=1}^{\infty}\left(e^{-(t-T)(n^2+m^2)}g_{mn} - \int_t^T e^{-(t-s)(n^2+m^2)} f_{mn}(s)\mathrm{d}s\right)\sin(mx)\sin(ny).$$

Note that if the exact solution u is smooth then the exact data (f,g) is smooth also. However, the obtained data, which come from practical measure, is often discrete and non-smooth. We shall therefore always assume that $f \in L^2((0,T); L^2(\Omega))$ and $g \in L^2(\Omega)$ and the error of the data is given on L^2 only. Note that the expression (7) is the solution of Problem (1) if it exists. In the following Theorem, we provide a condition of its existence.

THEOREM 1. *The Problem* (1) *has a unique solution* u *if and only if*

$$\sum_{n,m=1}^{\infty}\left(e^{T(n^2+m^2)}g_{mn} - \int_0^T e^{s(n^2+m^2)} f_{mn}(s)\mathrm{d}s\right)^2 < \infty. \tag{8}$$

Proof. Suppose the Problem (1) has a solution $u \in C([0,T]; H_0^1(\Omega)) \cap C^1((0,T); L^2(\Omega))$, then u can be given in (7). This implies that

$$u_{mn}(0) = e^{T(n^2+m^2)} g_{mn} - \int_0^T e^{s(n^2+m^2)} f_{mn}(s) \mathrm{d}s. \tag{9}$$

Then

$$\|u(.,.,0)\|^2_{L_2(\Omega)} = \sum_{n,m=1}^{\infty} \left(e^{T(n^2+m^2)} g_{mn} - \int_0^T e^{s(n^2+m^2)} f_{mn}(s) \mathrm{d}s \right)^2 < \infty.$$

If (8) holds, then define $v(x,y)$ as the function

$$v(x,y) = \sum_{n,m=1}^{\infty} \left(e^{T(n^2+m^2)} g_{mn} - \int_0^T e^{s(n^2+m^2)} f_{mn}(s) \mathrm{d}s \right) \sin(mx) \sin(ny) \in L^2(\Omega).$$

Consider the problem

$$\begin{cases} u_t - u_{xx} - u_{yy} = f(x,y,t), \\ u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0, \quad t \in (0,T) \\ u(x,y,0) = v(x,y), \quad (x,y) \in (0,\pi) \times (0,\pi). \end{cases} \tag{10}$$

Problem (10) has a unique solution u (See [7]). We have

$$\begin{aligned} u(x,y,t) &= \sum_{n,m=1}^{\infty} \Bigg(e^{-t(n^2+m^2)} \langle v(x,y), \sin(mx)\sin(ny) \rangle \\ &\quad + \int_0^t e^{(s-t)(n^2+m^2)} f_{mn}(s) \mathrm{d}s \Bigg) \sin(mx) \sin(ny). \end{aligned} \tag{11}$$

Let $t = T$ in (11), we have

$$\begin{aligned} u(x,y,T) &= \\ &= \sum_{n,m=1}^{\infty} \Bigg(e^{-T(n^2+m^2)} \left(e^{T(n^2+m^2)} g_{mn} - \int_0^T e^{s(n^2+m^2)} f_{mn}(s) \mathrm{d}s \right) \\ &\quad + \int_0^T e^{(s-T)(n^2+m^2)} f_{mn}(s) \mathrm{d}s \Bigg) \sin(mx) \sin(ny) \\ &= \sum_{n,m=1}^{\infty} g_{mn} \sin(mx) \sin(ny) = g(x,y). \end{aligned}$$

Hence, u is the unique solution of (1). $\square$

THEOREM 2. *Problem* (1) *has at most one solution in* $C([0,T];H_0^1(\Omega))\cap C^1((0,T);H^2(\Omega))$.

Proof. The proof can be found in [7]. □

In spite of the uniqueness, Problem (1) is still ill-posed and a regularization is necessary. In the next Section, we establish the approximation for the problem.

3. REGULARIZATION AND ERROR ESTIMATE

In this section, we introduce a regularized problem and investigate the error estimate between the regularized solution and the exact one. Assume that f_ϵ and g_ϵ are measured data satisfying

$$\|f_\epsilon - f\|_{L^2(0,T;L^2(\Omega))} \le \epsilon, \quad \|g_\epsilon - g\|_{L^2(\Omega)} \le \epsilon.$$

Let us define a regularization solution of problem for noisy data g_ϵ as follows:

$$u^\epsilon(x,y,t) = \sum_{m,n\ge 1}^{m^2+n^2\le M_\epsilon} u^\epsilon_{mn}(t)\sin(mx)\sin(ny) \tag{12}$$

where

$$u^\epsilon_{mn}(t) = e^{(T-t)(m^2+n^2)}(g_\epsilon)_{mn} - \int_t^T e^{(s-t)(m^2+n^2)}\left(f_\epsilon\right)_{mn}(s)\mathrm{d}s$$

and

$$\begin{aligned}
(g_\epsilon)_{mn} &= \frac{4}{\pi^2}\langle g_\epsilon(x,y),\sin(mx)\sin(ny)\rangle_{L^2(\Omega)},\\
(f_\epsilon)_{mn} &= \frac{4}{\pi^2}\langle f_\epsilon(x,y,t),\sin(mx)\sin(ny)\rangle_{L^2(\Omega)}.
\end{aligned}$$

THEOREM 3. *The solution of Problem* (12) *depends continuously on* $(g_\epsilon, f_\epsilon)\in L^2(\Omega)\times L^2(0,T;L^2(\Omega))$.

Proof. Let u and v be two solutions of (12) corresponding to the values $(g^1_\epsilon, f^1_\epsilon)$ and $(g^2_\epsilon, f^2_\epsilon)$. We have

$$\begin{aligned}
u(x,y,t) &= \sum_{m,n\ge 1}^{m^2+n^2\le M_\epsilon}\Big(e^{(T-t)(m^2+n^2)}(g^1_\epsilon)_{mn}\\
&\quad - \int_t^T e^{(s-t)(m^2+n^2)}\left(f^1_\epsilon\right)_{mn}(s)\mathrm{d}s\Big)\sin(mx)\sin(ny)
\end{aligned}$$

and

$$v(x,y,t) = \sum_{m,n\geq 1}^{m^2+n^2\leq M_\epsilon} \Big(e^{(T-t)(m^2+n^2)} (g_\epsilon^2)_{mn} - \int_t^T e^{(s-t)(m^2+n^2)} \left(f_\epsilon^2\right)_{mn}(s)\mathrm{d}s \Big) \sin(mx)\sin(ny).$$

Using the inequality $(a-b)^2 \leq 2(a^2+b^2)$, we have

$$\|u(.,.,t)-v(.,.,t)\|^2_{L_2(\Omega)} =$$

$$= \frac{\pi^2}{4} \sum_{m,n\geq 1, m^2+n^2\leq M_\epsilon} \Big| e^{(T-t)(m^2+n^2)} \Big((g_\epsilon^1)_{mn} - (g_\epsilon^2)_{mn} \Big) - \int_t^T e^{(s-t)(m^2+n^2)} \left(f_\epsilon^1 - f_\epsilon^2\right)_{mn}(s) \Big) \mathrm{d}s \Big|^2$$

$$\leq \frac{\pi^2}{2} \sum_{m,n\geq 1, m^2+n^2\leq M_\epsilon} \Big| e^{(T-t)M_\epsilon} \Big((g_\epsilon^1)_{mn} - (g_\epsilon^2)_{mn} \Big) \Big|^2 + \int_t^T \Big| e^{(s-t)M_\epsilon} \Big(\left(f_\epsilon^1 - f_\epsilon^2\right)_{mn}(s) \Big) \Big|^2 \mathrm{d}s$$

$$\leq \frac{\pi^2}{2} e^{2(T-t)M_\epsilon} \sum_{m,n\geq 1}^{\infty} \Big(\big| (g_\epsilon^1)_{mn} - (g_\epsilon^2)_{mn} \big|^2 + \int_t^T \big| \left(f_\epsilon^1 - f_\epsilon^2\right)_{mn}(s) \big|^2 \mathrm{d}s \Big).$$

Then, we obtain

$$\|u(.,.,t) - v(.,.,t)\|^2_{L_2(\Omega)} \leq e^{2(T-t)M_\epsilon} \Big(\|g_\epsilon^1 - g_\epsilon^2\|^2_{L_2(\Omega)} + \left\| f_\epsilon^1 - f_\epsilon^2 \right\|^2_{L_2(0,T;L_2(\Omega))} \Big). \quad \square$$

Remark 1. If $M_\epsilon = \frac{1}{T} \ln\left(\frac{T}{\epsilon + \epsilon \ln(\frac{T}{\epsilon})} \right)$ then the stability magnitude of the method is of order

$$U_1(\epsilon) = e^{TM_\epsilon} = \epsilon^{-1} \left(\frac{1 + \ln(\frac{T}{\epsilon})}{T} \right)^{-1}.$$

Note that the stability magnitude of methods given in [4, 27] is

$$U_2(\epsilon) = D\epsilon^{-1}$$

and in [6, 28], it is

$$U_3(\epsilon) = \frac{T}{\epsilon + \epsilon \ln(\frac{T}{\epsilon})}.$$

We note that $U_1(\epsilon) < U_2(\epsilon)$ and $U_1(\epsilon) < U_3(\epsilon)$ for all $t \in [0,T]$. Hence, the stability magnitude of our well-posed problem is much better, especially when $t = 0$, than the stability magnitudes given by quasi-reversibility method and quasi-boundary value method.

THEOREM 4. *For each $h \in L_2(\Omega)$, we put*

$$\begin{aligned} h_{mn} &= \frac{4}{\pi^2} \int\limits_{\Omega} h(x,y)\sin(mx)\sin(ny)\mathrm{d}x\mathrm{d}y, \\ \Gamma_M(h)(x,y) &= \sum_{m,n\geq 1}^{m^2+n^2\leq M} h_{mn}\sin(mx)\sin(ny). \end{aligned}$$

Then, we have:

(i) *If $h \in L_2(\Omega)$ then* $\lim\limits_{M\to+\infty} \|\Gamma_M(h) - h\|^2_{L_2(\Omega)} = 0.$

(ii) *If $h \in H_0^1(\Omega)$ then* $\lim\limits_{M\to+\infty} \|\Gamma_M(h) - h\|^2_{H_0^1(\Omega)} = 0$ *and*

$$\|\Gamma_M(h) - h\|_{L_2(\Omega)} \leq \frac{4}{\pi^2\sqrt{M}} \|h\|_{H_0^1(\Omega)}.$$

(iii) *If $h \in H_0^1(\Omega)$ and $\Delta h \in L_2(\Omega)$ then* $\lim\limits_{M\to+\infty} \|\Delta\Gamma_M(h) - \Delta h\|^2_{L_2(\Omega)} = 0$ *and*

$$\begin{aligned} \|\Gamma_M(h) - h\|_{L_2(\Omega)} &\leq \frac{4}{\pi^2 M} \|\Delta h\|_{L_2(\Omega)}, \\ \|\Gamma_M(h) - h\|_{H_0^1(\Omega)} &\leq \frac{4}{\pi^2\sqrt{M}} \|\Delta h\|_{L_2(\Omega)}. \end{aligned}$$

Proof. (i) By Parseval equality we get

$$\sum_{m,n\geq 1} |h_{mn}|^2 = \frac{4}{\pi^2} \|h\|^2_{L_2(\Omega)}.$$

We have

$$\begin{aligned} \|\Gamma_M(h) - h\|^2_{L_2(\Omega)} &= \left\| \sum_{m,n\geq 1; m^2+n^2>M} h_{mn}\sin(mx)\sin(ny) \right\|^2_{L_2(\Omega)} \\ &= \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2>M} |h_{mn}|^2, \end{aligned}$$

and it implies that $\lim\limits_{M\to+\infty} \|h_M - h\|^2_{L_2(\Omega)} = 0.$

(ii) Assume that $h \in H_0^1(\Omega)$. Using integration by parts, we have

$$\int\limits_{\Omega} h(x,y)\sin(mx)\sin(ny)\mathrm{d}x\mathrm{d}y = -\frac{1}{m}h(x,y)\sin(ny)\cos(mx)|_{\partial\Omega}$$

$$+\frac{1}{m}\int\limits_{\Omega} h_x(x,y)\cos(mx)\sin(ny)\mathrm{d}x\mathrm{d}y = \frac{1}{m}\int\limits_{\Omega} h_x(x,y)\cos(mx)\sin(ny)\mathrm{d}x\mathrm{d}y$$

and

$$\int\limits_{\Omega} h(x,y)\sin(mx)\sin(ny)\mathrm{d}x\mathrm{d}y = -\frac{1}{n}h(x,y)\sin(mx)\cos(ny)|_{\partial\Omega}$$

$$+\frac{1}{n}\int\limits_{\Omega} h_y(x,y)\sin(mx)\cos(ny)\mathrm{d}x\mathrm{d}y = \frac{1}{n}\int\limits_{\Omega} h_y(x,y)\sin(mx)\cos(ny)\mathrm{d}x\mathrm{d}y.$$

Therefore, we get

$$\begin{aligned} h_{mn} &= \frac{4}{\pi^2 m}\int\limits_{\Omega} h_x(x,y)\cos(mx)\sin(ny)\mathrm{d}x\mathrm{d}y \\ &= \frac{4}{\pi^2 n}\int\limits_{\Omega} h_y(x,y)\sin(mx)\cos(ny)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

It follows that

$$\begin{aligned} (m^2+n^2)\left|h_{mn}\right|^2 &= \frac{16}{\pi^4}\left(\int\limits_{\Omega} h_x(x,y)\cos(mx)\sin(ny)\mathrm{d}x\mathrm{d}y\right)^2 \\ &\quad +\frac{16}{\pi^4}\left(\int\limits_{\Omega} h_y(x,y)\sin(mx)\cos(ny)\mathrm{d}x\mathrm{d}y\right)^2, \end{aligned}$$

and then

$$\sum_{m,n\geq 1}(m^2+n^2)\left|h_{mn}\right|^2 \leq \frac{64}{\pi^6}\left\|h_x\right\|^2_{L_2(\Omega)} + \frac{64}{\pi^6}\left\|h_y\right\|^2_{L_2(\Omega)} = \frac{64}{\pi^6}\left\|h\right\|^2_{H^1_0(\Omega)}.$$

We see that

$$\begin{aligned} \left\|\Gamma_M(h)-h\right\|^2_{H^1_0(\Omega)} &= \left\|\frac{\partial h_M}{\partial x}-\frac{\partial h}{\partial x}\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial h_M}{\partial y}-\frac{\partial h}{\partial y}\right\|^2_{L_2(\Omega)} \\ &= \left\|\sum_{m,n\geq 1; m^2+n^2\geq M} m h_{mn}\cos(mx)\sin(ny)\right\|^2_{L_2(\Omega)} \\ &\quad + \left\|\sum_{m,n\geq 1; m^2+n^2\geq M} n h_{mn}\sin(mx)\cos(ny)\right\|^2_{L_2(\Omega)} \end{aligned}$$

$$= \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2\geq M} (m^2+n^2)|h_{mn}|^2,$$

and it implies that $\lim\limits_{M\to+\infty} \|h_M - h\|_{H_0^1(\Omega)} = 0$.

Moreover,

$$\begin{aligned}
\|\Gamma_M(h) - h\|^2_{L_2(\Omega)} &= \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2\geq M} |h_{mn}|^2 \\
&\leq \frac{\pi^2}{4M} \sum_{m,n\geq 1; m^2+n^2\geq M} (m^2+n^2)|h_{mn}|^2 \leq \frac{16}{\pi^4 M} \|h\|^2_{H_0^1(\Omega)}.
\end{aligned}$$

(iii) Assume that $h \in H_0^1(\Omega)$ and $\Delta h \in L_2(\Omega)$. By a similar way, we get

$$\begin{aligned}
h_{mn} &= \frac{4}{\pi^2 m^2} \int_\Omega h_{xx}(x,y)\sin(mx)\sin(ny)\mathrm{d}x\mathrm{d}y \\
&= \frac{4}{\pi^2 n^2} \int_\Omega h_{yy}(x,y)\sin(mx)\sin(ny)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

It follows that

$$(m^2+n^2)h_{mn} = \frac{4}{\pi^2}(\Delta h)_{mn}$$

and then

$$\sum_{m,n\geq 1} |(m^2+n^2)h_{mn}|^2 = \frac{64}{\pi^6} \|\Delta h\|^2_{L_2(\Omega)}.$$

We have

$$\begin{aligned}
\|\Delta\Gamma_M(h) - \Delta h\|^2_{L_2(\Omega)} &= \left\| \sum_{m,n\geq 1; m^2+n^2\geq M} (m^2+n^2)h_{mn}\sin(mx)\sin(ny) \right\|^2_{L_2(\Omega)} \\
&= \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2\geq M} |(m^2+n^2)h_{mn}|^2
\end{aligned}$$

and it implies that $\lim\limits_{M\to+\infty} \|\Delta h_M - \Delta h\|^2_{L_2(\Omega)} = 0$.

In addition, we have

$$\begin{aligned}
\|\Gamma_M(h) - h\|^2_{L_2(\Omega)} &= \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2\geq M} |h_{mn}|^2 \\
&\leq \frac{\pi^2}{4M^2} \sum_{m,n\geq 1; m^2+n^2\geq M} |(m^2+n^2)h_{mn}|^2 \leq \frac{16}{\pi^2 M^2} \|\Delta h\|^2_{L_2(\Omega)},
\end{aligned}$$

and

$$\|\Gamma_M(h)-h\|^2_{H_0^1(\Omega)} = \frac{\pi^2}{4}\sum_{m,n\geq 1;m^2+n^2\geq M}(m^2+n^2)|h_{mn}|^2$$
$$\leq \frac{\pi^2}{4M}\sum_{m,n\geq 1;m^2+n^2\geq M}|(m^2+n^2)h_{mn}|^2 \leq \frac{16}{\pi^4 M}\|\Delta h\|^2_{L_2(\Omega)}.$$

Hence, the proof is completed. □

THEOREM 5. *Let $f\in L_2(0,T;L_2(\Omega))$ and $g\in L_2(\Omega)$ such that system* (1) *has a (unique) solution*

$$u\in C\left([0,T];H_0^1(\Omega)\right)\cap C^1((0,T);L_2(\Omega)).$$

If we select

$$M_\epsilon = \frac{\ln(\epsilon^{-1})}{2T},$$

then $\lim\limits_{\epsilon\to 0^+}\|u^\epsilon(.,.,0)-u(.,.,0)\|_{H_0^1(\Omega)}=0$ *and*

$$\|u^\epsilon(.,.,0)-u(.,.,0)\|_{L_2(\Omega)} \leq \frac{4\sqrt{2}}{\pi T}\sqrt[4]{\epsilon}+\frac{4\sqrt{2T}}{\pi^2}\|u(.,.,0)\|_{H_0^1(\Omega)}\cdot\frac{1}{\sqrt{\ln(\epsilon^{-1})}}$$

Moreover, if $\Delta u(.,.,0)\in L_2(\Omega)$ *then* $\lim\limits_{\epsilon\to 0^+}\|\Delta u^\epsilon(.,.,0)-\Delta u(.,.,0)\|_{L_2(\Omega)}=0$ *and*

$$\|u^\epsilon(.,.,0)-u(.,.,0)\|_{L_2(\Omega)} \leq \frac{4}{\pi\sqrt{T}}\sqrt[4]{\varepsilon}+\frac{8T}{\pi^2}\|\Delta u(.,.,0)\|_{L_2(\Omega)}\cdot\frac{1}{\ln(\epsilon^{-1})},$$
$$\|u^\epsilon(.,.,0)-u(.,.,0)\|_{H_0^1(\Omega)} \leq \frac{4\sqrt{2}}{\pi T}\sqrt[4]{\varepsilon}+\frac{4\sqrt{2T}}{\pi^2}\|u(.,.,0)\|_{H_0^1(\Omega)}\cdot\frac{1}{\sqrt{\ln(\epsilon^{-1})}}.$$

Proof. Denote $u_\epsilon = u^\epsilon(x,y,0)$ and recall $(u_\epsilon)_{mn}=\langle u^\epsilon(x,y,0),\sin(mx)\sin(ny)\rangle$. By Theorem 4, we can approximate $u(.,.,0)$ by $\Gamma_{M_\varepsilon}(u(.,.,0))$. Therefore, now we just need to estimate the error between u_ε and $\Gamma_{M_\varepsilon}(u(.,.,0))$.

Step 1. Estimate $|(u_\varepsilon)_{mn}-(\Gamma_{M_\varepsilon}(u(.,.,0)))_{mn}|$.

The error vanishes when $m^2+n^2>M_\varepsilon$. In the case $m^2+n^2\leq M_\varepsilon$ one has

$$|(u_\varepsilon)_{mn}-(\Gamma_{M_\varepsilon}(u(.,.,0)))_{mn}| = |u_{\varepsilon,m,n}-(u(x,y,0))_{mn}| =$$
$$=\left|e^{(m^2+n^2)T}(g_\varepsilon-g)_{mn}-\int_0^T e^{(m^2+n^2)t}\left(f_\varepsilon(x,y,t)-f(x,y,t)\right)_{mn}\mathrm{d}t\right|$$

$$\leq e^{(m^2+n^2)T}\left|(g_\varepsilon-g)_{mn}\right| + \int_0^T e^{(m^2+n^2)t}\left|(f_\varepsilon(x,y,t)-f(x,y,t))_{mn}\right|\mathrm{d}t$$

$$\leq \frac{4}{\pi^2}e^{(m^2+n^2)T}\left\|g_\varepsilon-g\right\|_{L^1(\Omega)} + \frac{4}{\pi^2}e^{(m^2+n^2)T}\int_0^T \left\|f_\varepsilon(.,.,t)-f(.,.,t)\right\|_{L^1(\Omega)}\mathrm{d}t$$

$$\leq \frac{8}{\pi^2}e^{(m^2+n^2)T}\epsilon \leq \frac{8}{\pi^2}e^{M_\varepsilon T}\varepsilon = \frac{8}{\pi^2}\sqrt{\epsilon}.$$

Step 2. Estimate errors between u_ϵ and $\Gamma_{M_\epsilon}(u(.,.,0))$.

Notice that $\epsilon^{-1/2} = e^{M_\epsilon T} > (M_\epsilon T)^k/k!$ for $k = 2,3,4$. We have

$$\|u_\epsilon - \Gamma_{M_\epsilon}(u(.,.,0))\|^2_{L_2(\Omega)} =$$

$$= \left\|\sum_{m,n\geq 1; m^2+n^2\leq M_\epsilon}((u_\epsilon)_{mn}-(u(.,.,0))_{mn})\sin(mx)\sin(ny)\right\|^2_{L_2(\Omega)}$$

$$= \frac{\pi^2}{4}\sum_{m,n\geq 1; m^2+n^2\leq M_\epsilon}|(u_\epsilon)_{mn}-(u(.,.,0))_{mn}|^2. \tag{13}$$

Moreover, since Step 1, if (m,n) satisfy $m^2+n^2\leq M_\epsilon$ then we have

$$|(u_\epsilon)_{mn}-(u(.,.,0))_{mn}|^2 \leq \left(\frac{8}{\pi^2}\sqrt{\epsilon}\right)^2 = \frac{64}{\pi^4}\epsilon.$$

Hence,

$$\sum_{m,n\geq 1; m^2+n^2\leq M_\epsilon}|(u_\epsilon)_{mn}-(u(.,.,0))_{mn}|^2 \leq (m^2+n^2)\frac{64}{\pi^4}\epsilon \leq M_\epsilon\frac{64}{\pi^4}\epsilon. \tag{14}$$

Combining (13) and (14) and $M_\epsilon T \leq e^{M_\epsilon T} = \epsilon^{-1/2}$ we obtain

$$\|u_\epsilon - \Gamma_{M_\epsilon}(u(.,.,0))\|^2_{L_2(\Omega)} \leq \frac{\pi^2}{4}M_\epsilon\frac{64}{\pi^4}\epsilon = \frac{16}{\pi^2}M_\epsilon\epsilon \leq \frac{16}{\pi^2 T}\sqrt{\epsilon}.$$

By a similar way and using (14), we get

$$\|u_\epsilon - \Gamma_{M_\epsilon}(u(.,.,0))\|^2_{H_0^1(\Omega)} =$$

$$= \left\|\frac{\partial u_\epsilon}{\partial x} - \frac{\partial}{\partial x}\Gamma_{M_\epsilon}(u(.,.,0))\right\|^2_{L_2(\Omega)} + \left\|\frac{\partial u_\epsilon}{\partial y} - \frac{\partial}{\partial y}\Gamma_{M_\epsilon}(u(.,.,0))\right\|^2_{L_2(\Omega)}$$

$$= \left\|\sum_{m,n\geq 1; m^2+n^2\leq M_\epsilon} m((u_\epsilon)_{mn}-(u(.,.,0))_{mn})\cos(mx)\sin(ny)\right\|^2_{L_2(\Omega)}$$

$$+\left\| \sum_{m,n\geq 1; m^2+n^2\leq M_\epsilon} n((u_\epsilon)_{mn} - (u(.,.,0))_{mn}) \sin(mx)\cos(ny) \right\|^2_{L_2(\Omega)}$$

$$= \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2\leq M_\epsilon} (m^2+n^2)|(u_\epsilon)_{mn} - (u(.,.,0))_{mn}|^2$$

$$\leq \frac{\pi^2}{4} M_\epsilon \sum_{m,n\geq 1; m^2+n^2\leq M_\epsilon} |(u_\epsilon)_{mn} - (u(.,.,0))_{mn}|^2 \leq \frac{\pi^2}{4} M_\epsilon M_\epsilon \frac{64}{\pi^4}\epsilon = \frac{16}{\pi^2} M_\epsilon^2 \epsilon.$$

Using the inequality $\epsilon^{-1/2} = e^{M_\epsilon T} > (M_\epsilon T)^2/2$, we get

$$\|u_\epsilon - \Gamma_{M_\epsilon}(u(.,.,0))\|^2_{H_0^1(\Omega)} \leq \frac{32}{\pi^2 T^2}\sqrt{\epsilon} \tag{15}$$

and

$$\|\Delta u_\epsilon - \Delta\Gamma_{M_\epsilon}(u(.,.,0))\|^2_{L_2(\Omega)} =$$

$$= \left\| \sum_{m,n\geq 1; m^2+n^2\leq M_\varepsilon} (m^2+n^2)((u_\epsilon)_{mn} - (u(.,.,0))_{mn}) \sin(mx)\sin(ny) \right\|^2_{L_2(\Omega)}$$

$$= \frac{\pi^2}{4} \sum_{m,n\geq 1; m^2+n^2\leq M_\epsilon} (m^2+n^2)^2|(u_\epsilon)_{mn} - (u(.,.,0))_{mn}|^2 \leq \frac{16}{\pi^2} M_\epsilon^3 \epsilon \leq \frac{96}{\pi^2 T^3}\sqrt{\epsilon}.$$

Step 3. Estimate errors between u_ϵ and u.

Using Lemma 1, we have

$$\begin{aligned} &\|u_\epsilon - u(.,.,0)\|_{H_0^1(\Omega)} = \\ &\leq \|u_\epsilon - \Gamma_{M_\epsilon}(u(.,.,0))\|_{H_0^1(\Omega)} + \|\Gamma_{M_\epsilon}(u(.,.,0)) - u(.,.,0)\|_{H_0^1(\Omega)} \\ &\leq \frac{4\sqrt{2}}{\pi T}\sqrt[4]{\epsilon} + \|\Gamma_{M_\epsilon}(u(.,.,0)) - u(.,.,0)\|_{H_0^1(\Omega)} \to 0, \end{aligned}$$

and

$$\begin{aligned} &\|u_\epsilon - u(.,.,0)\|_{L_2(\Omega)} = \\ &\leq \|u_\epsilon - \Gamma_{M_\epsilon}(u(.,.,0))\|_{L_2(\Omega)} + \|\Gamma_{M_\epsilon}(u(.,.,0)) - u(.,.,0)\|_{L_2(\Omega)} \\ &\leq \frac{4}{\pi\sqrt{T}}\sqrt[4]{\epsilon} + \frac{4}{\pi^2\sqrt{M_\epsilon}} \|u(.,.,0)\|_{H_0^1(\Omega)}. \end{aligned}$$

Now, we assume that $\Delta u(.,.,0) \in L_2(\Omega)$. Then

$$\begin{aligned} &\|\Delta u_\epsilon - \Delta u(.,.,0)\|_{L_2(\Omega)} = \\ &\leq \|\Delta u_\epsilon - \Delta\Gamma_{M_\epsilon}(u(.,.,0))\|_{L_2(\Omega)} + \|\Delta\Gamma_{M_\epsilon}(u(.,.,0)) - \Delta u(.,.,0)\|_{L_2(\Omega)} \\ &\leq \frac{4\sqrt{6}}{\pi T\sqrt{T}}\sqrt[4]{\epsilon} + \|\Delta\Gamma_{M_\epsilon}(u(.,.,0)) - \Delta u(.,.,0)\|_{L_2(\Omega)} \to 0. \end{aligned}$$

Moreover,

$$\begin{aligned}
&\|u_\varepsilon - u(.,.,0)\|_{L_2(\Omega)} = \\
&\le \|u_\varepsilon - \Gamma_{M_\varepsilon}(u(.,.,0))\|_{L_2(\Omega)} + \|\Gamma_{M_\varepsilon}(u(.,.,0)) - u(.,.,0)\|_{L_2(\Omega)} \\
&\le \frac{4}{\pi\sqrt{T}}\sqrt[4]{\varepsilon} + \frac{4}{\pi^2 M_\varepsilon}\|\Delta u(.,.,0)\|_{L_2(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
&\|u_\varepsilon - u(.,.,0)\|_{H_0^1(\Omega)} = \\
&\le \|u_\varepsilon - \Gamma_{M_\varepsilon}(u(.,.,0))\|_{H_0^1(\Omega)} + \|\Gamma_{M_\varepsilon}(u(.,.,0)) - u(.,.,0)\|_{H_0^1(\Omega)} \\
&\le \frac{4\sqrt{2}}{\pi T}\sqrt[4]{\varepsilon} + \frac{4}{\pi^2\sqrt{M_\varepsilon}}\|\Delta u(.,.,0)\|_{L_2(\Omega)}.
\end{aligned}$$

The proof is completed. □

Remark 2. In [27], the error $\|u_\epsilon(.,.,0) - u(.,.,0)\|_{L_2(\Omega)}$ is not given. In [4, 6, 10, 19, 26–28], the error $\|u_\epsilon(.,.,0) - u(.,.,0)\|_{H_0^1(\Omega)}$ is not given. In this case, the error $\|u_\epsilon(.,.,0) - u(.,.,0)\|_{H_0^1(\Omega)}$ is established. This is a strong point of our method.

THEOREM 6. *Let Problem* (1) *have a (unique) solution*

$$u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1((0,T); L_2(\Omega)).$$

satisfying the condition

$$\frac{\pi^2}{4}\sum_{n,m=1}^{\infty}(n^2+m^2)^k u_{mn}^2 < A_1^2$$

for $k > 0$ is a positive number. If we select

$$M_\epsilon = \frac{1}{T}\ln\left(\frac{T(\ln\frac{T}{\epsilon})^r}{\epsilon + \epsilon\ln(\frac{T}{\epsilon})}\right), \quad (r > 0)$$

then the following estimate holds

$$\begin{aligned}
&\|u^\epsilon(.,.,t) - u(.,.,t)\|_{L_2(\Omega)} \le \\
&\le \sqrt{6\epsilon^{\frac{2t}{T}}\left(\frac{1+\ln(T\epsilon^{-1}}{T})\right)^{\frac{2t}{T}-2} + (\ln\frac{T}{\epsilon})^{2r(\frac{t}{T}-1)}\left[T\ln\left(\frac{\epsilon+\epsilon\ln(\frac{T}{\epsilon})}{T(\ln\frac{T}{\epsilon})^r}\right)\right]^k A_1^2}.
\end{aligned}$$

Proof. We have

$$u(x,y,t) = \sum_{m,n=1}^{\infty} u_{mn}(t)\sin(mx)\sin(ny) \tag{16}$$

$$u^\epsilon(x,y,t) = \sum_{m,n\geq 1}^{m^2+n^2\leq M_\epsilon} u^\epsilon_{mn}(t)\sin(mx)\sin(ny) \tag{17}$$

where

$$\begin{aligned} u^\epsilon_{mn}(t) &= e^{(T-t)(m^2+n^2)}(g_\epsilon)_{mn} - \int_t^T e^{(s-t)(m^2+n^2)}\left(f_\epsilon(x,y,s)\right)_{mn}\mathrm{d}s \\ u_{mn}(t) &= e^{(T-t)(m^2+n^2)}g_{mn} - \int_t^T e^{(s-t)(m^2+n^2)}f_{mn}(s)\mathrm{d}s. \end{aligned}$$

By direct transform, we get

$$\begin{aligned} \|u^\epsilon(.,.,t)-u(.,.,t)\|^2_{L_2(\Omega)} &\leq 3\frac{\pi^2}{4}\sum_{m^2+n^2\leq M_\epsilon}^{m,n\geq 1} e^{2(T-t)(n^2+m^2)}((g_\epsilon)_{mn}-g_{mn})^2 \\ &+ 3\frac{\pi^2}{4}\sum_{m^2+n^2\leq M_\epsilon}^{m,n\geq 1}\Big(\int_t^T e^{(s-t)(m^2+n^2)}((f_\epsilon(x,y,s))_{mn}-f_{nm}(s)\mathrm{d}s\Big)^2 \\ &+ 3\frac{\pi^2}{4}\sum_{m^2+n^2\geq M_\epsilon}^{m,n\geq 1} u^2_{mn}(t). \end{aligned}$$

It follows that

$$\begin{aligned} \|u^\epsilon(.,.,t)-u(.,.,t)\|^2_{L_2(\Omega)} \;\leq\; & 3e^{2(T-t)M_\epsilon}\|g_\epsilon - g\|^2_{L_2(\Omega)} \\ & + 3e^{2(T-t)M_\epsilon}\|f_\epsilon - f\|^2_{L_2(0,T;L_2(\Omega))} \\ & + 3\frac{\pi^2}{4}\sum_{m^2+n^2\geq M_\epsilon}^{m,n\geq 1}(n^2+m^2)^{-k}(m^2+n^2)^k u^2_{mn}(t). \end{aligned}$$

Hence, we get

$$\begin{aligned} \|u^\epsilon(.,.,t)-u(.,.,t)\|^2_{L_2(\Omega)} &\leq 6e^{2(T-t)M_\epsilon}\epsilon^2 + \frac{1}{M_\epsilon^k}\sum_{n,m=1}^{\infty}(m^2+n^2)^k u^2_{mn} \\ &\leq 6\epsilon^{\frac{2t}{T}}\left(\frac{1+\ln(T\epsilon^{-1}}{T})\right)^{\frac{2t}{T}-2}(\ln\frac{T}{\epsilon})^{2r(\frac{t}{T}-1)} \\ &+\Big[T\ln\left(\frac{\epsilon+\epsilon\ln(\frac{T}{\epsilon})}{T(\ln\frac{T}{\epsilon})^r}\right)\Big]^k\frac{\pi^2}{4}\sum_{n,m=1}^{\infty}(m^2+n^2)^k u^2_{mn}(t). \quad \square \end{aligned}$$

THEOREM 7. *Assume that there exists the positive numbers β, A_2 such that*

$$\frac{\pi^2}{4}\sum_{m,n=1}^{\infty} e^{2\beta(m^2+n^2)}u_{mn}^2(t) < A_2^2. \tag{18}$$

Let us choose

$$M_\epsilon = \sqrt{\frac{1}{T+\beta}\ln(\frac{1}{\epsilon})}$$

then the following convergence estimate holds

$$\|u^\epsilon(.,.,t) - u(.,.,t)\|_{L_2(\Omega)} \le \left(A_2 + 2\epsilon^{\frac{t}{T+\beta}}\right)\epsilon^{\frac{\beta}{T+\beta}} \tag{19}$$

for every $t \in [0,T]$.

Proof. Let v^ϵ be the function defined by

$$v^\epsilon(x,y,t) = \sum_{m,n\ge 1}^{m^2+n^2\le M_\epsilon} v_{mn}^\epsilon(t)\sin(mx)\sin(ny) \tag{20}$$

where

$$v_{nm}^\epsilon(t) = e^{(T-t)(n^2+m^2)}g_{mn} - \int_t^T e^{(s-t)(n^2+m^2)}\left(f_\epsilon(x,y,s)\right)_{mn}\mathrm{d}s.$$

Since (20), we have

$$u(x,y,t) - v^\epsilon(x,y,t) =$$

$$= \sum_{m^2+n^2>M_\epsilon}^{\infty}\left(e^{-(t-T)(m^2+n^2)}g_{mn} - \int_t^T e^{-(t-s)(m^2+n^2)}f_{mn}(s)\mathrm{d}s\right)\sin(mx)\sin(ny)$$

$$= \sum_{m^2+n^2>M_\epsilon}^{\infty} < u(x,y,t), \sin(mx)\sin(ny) > \sin(mx)\sin(ny).$$

Therefore, we have

$$\begin{aligned}
\|u(.,.,t) - v^\epsilon(.,.,t)\|_{L_2(\Omega)}^2 &= \frac{\pi^2}{4}\sum_{m^2+n^2>M_\epsilon}^{\infty} e^{-2\beta(m^2+n^2)}e^{2\beta(m^2+n^2)}u_{mn}^2(t)\\
&\le \frac{\pi^2}{4}e^{-2\beta M_\epsilon}\sum_{m^2+n^2>M_\epsilon}^{\infty} e^{2\beta(m^2+n^2)}u_{mn}^2(t)\\
&\le e^{-2\beta M_\epsilon}\frac{\pi^2}{4}\sum_{m,n=1}^{\infty} e^{2\beta(m^2+n^2)}u_{mn}^2(t) \qquad (21)\\
&\le e^{-2\beta M_\epsilon}A_2^2. \qquad (22)
\end{aligned}$$

On the other hand, we have

$$\|u^\epsilon(.,.,t) - v^\epsilon(.,.,t)\|^2_{L_2(\Omega)} =$$

$$= \sum_{m,n=1}^{\infty} \left| e^{(m^2+n^2)(T-t)}(g_\varepsilon - g)_{mn} - \int_t^T e^{(m^2+n^2)(s-t)} \left(f_\varepsilon(x,y,s) - f(x,y,s)\right)_{mn} \mathrm{d}s \right|$$
$$\le 2e^{2(T-t)M_\epsilon} \left(\|u\|g_\epsilon - g^2_{L^1(\Omega)} + \|f_\varepsilon(.,.,t) - f(.,.,t)\|^2_{L^1(0,T;L^1(\Omega))} \right)$$
$$\le 4e^{2(T-t)M_\epsilon}\epsilon^2. \tag{23}$$

Combining (22) and (23), we get

$$\|u^\epsilon(.,.,t) - u(.,.,t)\|_{L_2(\Omega)} \le \|u^\epsilon(.,.,t) - v^\epsilon(.,.,t)\|_{L_2(\Omega)} + v^\epsilon(.,.,t) - u(.,.,t)\|_{L_2(\Omega)}$$
$$\le e^{-\beta M_\epsilon^2} A_2 + 2e^{(T-t)M_\epsilon}\epsilon.$$

From

$$M_\epsilon = \frac{1}{T+\beta}\ln(\frac{1}{\epsilon})$$

the following convergence estimate holds

$$\|u^\epsilon(.,.,t) - u(.,.,t)\|_{L_2(\Omega)} \le \epsilon^{\frac{\beta}{T+\beta}} A_2 + 2\epsilon^{\frac{t+\beta}{T+\beta}} = \epsilon^{\frac{\beta}{T+\beta}} \left(A_2 + 2\epsilon^{\frac{t}{T+\beta}} \right). \quad \square$$

Remark 3. 1. Condition (18) is not verifiable. We can check it by replacing the conditions of f and g. We have

$$\sum_{m,n=1}^{\infty} e^{2\beta(m^2+n^2)} u^2_{mn}(t)$$
$$= \sum_{m,n=1}^{\infty} e^{2\beta(m^2+n^2)} \left(e^{-(t-T)(m^2+n^2)} g_{mn} - \int_t^T e^{-(t-s)(m^2+n^2)} f_{mn}(s)\mathrm{d}s \right)^2.$$

Hence, we can replace (18) by the different conditions

$$\sum_{m,n=1}^{\infty} e^{2(T+\beta)(m^2+n^2)} g_{mn} < \infty, \quad \sum_{m,n=1}^{\infty} \int_0^T e^{2(s+\beta)(m^2+n^2)} f^2_{mn}(s)\mathrm{d}s < \infty.$$

2. We notice that the error estimate (19) ($\beta > 0$) is of Hölder type for all $t \in [0,T]$. It is easy to see that the convergence rate of $\epsilon^p, (0 < p)$ is more quickly than the logarithmic order $\left(\ln(\frac{1}{\epsilon})\right)^{-q}$ $(q > 0)$ when $\epsilon \to 0$. Comparing error (19) with the results in [4, 6, 10, 19, 26–31], we can see that our method is very effective.

4. NUMERICAL EXAMPLE

In this section, we show the validity of the proposed methods by a simple numerical example. We consider the problem:
(24)

$$\begin{cases} u_t - \Big(u_{xx} + u_{yy}\Big) = f(x,y,t), \quad (x,y,t) \in (0,\pi)^2 \times (0,1), \\ u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0, \quad (x,y,t) \in (0,\pi)^2 \times (0,1), \\ u(x,y,1) = g(x,y), \quad (x,y) \in (0,\pi)^2, \end{cases}$$

where

$$\begin{aligned} g(x,y) &= \sum_{k=1}^{15} \frac{e^{-1}}{k^2} \sin(2kx)\sin(ky), \\ f(x,y,t) &= e^{-t}\sum_{k=1}^{15} \left(5 - \frac{1}{k^2}\right) \sin(2kx)\sin(ky). \end{aligned}$$

The problem has an exact solution:

$$u(x,y,t) = e^{-t}\sum_{k=1}^{15} \frac{1}{k^2} \sin(2kx)\sin(ky).$$

For the measured noised data g^ϵ such that $\|g - g^\epsilon\| \le \epsilon$, we have

$$g^\epsilon(x,y) = g(x,y) + \frac{\epsilon}{\pi}\cdot \texttt{rand}(), \quad \texttt{rand}() \in (-1,1)$$

In this example, we choose the computation grid $(I,J) = (128,128)$, $x_i = i\pi/I,\ y_j = j\pi/J\ (i,j = 0...128)$. Table 1 shows the value of $M_\epsilon = M_\epsilon^i,\ \epsilon = 10^{-k}, k = 3,5,7$ as chosen in Remark 1 $(i = 1)$ and Theorems 5, 6, 7 $(i = 2,3,4)$, respectively. Note that for M_ϵ^4 from Theorem 7, the value of β was set to 1/2.

Table 1
The value of M_ϵ^i as chosen in Remark 1 $(i = 1)$
and Theorem 5,6,7 $(i = 2,3,4)$, respectively

	$\epsilon = 10^{-3}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-7}$
M_ϵ^1	4.840	8.986	13.28
M_ϵ^2	3.454	5.756	8.059
M_ϵ^3	8.705	13.87	18.84
M_ϵ^4	2.146	2.770	3.278

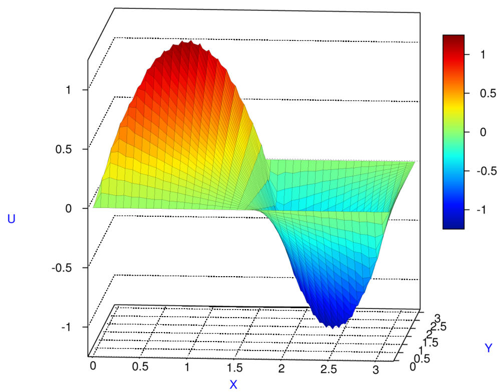
With the error magnitude of the measured data $\epsilon = 10^{-k}, k = 3,5,7$; $t = j/10, j = 0..9$, Table 2 shows the error estimations of regularized solutions comparing to the exact solution in $L_2\left((0,\pi)^2\right)$ with different values of M_ϵ.

Table 2
Error estimations between exact solution and regularized solutions in $L_2\left((0,\pi)^2\right)$ by different values $M_\epsilon = M_\epsilon^i$, $i = 1,2,3,4$ and $\epsilon = 10^{-k}, k = 3,5,7$

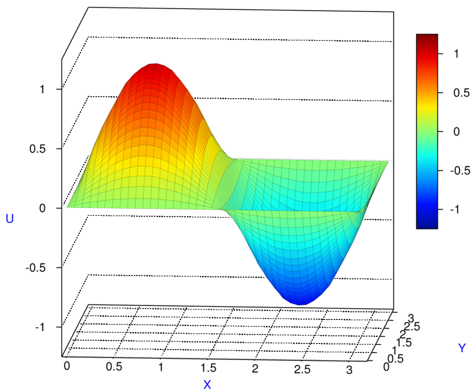
	$\epsilon = 10^{-3}$				$\epsilon = 10^{-5}$				$\epsilon = 10^{-7}$			
t	M_ϵ^1	M_ϵ^2	M_ϵ^3	M_ϵ^4	M_ϵ^1	M_ϵ^2	M_ϵ^3	M_ϵ^4	M_ϵ^1	M_ϵ^2	M_ϵ^3	M_ϵ^4
0.0	5.161E-01	5.161E-01	1.423E-01	5.161E-01	1.423E-01	1.423E-01	1.424E-01	5.161E-01	1.423E-01	1.423E-01	1.424E-01	5.161E-01
0.1	4.670E-01	4.670E-01	1.288E-01	4.670E-01	1.287E-01	1.287E-01	1.287E-02	4.670E-01	1.287E-01	1.287E-01	1.288E-01	4.670E-01
0.2	4.226E-01	4.226E-01	1.165E-01	4.226E-01	1.165E-01	1.165E-01	1.165E-01	4.226E-01	1.165E-01	1.165E-01	1.165E-01	4.226E-01
0.3	3.824E-01	3.824E-01	1.054E-01	3.824E-01	1.054E-01	1.054E-01	1.054E-01	3.824E-01	1.054E-01	1.054E-01	1.054E-01	3.824E-01
0.4	3.460E-01	3.460E-01	9.537E-02	3.460E-01	9.537E-02	9.537E-02	9.537E-02	3.460E-01	9.537E-02	9.537E-02	9.537E-02	3.460E-01
0.5	3.130E-01	3.130E-01	8.629E-02	3.130E-01	8.629E-02	8.629E-02	8.629E-02	3.130E-01	8.629E-02	8.629E-02	8.629E-02	3.130E-01
0.6	2.833E-01	2.833E-01	7.808E-02	2.833E-01	7.808E-02	7.808E-02	7.808E-02	2.833E-01	7.808E-02	7.808E-02	7.808E-02	2.833E-01
0.7	2.563E-01	2.563E-01	7.065E-02	2.563E-01	7.065E-02	7.065E-02	7.065E-02	2.563E-01	7.065E-02	7.065E-02	7.065E-02	2.563E-01
0.8	2.319E-01	2.319E-01	6.393E-02	2.319E-01	6.393E-02	6.393E-02	6.393E-02	2.319E-01	6.393E-02	6.393E-02	6.393E-02	2.319E-01
0.9	2.098E-01	2.098E-01	5.784E-02	2.098E-01	5.784E-02	5.784E-02	5.784E-02	2.098E-01	5.784E-02	5.784E-02	5.784E-02	2.098E-01

From the table, we see that the bigger the value M_ϵ, the better regularized solutions we obtain. Because the function $u(x, y, t)$ has a finite series of Fourier coefficients, we can easily verify that it satisfies all the conditions when choosing value M_ϵ^i, $i = 1..4$.

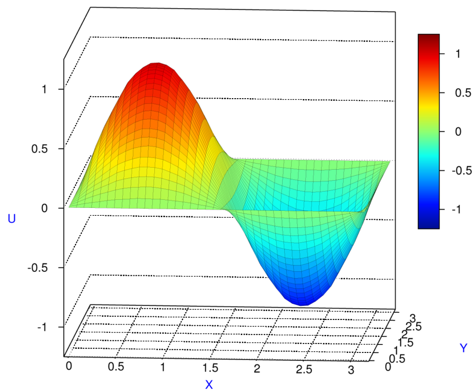
For better illustration, we present some 3D-graphs of the exact solution and regularized solutions by the proposed method in Figure 1 with chosen $M_\epsilon = M_\epsilon^1$ and $M_\epsilon = M_\epsilon^3$. Furthermore, in Figure 2, we show the section cut graphs of them at $y = \frac{51}{128}\pi$, $y = \frac{1}{2}\pi$ and $y = \frac{51}{64}\pi$ when $t = 0$ and $\epsilon = 10^{-5}$.



(a) Exact solution $u(x, y, t)$ at $t = 0$

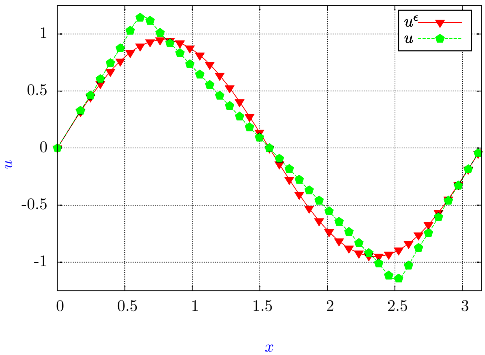


(b) $u^\epsilon(x, y, 0)$, $\epsilon = 10^{-5}$, $M_\epsilon = M_\epsilon^1$

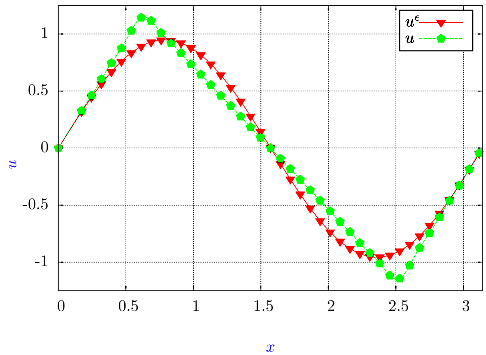


(c) $u^\epsilon(x, y, 0)$, $\epsilon = 10^{-5}$, $M_\epsilon = M_\epsilon^3$

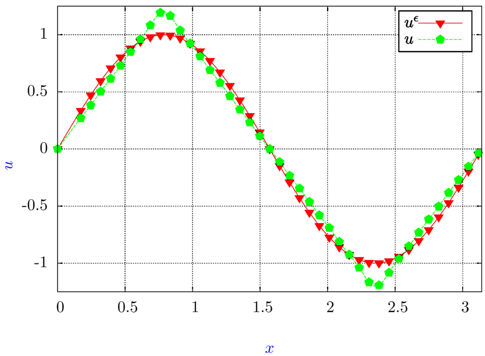
Fig. 1 – Exact solution and regularized solutions when $t = 0$.



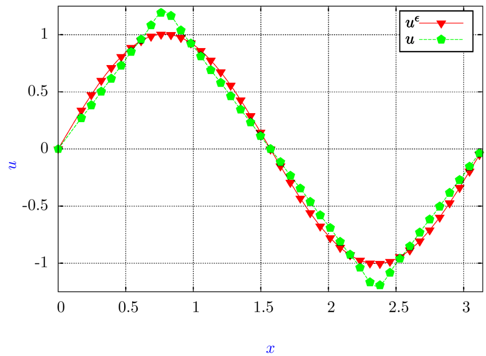
(a) $M_\epsilon = M_\epsilon^1, \quad y = \frac{51}{128}\pi$



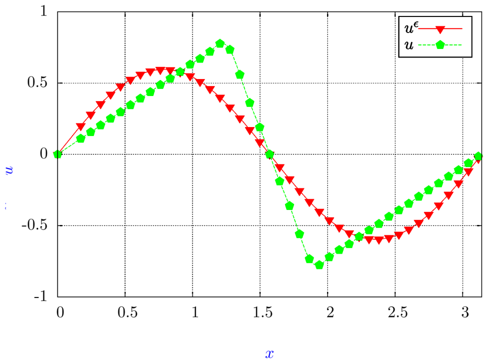
(b) $M_\epsilon = M_\epsilon^3, \quad y = \frac{51}{128}\pi$



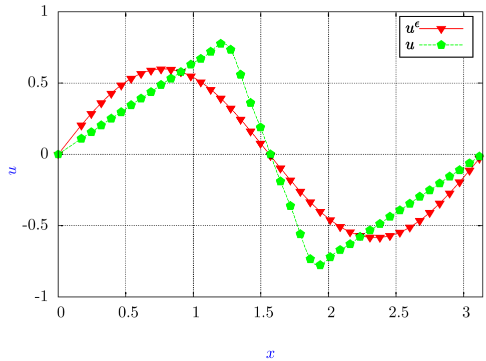
(c) $M_\epsilon = M_\epsilon^1, \quad y = \frac{1}{2}\pi$



(d) $M_\epsilon = M_\epsilon^3, \quad y = \frac{1}{2}\pi$



(e) $M_\epsilon = M_\epsilon^1, \quad y = \frac{51}{64}\pi$



(f) $M_\epsilon = M_\epsilon^3, \quad y = \frac{51}{64}\pi$

Fig. 2 – Section cut graphs of exact solution u and regularized solutions u^ϵ at $y = \frac{51}{128}\pi, \frac{1}{2}\pi, \frac{51}{64}\pi$ when $t = 0$.

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REFERENCES

[1] K.A. Ames and R.J. Hughes, *Structural stability for ill-posed problems in Banach space.* Semigroup Forum, **70** (2005), *1*, 127–145.

[2] F. Black and M. Scholes, *The pricing of options and corporate liabilities.* J. Polit. Econ. **81**(3) (1973), 637–654.

[3] A.S. Carraso, *Logarithmic convexity and the "slow evolution" constraint in ill-posed initial value problems.* SIAM J. Math. Anal. **30** (1999), *3*, 479–496.

[4] G.W. Clark and S.F. Oppenheimer, *Quasireversibility methods for non-well posed problems.* Electron. J. Differ. Equ. **1994** (1994), *8*, 1–9.

[5] Chu-Li Fu, Zhi Qian and Rui Shi, *A modified method for a backward heat conduction problem.* Appl. Math. Comput. **185** (2007), 564–573.

[6] M. Denche and K. Bessila, *A modified quasi-boundary value method for ill-posed problems.* J. Math. Anal. Appl. **301** (2005), 419–426.

[7] Lawrence C. Evans, *Partial Differential Equation.* American Mathematical Society **19** (1997), Providence, Rhode Island.

[8] X.L. Feng, Z. Qian and C.L. Fu, *Numerical approximation of solution of nonhomogeneous backward heat conduction problem in bounded region.* Math. Comput. Simulation **79** (2008), *2*, 177–188.

[9] C.L-Fu, X.T. Xiong and Z. Qian, *Fourier regularization for a backward heat equation.* J. Math. Anal. Appl. **331** (2007), *1*, 472–480.

[10] H. Gajewski and K. Zaccharias, *Zur regularisierung einer klass nichtkorrekter probleme bei evolutiongleichungen.* J. Math. Anal. Appl. **38** (1972), 784–789.

[11] J. Hadamard, *Lecture note on Cauchy's problem in linear partial differential equations.* Yale Uni Press, New Haven, 1923.

[12] D.N. Hao, *A mollification method for ill-posed problems.* Numer. Math. **68** (1994), 469-506.

[13] Y. Huang and Q. Zhneg, *Regularization for ill-posed Cauchy problems associated with generators of analytic semigroups.* J. Differential Equations **203** (2004), *1*, 38–54.

[14] Y. Huang and Q. Zhneg, *Regularization for a class of ill-posed Cauchy problems.* Proc. Amer. Math. Soc. **133** (2005), 3005–3012.

[15] V.K. Ivanov, I.V. Mel'nikova and F.M. Filinkov, *Differential-Operator Equations and Ill-Posed problems.* Nauka, Moscow, 1995 (Russian).

[16] F. John, *Continuous dependence on data for solutions of partial differential equations with a prescribed bound.* Comm. Pure Appl. Math. **13** (1960), 551–585.

[17] R. Lattès and J.-L. Lions, *Méthode de Quasi-réversibilité et Applications.* Dunod, Paris, 1967.

[18] S.M. Kirkup and M.Wadsworth, *Solution of inverse diffusion problems by operator-splitting methods.* Appl. Math. Modell. **26** (2002), 1003–1018.

[19] K. Miller, *Stabilized quasi-reversibility and other nearly-best-possible methods for non-well posed problems.* Symposium on Non-Well Posed Problems and Logarithmic Convexity, Lecture Notes in Mathematics **316** (1973), Springer-Verlag, Berlin 161–176.

[20] I.V. Mel'nikova and S.V. Bochkareva, *C-semigroups and regularization of an ill-posed Cauchy problem.* Dok. Akad. Nauk. **329** (1993), 270–273.

[21] P.T. Nam, D.D. Trong and N.H. Tuan, *The truncation method for a two-dimensional nonhomogeneous backward heat problem.* Appl. Math. Comput. **216** (2010), *12*, 3423–3432.

[22] L.E. Payne, *Some general remarks on improperly posed problems for partial differential equations.* Symposium on Non-Well Posed Problems and Logarithmic Convexity, Lecture Notes in Mathematics **316** (1973), Springer-Verlag, Berlin, 1–30.

[23] R.E. Showalter, *The final value problem for evolution equations.* J. Math. Anal. Appl. **47** (1974), 563–572.

[24] R.E. Showalter, *Cauchy problem for hyper-parabolic partial differential equations.* In Trends in the Theory and Practice of Non-Linear Analysis, Elsevier 1983.

[25] A.N. Tikhonov and V.Y. Arsenin, *Solutions of Ill-Posed Problems.* Winston, Washington, 1977.

[26] D.D. Trong and N.H. Tuan, *Regularization and error estimates for nonhomogeneous backward heat problems.* Electron. J. Differ. Equ. **2006** (2006), *04*, 1–10.

[27] D.D. Trong, P.H. Quan, T.V. Khanh and N.H. Tuan, *A nonlinear case of the 1-D backward heat problem: regularization and error estimate.* Zeitschrift Analysis und ihre Anwendungen **26** (2007), *2*, 231–245.

[28] D.D. Trong and T.N. Lien.(2007), *Regularization of a discrete backward problem using coefficients of truncated Lagrange polynomials. Electron. J. Differ. Equ.,* **2007** (2007), *51*, 1–14.

[29] D.D. Trong and N.H. Tuan, *A nonhomogeneous backward heat problem: regularization and error estimates.* Electron. J. Differ. Equ., **2008** , *33*, 1–14.

[30] D.D. Trong and N.H. Tuan, *Regularization and error estimate for the nonlinear backward heat problem using a method of integral equation.* Nonlinear Anal., **71** (2009), *9*, 4167–4176.

[31] N.H. Tuan and D.D. Trong, *A new regularized method for two dimensional nonhomogeneous backward heat problem.* Appl. Math. Comput. **215** (2009), *3*, 873–880.

[32] P.T. Nam, D.D. Trong and N.H. Tuan, *The truncation method for a two-dimensional nonhomogeneous backward heat problem.* Appl. Math. Comput. **216** (2010), *12*, 3423–3432.

[33] T. Schrsoter and U. Tautenhahn, *On optimal regularization methods for the backward heat equation.* Z. Anal. Anwed. **15** (1996), 475–493.

[34] B. Yildiz and M. Ozdemir, *Stability of the solution of backward heat equation on a weak conpactum.* Appl. Math. Comput. **111** (2000), 1–6.

[35] B. Yildiz, H. Yetis and A.Sever, *A stability estimate on the regularized solution of the backward heat problem.* Appl. Math. Comput. **135** (2003), 561–567.

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