CONVERGENCE RESULTS FOR CONTACT PROBLEMS WITH MEMORY TERM

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In this paper, we consider two quasistatic contact problems. The material's behavior is modelled with an elastic constitutive law for the first problem and a viscoplastic constitutive law for the second problem. The novelty arises in the fact that the contact is frictionless and is modelled with a condition which involves normal compliance and memory term. Moreover, for the second problem we consider a condition with unilateral constraint. For each problem we derive a variational formulation of the model and prove its unique solvability. Also, we analyze the dependence of the solution with respect to the data.

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1. INTRODUCTION

The first aim of this paper is to study a quasistatic frictionless contact problem for elastic materials, within the framework of the Mathematical Theory of Contact Mechanics. We model the behavior of the material with a constitutive law of the form

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}), \tag{1.1}$$

where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress field, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor and $\mathcal{F}$ is a fourth order tensor which describes the elastic properties of the material. The contact is modelled with a condition which involves normal compliance, memory term and infinite penetration. We prove the unique solvability of this model by using new arguments on history-dependent variational inequalities presented in [6]. Also, we state and prove the dependence of the solution with respect to the data.

The second aim is to study the continuous dependence of the solution of a quasistatic frictionless contact problem for rate-type viscoplastic materials. We model the behavior of the material with a constitutive law of the form

$$\dot{\boldsymbol{\sigma}} = \mathcal{F}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})). \tag{1.2}$$

Here, $\mathcal{G}$ is a nonlinear constitutive function which describes the viscoplastic properties of the material. In (1.2) and everywhere in this paper the dot above a variable represents derivative with respect to the time variable t. The second part represents a continuation of [3] where a contact problem for viscoplastic materials of the form (1.2) was considered. The process was assumed to be quasistatic and the contact was modelled by using the normal compliance condition, finite penetration and memory term. The unique solvability of the solution was obtained. Also, the convergence of the solution of the problem with infinite penetration to the solution of the problem with finite penetration as the stiffness coefficient converges to infinity was proved. In the present paper, we analyse the dependence of the solution of the viscoplastic contact problem in [3] with respect to the data.

The rest of the paper is structured as follows. In Section 2, we provide the notation we shall use as well as some preliminary material. In Section 3, we present the classical formulation of the first problem, list the assumption on the data and derive the variational formulation. Then we state and prove the unique weak solvability of the problem, Theorem 3.1, and a convergence result, Theorem 3.2. In Section 4, we introduce the classical formulation of the second problem and resume the results on its unique weak solvability obtained in [3]. Then we state and prove a convergence result, Theorem 4.3.

2. NOTATION AND PRELIMINARIES

Everywhere in this paper, we use the notation $\mathbb{N}^*$ for the set of positive integers and $\mathbb{R}_+$ will represent the set of nonnegative real numbers, *i.e.* $\mathbb{R}_+ = [0, +\infty)$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i \ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Let Ω be a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ and let Γ_1 be a measurable part of Γ such that meas $(\Gamma_1) > 0$. We use the notation $\boldsymbol{x} = (x_i)$ for a typical point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, *e.g.* $u_{i,j} = \partial u_i / \partial x_j$. We consider the spaces

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1 \,\}, \quad Q = \{\, \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d \times d} : \tau_{ij} = \tau_{ji} \,\}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u},\boldsymbol{v})_V=\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x,\qquad (\boldsymbol{\sigma},\boldsymbol{\tau})_Q=\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,\mathrm{d}x,$$

and the associated norms $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here $\boldsymbol{\varepsilon}$ represents the deformation operator given by

$$\boldsymbol{\varepsilon}(\boldsymbol{v})=(\varepsilon_{ij}(\boldsymbol{v})),\quad \varepsilon_{ij}(\boldsymbol{v})=\frac{1}{2}\,(v_{i,j}+v_{j,i})\quad \forall\,\boldsymbol{v}\in H^1(\Omega)^d.$$

Also, we define the space

$$Q_1=\{\,\boldsymbol{\tau}\in Q:\ \mathrm{Div}\,\boldsymbol{\tau}\in L^2(\Omega)^d\,\},\tag{2.1}$$

which is a Hilbert space endowed with the inner product

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{Q_1}=(\boldsymbol{\sigma},\boldsymbol{\tau})_Q+(\mathrm{Div}\,\boldsymbol{\sigma},\mathrm{Div}\,\boldsymbol{\tau})_{L^2(\Omega)^d},$$

and the associated norm $\|\cdot\|_{Q_1}$. Here Div represents the divergence operator given by $\mathrm{Div}\,\boldsymbol{\sigma}=(\sigma_{ij,j})$.

The assumption $\mathrm{meas}(\Gamma_1)>0$ allows the use of Korn's inequality which involves the completeness of the space $(V,\|\cdot\|_V)$. For an element $\boldsymbol{v}\in V$ we still write $\boldsymbol{v}$ for its trace and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ given by $v_\nu=\boldsymbol{v}\cdot\boldsymbol{\nu}$, $\boldsymbol{v}_\tau=\boldsymbol{v}-v_\nu\boldsymbol{\nu}$. Let Γ_3 be a measurable part of Γ. Then, by the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d}\le c_0\,\|\boldsymbol{v}\|_V\qquad \forall\,\boldsymbol{v}\in V.\tag{2.2}$$

Also, for a regular stress function $\boldsymbol{\sigma}$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential components, *i.e.* $\sigma_\nu=(\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau=\boldsymbol{\sigma}\boldsymbol{\nu}-\sigma_\nu\boldsymbol{\nu}$. For the convenience of the reader we recall the following Green's formula:

$$\int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x+\int_\Omega \mathrm{Div}\,\boldsymbol{\sigma}\cdot\boldsymbol{v}\,\mathrm{d}x=\int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,\mathrm{d}a.\tag{2.3}$$

We denote by $\mathbf{Q}_\infty$ the space of fourth order tensor fields given by

$$\mathbf{Q}_\infty=\{\ \mathcal{E}=(\mathcal{E}_{ijkl})\ :\ \mathcal{E}_{ijkl}=\mathcal{E}_{jikl}=\mathcal{E}_{klij}\in L^\infty(\Omega),\quad 1\le i,j,k,l\le d\ \},$$

and $\mathbf{Q}_\infty$ is a real Banach space with the norm $\|\mathcal{E}\|_{\mathbf{Q}_\infty}=\displaystyle\sum_{1\le i,j,k,l\le d}\|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}$.
Moreover, a simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q\le d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty}\|\boldsymbol{\tau}\|_Q\qquad \forall\,\mathcal{E}\in\mathbf{Q}_\infty,\ \boldsymbol{\tau}\in Q.\tag{2.4}$$

For each Banach space X we use the notation $C(\mathbb{R}_+;X)$ for the space of continuous functions defined on $\mathbb{R}_+$ with values on X. It is well known that $C(\mathbb{R}_+;X)$ can be organized in a canonical way as a Fréchet space. Details can be found in [1] and [5], for instance. Here we restrict ourseleves to recall that the convergence of a sequence $(x_m)_m$ to the element x, in the space $C(\mathbb{R}_+;X)$, can be described as follows:

$$(2.5)\qquad \begin{cases} x_m \to x \quad \text{in } C(\mathbb{R}_+;X) \ \text{ as } \ m\to\infty \ \text{ if and only if} \\[2ex] \displaystyle\max_{r\in[0,n]} \|x_m(r)-x(r)\|_X \to 0 \ \text{ as } m\to\infty, \quad \text{for all } \ n\in\mathbb{N}^*. \end{cases}$$

Let X be a real Hilbert space with inner product $(\cdot,\cdot)_X$ and associated norm $\|\cdot\|_X$. Let K be a subset of X and consider the operators $A:K\to X$, $\mathcal{R}:C(\mathbb{R}_+;X)\to C(\mathbb{R}_+;X)$ and function $f:\mathbb{R}_+\to X$. We are interested in the problem of finding a function $u\in C(\mathbb{R}_+;X)$ such that $u(t)\in K$, for all $t\in\mathbb{R}_+$, and inequality below holds

$$(2.6)\qquad \begin{aligned} &(Au(t),v-u(t))_X + (\mathcal{R}u(t),v)_X - (\mathcal{R}u(t),u(t))_X \\ &\qquad \geq (f(t),v-u(t))_X,\ \forall\, v\in K. \end{aligned}$$

In the study of (2.6) we assume that

$$(2.7)\qquad K \text{ is a nonempty, closed, convex subset of } X$$

and $A:K\to X$ is a strongly monotone Lipschitz continuous operator, *i.e.*

$$(2.8)\qquad \begin{cases} \text{(a) There exists } m>0 \text{ such that} \\ \qquad (Au_1-Au_2,u_1-u_2)_X \geq m\,\|u_1-u_2\|_X^2 \quad \forall\, u_1,\,u_2\in K. \\ \text{(b) There exists } M>0 \text{ such that} \\ \qquad \|Au_1-Au_2\|_X \leq M\,\|u_1-u_2\|_X \quad \forall\, u_1,\,u_2\in K. \end{cases}$$

The operator $\mathcal{R}:C(\mathbb{R}_+;X)\to C(\mathbb{R}_+;X)$ satisfies

$$(2.9)\qquad \begin{cases} \text{For every } n\in\mathbb{N}^* \text{ there exists } r_n>0 \text{ such that for all } t\in[0,n] \\[1ex] \displaystyle \|\mathcal{R}u_1(t)-\mathcal{R}u_2(t)\|_X \leq r_n\int_0^t \|u_1(s)-u_2(s)\|_X\,\mathrm{d}s, \end{cases}$$

for all $u_1,\,u_2\in C(\mathbb{R}_+;X)$, and, finally, we assume that

$$(2.10)\qquad f\in C(\mathbb{R}_+;X).$$

The next results, proved in [7], will be used in the rest of this paper.

Theorem 2.1. *Let X be Hilbert space and assume that* (2.7)–(2.10) *hold. Then, the inequality* (2.6) *has a unique solution $u\in C(\mathbb{R}_+;K)$.*

Corollary 2.2. *Let X be a Hilbert space and assume that* (2.8)–(2.10) *hold. Then there exists a unique function $u\in C(\mathbb{R}_+;X)$ such that*

$$(2.11)\qquad (Au(t),v)_X + (\mathcal{R}u(t),v)_X = (f(t),v)_X \quad \forall\, v\in X, \quad \forall\, t\in\mathbb{R}_+.$$

To avoid any confusion, we note that here and below the notation $Au(t)$ and $\mathcal{R}u(t)$ are short hand notation for $A(u(t))$ and $(\mathcal{R}u)(t)$, for all $t \in \mathbb{R}_+$. We end this section with a short description of the physical setting of the two contact problems.

An elastic body in the first problem and a viscoplastic body in the second problem occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that $\text{meas}(\Gamma_1) > 0$. The body is subject to the action of body forces of density $\boldsymbol{f}_0$. We also assume that it is fixed on Γ_1 and surface tractions of density $\boldsymbol{f}_2$ act on Γ_2. On Γ_3, the body is in frictionless contact with a deformable obstacle, the so-called foundation. We assume that process is quasistatic and is studied in the interval of time $\mathbb{R}_+$.

3. ANALYSIS OF AN ELASTIC CONTACT PROBLEM

The classical formulation of the first contact problem is the following.

Problem $\mathcal{P}_1$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that, for each* $t \in \mathbb{R}_+$,

$$\boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\text{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{3.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{3.4}$$

$$\sigma_\nu(t) + p(u_\nu(t)) + \int_0^t b(t-s)u_\nu^+(s)\mathrm{d}s = 0 \quad \text{on} \quad \Gamma_3, \tag{3.5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3. \tag{3.6}$$

Here and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\boldsymbol{x}$ or t. Equation (3.1) represents the elastic constitutive law of the material. Equation (3.2) is the equation of equilibrium, conditions (3.3), (3.4) represent the displacement and traction boundary conditions, respectively, and condition (3.5) shows that the contact follows a normal compliance condition with memory term. At the moment t, the reaction of the foundation depends both on the current value of the penetration (represented by the term $p(u_\nu(t))$) as well as on the history of the penetration (represented by the integral term). Finally, (3.6) is the frictionless condition.

We assume that the body forces and surface tractions have the regularity

$$\boldsymbol{f}_0 \in C(\mathbb{R}_+; L^2(\Omega)^d), \quad \boldsymbol{f}_2 \in C(\mathbb{R}_+; L^2(\Gamma_2)^d). \tag{3.7}$$

Also, we assume that the normal compliance function p verifies

$$(3.8)\qquad \left\{\begin{array}{l} \text{(a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \\ \text{(b) There exists } L_p > 0 \text{ such that} \\ \qquad |p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)| \leq L_p\, |r_1 - r_2| \\ \qquad\quad \forall\, r_1,\, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3 \\ \text{(c) } (p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2))(r_1 - r_2) \geq 0 \\ \qquad\quad \forall\, r_1,\, r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3 \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x}, r) \text{ is measurable on } \Gamma_3,\ \forall\, r \in \mathbb{R} \\ \text{(e) } p(\boldsymbol{x}, r) = 0 \text{ for all } r \leq 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3 \end{array}\right.$$

and the surface memory function satisfies

$$(3.9)\qquad b \in C(\mathbb{R}_+; L^\infty(\Gamma_3)).$$

Further details on the contact condition (3.5), normal compliance function p and surface memory function b can be found in [4] or [8].

We turn now to the variational formulation of Problem $\mathcal{P}_1$. To this end, we assume in what follows that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (3.1)–(3.6). Let $\boldsymbol{v} \in V$ and $t \in \mathbb{R}_+$ be given. We use Green's formula (2.3), equation of equilibrium (3.2), we split the boundary integral over Γ_1, Γ_2 and Γ_3 and, since $\boldsymbol{v} = \boldsymbol{0}$ on $\Gamma_1 \times \mathbb{R}_+$ and $\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t)$ on $\Gamma_2 \times \mathbb{R}_+$, it follows that

$$(3.10)\quad \int_\Omega \boldsymbol{\sigma}(t) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}x = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\,\mathrm{d}a + \int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \boldsymbol{v}\,\mathrm{d}a.$$

Moreover, since $\boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \boldsymbol{v} = \sigma_\nu(t) v_\nu + \boldsymbol{\sigma}_\tau(t) \cdot \boldsymbol{v}_\tau$ on Γ_3, condition (3.6) implies that $\int_{\Gamma_3} \boldsymbol{\sigma}(t)\boldsymbol{\nu} \cdot \boldsymbol{v}\,\mathrm{d}a = \int_{\Gamma_3} \sigma_\nu(t) v_\nu\,\mathrm{d}a$. We use the contact condition (3.5) to see that

$$\sigma_\nu(t) v_\nu = -\Big(p(u_\nu(t)) + \int_0^t b(t-s)\, u_\nu^+(s)\,\mathrm{d}s\Big) v_\nu \quad \text{on} \quad \Gamma_3.$$

We combine the above relations to deduce that

$$(3.11)\quad (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (p(u_\nu(t)), v_\nu)_{L^2(\Gamma_3)} + \Big(\int_0^t b(t-s)\, u_\nu^+(s)\,\mathrm{d}s, v_\nu\Big)_{L^2(\Gamma_3)}$$
$$= (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_2)^d}.$$

We use now (3.1) and (3.11) to derive the following variational formulation of the frictionless contact problem (3.1)–(3.6).

Problem $\mathcal{P}_1^V$. *Find a displacement field* $\boldsymbol{u} : \mathbb{R}_+ \to V$ *such that* $\boldsymbol{u}(t) \in V$, *for all* $t \in \mathbb{R}_+$, *and the equality below holds*

$$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (p(u_\nu(t)), v_\nu)_{L^2(\Gamma_3)} + \Big(\int_0^t b(t-s) u_\nu^+(s)\, \mathrm{d}s, v_\nu \Big)_{L^2(\Gamma_3)}$$

$$= (\boldsymbol{f}_0(t), \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_2)^d} \quad \forall\, \boldsymbol{v} \in V. \tag{3.12}$$

Next, we prove the unique weak solvability of the variational problem $\mathcal{P}_1^V$. To this end, we assume that the elasticity tensor $\mathcal{F}$ satisfies the following conditions.

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) There exists } L_{\mathcal{F}} > 0 \text{ such that} \\ \qquad \|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \qquad\quad \forall\, \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\ \qquad (\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \qquad\quad \forall\, \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{0}_{\mathbb{S}^d}) \text{ belongs to } Q. \end{array} \right. \tag{3.13}$$

We have the following existence and uniqueness result.

THEOREM 3.1. *Assume that* (3.13), (3.7)–(3.9) *hold. Then, Problem* $\mathcal{P}_1^V$ *has a unique solution which satisfies* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$.

Proof. We start by providing an equivalent form to Problem $\mathcal{P}_1^V$. To this end, we use the Riesz representation Theorem to define the operators $P : V \to V$, $\mathcal{B} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ and the function $\boldsymbol{f} : \mathbb{R}_+ \to V$ by equalities

$$(P\boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu) v_\nu \, \mathrm{d}a \quad \forall\, \boldsymbol{u},\ \boldsymbol{v} \in V, \tag{3.14}$$

$$(\mathcal{B}\boldsymbol{u}(t), \boldsymbol{v})_V = \Big(\int_0^t b(t-s) u_\nu^+(s)\, \mathrm{d}s, v_\nu \Big)_{L^2(\Gamma_3)} \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V), \boldsymbol{v} \in V \tag{3.15}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, \mathrm{d}a \quad \forall\, \boldsymbol{v} \in V, \quad t \in \mathbb{R}_+. \tag{3.16}$$

Then, it is easy to see that Problem $\mathcal{P}_1^V$ is equivalent to the problem of finding a function $\boldsymbol{u} : \mathbb{R}_+ \to V$ such that for all $t \in \mathbb{R}_+$ and $\boldsymbol{v} \in V$ the equality below holds

$$\boldsymbol{u}(t) \in V, \quad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u}(t), \boldsymbol{v})_V + (\mathcal{B}\boldsymbol{u}(t), \boldsymbol{v})_V = (\boldsymbol{f}(t), \boldsymbol{v})_V. \tag{3.17}$$

To solve the variational equation (3.17) we use Corollary 2.2 with $X = V$. To this end, we consider the operator $A : V \to V$ defined by

$$(Au, v)_V = (\mathcal{F}\varepsilon(u), \varepsilon(v))_Q + (Pu, v)_V \quad \forall\, u,\, v \in V. \tag{3.18}$$

Then, for all $t \in \mathbb{R}_+$ the equality (3.17) can be written as

$$u(t) \in V, \quad (Au(t), v)_V + (\mathcal{B}u(t), v)_V = (f(t), v)_V \quad \forall\, v \in V.$$

Using (3.13), (3.8) and the definition of the operator P we deduce that the operator A is strongly monotone and Lipschitz continuous, *i.e.* it verifies (2.8).

Let $n \in \mathbb{N}^*$. Then, a simple calculation based on assumption (3.9) and inequality (2.2) shows that $\forall\, u_1,\, u_2 \in C(\mathbb{R}_+; V)$, $\forall t \in [0, n]$ the following inequality holds:

$$\|\mathcal{B}u_1(t) - \mathcal{B}u_2(t)\|_V \le c_0^2 \max_{r\in[0,n]} \|b(r)\|_{L^\infty(\Gamma_3)} \int_0^t \|u_1(s) - u_2(s)\|_V \, \mathrm{d}s. \tag{3.19}$$

This inequality implies that the operator $\mathcal{B}$ given by (3.15) satisfies (2.9) with

$$r_n = c_0^2 \max_{r\in[0,n]} \|b(r)\|_{L^\infty(\Gamma_3)}. \tag{3.20}$$

Finally, using (3.7) and (3.16) we deduce that $f \in C(\mathbb{R}_+; V)$ and, therefore, (2.10) holds. It follows now from Corollary 2.2 that there exists a unique function $u \in C(\mathbb{R}_+; V)$ which satisfies the equation

$$(Au(t), v)_V + (\mathcal{B}u(t), v)_V = (f(t), v)_V \quad \forall\, v \in V \quad t \in \mathbb{R}_+.$$

And, using (3.15), (3.16) and (3.18) we deduce that there exists a unique solution $u \in C(\mathbb{R}_+; V)$ to the equality (3.12) for all $t \in \mathbb{R}_+$, which concludes the proof. □

Let σ be the function defined by (3.1). Then, it follows from (3.13) that $\sigma \in C(\mathbb{R}_+; Q)$. Moreover, it is easy to see that (3.11) holds for all $t \in \mathbb{R}_+$ and, using standard arguments, it results from here that

$$\mathrm{Div}\, \sigma(t) + f_0(t) = \mathbf{0} \qquad \forall t \in \mathbb{R}_+. \tag{3.21}$$

Therefore, using the regularity $f_0 \in C(\mathbb{R}_+; L^2(\Omega)^d)$ in (3.7) we deduce that $\mathrm{Div}\, \sigma \in C(\mathbb{R}_+; L^2(\Omega)^d)$ which implies that $\sigma \in C(\mathbb{R}_+; Q_1)$. A couple of functions (u, σ) which satisfies (3.1), (3.12) for all $t \in \mathbb{R}_+$ is called a *weak solution* to the contact problem $\mathcal{P}_1$. We conclude that Theorem 3.1 provides the unique weak solvability of Problem $\mathcal{P}_1$. Moreover, the regularity of the weak solution is $u \in C(\mathbb{R}_+; V)$, $\sigma \in C(\mathbb{R}_+; Q_1)$.

We study now the dependence of the solution of Problem $\mathcal{P}_1^V$ with respect to perturbations of the data. To this end, we assume in what follows that (3.13), (3.7)–(3.9) hold and we denote by $\boldsymbol{u}$ the solution of Problem $\mathcal{P}_1^V$ obtained in Theorem 3.1. For each $\rho > 0$ let $\boldsymbol{f}_{0\rho}$, $\boldsymbol{f}_{2\rho}$, p_ρ and b_ρ be perturbations of $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, p and b which satisfy conditions (3.7)–(3.9). We consider the following variational problem.

Problem $\mathcal{P}_{1\rho}^V$. *Find a displacement field* $\boldsymbol{u}_\rho : \mathbb{R}_+ \to V$ *such that* $\boldsymbol{u}_\rho(t) \in V$*, for all* $t \in \mathbb{R}_+$*, and the equality below holds for all* $\boldsymbol{v} \in V$:

$$(3.22)\qquad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (p_\rho(u_{\rho\nu}(t)), v_\nu)_{L^2(\Gamma_3)}$$
$$+\Big(\int_0^t b_\rho(t-s)u_{\rho\nu}^+(s)\,\mathrm{d}s, v_\nu\Big)_{L^2(\Gamma_3)} = (\boldsymbol{f}_{0\rho}(t), \boldsymbol{v})_{L^2(\Omega)^d} + (\boldsymbol{f}_{2\rho}(t), \boldsymbol{v})_{L^2(\Gamma_2)^d}.$$

Note that, here and below, $u_{\rho\nu}$ represents the normal component of the function $\boldsymbol{u}_\rho$.

It follows from Theorem 3.1 that, for each $\rho > 0$ Problem $\mathcal{P}_{1\rho}^V$ has a unique solution $\boldsymbol{u}_\rho \in C(\mathbb{R}_+; V)$. Consider now the following assumptions

$$(3.23)\qquad b_\rho \to b \quad \text{in} \quad C(\mathbb{R}_+; L^\infty(\Gamma_3)) \quad \text{as} \quad \rho \to 0,$$

$$(3.24)\qquad \boldsymbol{f}_{0\rho} \to \boldsymbol{f}_0 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Omega)^d) \quad \text{as} \quad \rho \to 0,$$

$$(3.25)\qquad \boldsymbol{f}_{2\rho} \to \boldsymbol{f}_2 \quad \text{in} \quad C(\mathbb{R}_+; L^2(\Gamma_2)^d) \quad \text{as} \quad \rho \to 0.$$

$$(3.26)\qquad \left\{\begin{array}{l} \text{There exists } G : \mathbb{R}_+ \to \mathbb{R}_+ \text{ and } \beta \in \mathbb{R}_+ \text{ such that} \\ \quad \text{(a) } |p_\rho(\boldsymbol{x}, r) - p(\boldsymbol{x}, r)| \le G(\rho)(|r| + \beta) \\ \qquad \forall\, r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3, \text{ for each } \rho > 0, \\ \quad \text{(b) } \lim\limits_{\rho \to 0} G(\rho) = 0. \end{array}\right.$$

We have the following convergence result.

Theorem 3.2. *Under assumptions* (3.23)–(3.26) *the solution* $\boldsymbol{u}_\rho$ *of Problem* $\mathcal{P}_{1\rho}^V$ *converges to the solution* $\boldsymbol{u}$ *of Problem* $\mathcal{P}_1^V$*, i.e.*

$$(3.27)\qquad \boldsymbol{u}_\rho \to \boldsymbol{u} \quad \textit{in} \quad C(\mathbb{R}_+; V) \quad \textit{as} \quad \rho \to 0.$$

Proof. Let $\rho > 0$. We use the Riesz representation Theorem to define the operators $P_\rho : V \to V$, $\mathcal{B}_\rho : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ and the function $\boldsymbol{f}_\rho : \mathbb{R}_+ \to V$ by equalities

$$(3.28)\ (P_\rho \boldsymbol{u}, \boldsymbol{v})_V = \int_{\Gamma_3} p_\rho(u_\nu) v_\nu\,\mathrm{d}a \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V,$$

$$(3.29)\ (\mathcal{B}_\rho \boldsymbol{u}(t), \boldsymbol{v})_V = \Big(\int_0^t b_\rho(t-s)u_\nu^+(s)\,\mathrm{d}s, v_\nu\Big)_{L^2(\Gamma_3)} \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V),\ \boldsymbol{v} \in V,$$

(3.30) $$(\boldsymbol{f}_\rho(t),\boldsymbol{v})_V=\int_\Omega \boldsymbol{f}_{0\rho}(t)\cdot\boldsymbol{v}\,\mathrm{d}x+\int_{\Gamma_2}\boldsymbol{f}_{2\rho}(t)\cdot\boldsymbol{v}\,\mathrm{d}a\quad\forall\boldsymbol{v}\in V,\ t\in\mathbb{R}_+.$$

It follows from the proof of Theorem 3.1 that $\boldsymbol{u}$ is a solution of Problem $\mathcal{P}_1^V$ iff $\boldsymbol{u}$ solves equality (3.17), for all $t\in\mathbb{R}_+$. In a similar way, $\boldsymbol{u}_\rho$ is a solution of Problem $\mathcal{P}_{1\rho}^V$ iff $\boldsymbol{u}_\rho(t)\in V$, for all $t\in\mathbb{R}_+$ and the following equality:

(3.31) $$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q+(P_\rho\boldsymbol{u}_\rho(t),\boldsymbol{v})_V+(\mathcal{B}_\rho\boldsymbol{u}_\rho(t),\boldsymbol{v})_V=(\boldsymbol{f}_\rho(t),\boldsymbol{v})_V,$$

holds for all $\boldsymbol{v}\in V$.

Let $n\in\mathbb{N}^*$ and let $t\in[0,n]$. We take $\boldsymbol{v}=\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)$ in (3.31) and $\boldsymbol{v}=\boldsymbol{u}(t)-\boldsymbol{u}_\rho(t)$ in (3.17) and add the resulting equalities to obtain

(3.32) $$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t))-\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t))-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q$$
$$=(P_\rho\boldsymbol{u}_\rho(t)-P\boldsymbol{u}(t),\boldsymbol{u}(t)-\boldsymbol{u}_\rho(t))_V+(\mathcal{B}_\rho\boldsymbol{u}_\rho(t)-\mathcal{B}\boldsymbol{u}(t),\boldsymbol{u}(t)-\boldsymbol{u}_\rho(t))_V$$
$$+(\boldsymbol{f}_\rho(t)-\boldsymbol{f}(t),\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t))_V.$$

Next, we use the definitions (3.28) and (3.14), the monotonicity of the function p_ρ and assumption (3.26) to see that

$$(P_\rho\boldsymbol{u}_\rho(t)-P\boldsymbol{u}(t),\boldsymbol{u}(t)-\boldsymbol{u}_\rho(t))_V\le\int_{\Gamma_3}G(\rho)(|u_\nu(t)|+\beta)\,|u_\nu(t)-u_{\rho\nu}(t)|\,\mathrm{d}a.$$

Therefore, using the trace inequality (2.2), after some elementary calculus we find that

(3.33) $$(P_\rho\boldsymbol{u}_\rho(t)-P\boldsymbol{u}(t),\boldsymbol{u}(t)-\boldsymbol{u}_\rho(t))_V$$
$$\le G(\rho)\big(c_0^2\,\|\boldsymbol{u}(t)\|_V+c_0\beta\,\mathrm{meas}\,(\Gamma_3)^{\frac12}\big)\,\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V.$$

On the other hand the operator $\mathcal{B}_\rho$ verifies (2.9), *i.e.*

(3.34) $$\|\mathcal{B}_\rho\boldsymbol{u}_\rho(t)-\mathcal{B}_\rho\boldsymbol{u}(t)\|_V\le c_0^2\max_{r\in[0,n]}\|b_\rho(r)\|_{L^\infty(\Gamma_3)}\int_0^t\|\boldsymbol{u}_\rho(s)-\boldsymbol{u}(s)\|_V\,\mathrm{d}s.$$

Using trace inequality we obtain

(3.35) $$\|\mathcal{B}_\rho\boldsymbol{u}(t)-\mathcal{B}\boldsymbol{u}(t)\|_V\le c_0^2\max_{r\in[0,n]}\|b_\rho(r)-b(r)\|_{L^\infty(\Gamma_3)}\int_0^t\|\boldsymbol{u}(s)\|_V\,\mathrm{d}s.$$

From (3.34) and (3.35) we conclude that

$$(\mathcal{B}_\rho\boldsymbol{u}_\rho(t)-\mathcal{B}\boldsymbol{u}(t),\boldsymbol{u}(t)-\boldsymbol{u}_\rho(t))_V\le\|\mathcal{B}_\rho\boldsymbol{u}_\rho(t)-\mathcal{B}\boldsymbol{u}(t)\|_V\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V$$

(3.36) $$\le\Big(\theta_{\rho n}\int_0^t\|\boldsymbol{u}_\rho(s)-\boldsymbol{u}(s)\|_V\,\mathrm{d}s+\omega_{\rho n}\int_0^t\|\boldsymbol{u}(s)\|_V\,\mathrm{d}s\Big)\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V,$$

where

(3.37) $$\theta_{\rho n}=c_0^2\max_{r\in[0,n]}\|b_\rho(r)\|_{L^\infty(\Gamma_3)},\quad\omega_{\rho n}=c_0^2\max_{r\in[0,n]}\|b_\rho(r)-b(r)\|_{L^\infty(\Gamma_3)}.$$

We also note that

$$(\boldsymbol{f}_\rho(t) - \boldsymbol{f}(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V \leq \delta_{\rho n} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \tag{3.38}$$

where

$$\delta_{\rho n} = \max_{r\in[0,n]} \|\boldsymbol{f}_\rho(r) - \boldsymbol{f}(r)\|_V. \tag{3.39}$$

Finally, using assumption (3.13) it follows that

$$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q \geq m_{\mathcal{F}} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V^2. \tag{3.40}$$

We combine (3.32), (3.33), (3.36), (3.38) and (3.40) to deduce that

$$\begin{aligned} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leq{}& \frac{G(\rho)}{m_{\mathcal{F}}} \big(c_0^2 \|\boldsymbol{u}(t)\|_V + c_0\beta \,\mathrm{meas}\,(\Gamma_3)^{\frac{1}{2}}\big) \\ &+ \frac{\theta_{\rho n}}{m_{\mathcal{F}}} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \,\mathrm{d}s + \frac{\omega_{\rho n}}{m_{\mathcal{F}}} \int_0^t \|\boldsymbol{u}(s)\|_V \,\mathrm{d}s + \frac{\delta_{\rho n}}{m_{\mathcal{F}}}. \end{aligned} \tag{3.41}$$

Denote $\xi_{n,u} = \max \frac{1}{m_{\mathcal{F}}} \Big\{ c_0^2 \max_{r\in[0,n]} \|\boldsymbol{u}(r)\|_V + c_0\beta \,\mathrm{meas}\,(\Gamma_3)^{\frac{1}{2}}, \int_0^t \|\boldsymbol{u}(s)\|_V \,\mathrm{d}s, 1 \Big\}$. Then, (3.41) yields

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leq \big(G(\rho) + \omega_{\rho n} + \delta_{\rho n}\big)\xi_{n,u} + \frac{\theta_{\rho n}}{m_{\mathcal{F}}} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \,\mathrm{d}s \tag{3.42}$$

and, using the Gronwall inequality we obtain that

$$\|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leq \big(G(\rho) + \omega_{\rho n} + \delta_{\rho n}\big)\xi_{n,u}\, e^{\frac{\theta_{\rho n}}{m_{\mathcal{F}}} t}. \tag{3.43}$$

We use assumption (3.23) to see that the sequence $(\theta_{\rho n})_\rho$ defined by (3.37) is bounded. Therefore, there exists $\zeta_n > 0$ which depends on n and is independent of ρ such that

$$0 \leq \theta_{\rho n} \leq \zeta_n \quad \text{for all} \quad \rho > 0. \tag{3.44}$$

We pass to the upper bound as $t \in [0, n]$ in (3.43) and use (3.44) to obtain

$$\max_{t\in[0,n]} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leq \big(G(\rho) + \omega_{\rho n} + \delta_{\rho n}\big)\xi_{n,u}\, e^{\frac{\zeta_n}{m_{\mathcal{F}}} n} \quad \text{for all} \quad \rho > 0. \tag{3.45}$$

We use now assumptions (3.23)–(3.25) and definitions (3.37), (3.39) to see that

$$\omega_{\rho n} \to 0 \quad \text{and} \quad \delta_{\rho n} \to 0 \quad \text{as } \rho \to 0. \tag{3.46}$$

We combine now (3.46) and (3.26)(b) with inequality (3.45) to obtain

$$\max_{t\in[0,n]} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \to 0 \quad \text{as } \rho \to 0. \tag{3.47}$$

The convergence (3.47) shows that (3.27) holds, which concludes the proof. □

Note that the convergence result in Theorem 3.2 can be easily extended to the corresponding stress functions. Indeed, let $\boldsymbol{\sigma}$ be the function defined by (3.1) and, for all $\rho > 0$, denote by $\boldsymbol{\sigma}_\rho$ the function given by

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \tag{3.48}$$

for all $t \in \mathbb{R}_+$. Then, it follows that $\boldsymbol{\sigma}_\rho \in C(\mathbb{R}_+; Q_1)$ and, moreover, (3.22) yields

$$\operatorname{Div} \boldsymbol{\sigma}_\rho(t) + \boldsymbol{f}_{0\rho}(t) = \boldsymbol{0} \qquad \forall\, t \in \mathbb{R}_+. \tag{3.49}$$

We combine now equalities (3.1), (3.21), (3.48) and (3.49), then we use the convergences (3.24) and (3.27) to see that

$$\boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \quad \text{in} \quad C(\mathbb{R}_+; Q_1) \quad \text{as } \rho \to 0. \tag{3.50}$$

4. A CONVERGENCE RESULT FOR A VISCOPLASTIC CONTACT PROBLEM

The classical formulation of the second contact problem is the following.

Problem $\mathcal{P}_2$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that for all* $t \in \mathbb{R}_+$

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t))) \quad \text{in} \quad \Omega, \tag{4.1}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{4.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{4.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \quad \text{on} \quad \Gamma_2, \tag{4.4}$$

$$\left\{ \begin{array}{l} u_\nu(t) \le g,\ \sigma_\nu(t) + p(u_\nu(t)) + \displaystyle\int_0^t b(t-s) u_\nu^+(s) \mathrm{d}s \le 0 \\ (u_\nu(t) - g)\Big(\sigma_\nu(t) + p(u_\nu(t)) + \displaystyle\int_0^t b(t-s) u_\nu^+(s) \mathrm{d}s\Big) = 0 \end{array} \right. \quad \text{on} \quad \Gamma_3, \tag{4.5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{4.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \text{in} \quad \Omega. \tag{4.7}$$

The difference between problems $\mathcal{P}_1$ and $\mathcal{P}_2$ consists in the fact that equation (4.1) represents the viscoplastic constitutive law of the material and condition (4.5) shows that the contact follows a normal compliance condition with memory term and unilateral constraint. Finally, (4.7) represents the initial conditions in which $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ denote the initial displacement and the initial stress field, respectively.

We assume that the elasticity tensor $\mathcal{F}$ and the nonlinear constitutive function $\mathcal{G}$ satisfy the following conditions:

$$
(4.8)\qquad \left\{\begin{array}{l}
\text{(a) } \mathcal{F} = (\mathcal{F}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) } \mathcal{F}_{ijkl} = \mathcal{F}_{klij} = \mathcal{F}_{jikl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d. \\
\text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that} \\
\qquad \mathcal{F}\boldsymbol{\tau}\cdot\boldsymbol{\tau} \ge m_{\mathcal{F}}\|\boldsymbol{\tau}\|^2 \ \ \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. in } \Omega.
\end{array}\right.
$$

$$
(4.9)\qquad \left\{\begin{array}{l}
\text{(a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\
\|\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{G}}\left(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\right) \\
\qquad\qquad \forall\, \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\sigma},\boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{G}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0}) \text{ belongs to } Q.
\end{array}\right.
$$

Also, as in the case of the first problem we assume that the normal compliance function p verifies (3.8), the surface memory function satisfies (3.9), the body forces and the surface tractions have the regularity (3.7) and, finally, we assume that the initial data verify

$$
(4.10)\qquad \boldsymbol{u}_0 \in U, \quad \boldsymbol{\sigma}_0 \in Q,
$$

where U denotes the set of admissible displacements defined by

$$
(4.11)\qquad U = \{\, \boldsymbol{v} \in V \,:\, v_\nu \le g \ \text{ a.e. } \Gamma_3 \,\}.
$$

The following existence and uniqueness result is proved in [2].

Lemma 4.1. *Assume that* (4.8), (4.9) *and* (4.10) *hold. Then, for each function* $\boldsymbol{u} \in C(\mathbb{R}_+;V)$ *there exists a unique function* $\mathcal{S}_1\boldsymbol{u} \in C(\mathbb{R}_+;Q)$ *such that*

$$
(4.12)\qquad \mathcal{S}_1\boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}_1\boldsymbol{u}(s) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s + \boldsymbol{\sigma}_0 - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_0),
$$

for all $t \in \mathbb{R}_+$. *Moreover, the operator* $\mathcal{S}_1 : C(\mathbb{R}_+;V) \to C(\mathbb{R}_+;Q)$ *satisfies the following property: for every* $n \in \mathbb{N}^*$ *there exists* $k_n > 0$ *such that* $\forall\, \boldsymbol{u}, \boldsymbol{v} \in C(\mathbb{R}_+;V)$ *and* $\forall\, t \in [0,n]$

$$
(4.13)\qquad \|\mathcal{S}_1\boldsymbol{u}(t) - \mathcal{S}_1\boldsymbol{v}(t)\|_Q \le k_n \int_0^t \|\boldsymbol{u}(s) - \boldsymbol{v}(s)\|_V\,\mathrm{d}s.
$$

We use (3.15), the Riesz's representation Theorem and the above lemma to define the operator $\mathcal{S} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ by equality

$$(\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v})_V = (\mathcal{S}_1\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathcal{B}\boldsymbol{u}(t), \boldsymbol{v})_V, \ \forall\, \boldsymbol{u} \in C(\mathbb{R}_+; V),\ \boldsymbol{v} \in V. \tag{4.14}$$

The variational formulation of Problem $\mathcal{P}_2$, derived in [3], is the following.

Problem $\mathcal{P}_2^V$. *Find a displacement field $\boldsymbol{u} : \mathbb{R}_+ \to V$ and a stress field $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$, such that*

$$\boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{S}_1\boldsymbol{u}(t), \tag{4.15}$$

$$\begin{aligned}(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (\mathcal{S}\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \\ +(P\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V \ \ \forall\, \boldsymbol{v} \in U\end{aligned} \tag{4.16}$$

hold, for all $t \in \mathbb{R}_+$.

In the study of the problem $\mathcal{P}_2^V$ we have the following existence and uniqueness result.

THEOREM 4.2. *Assume that* (3.7)–(3.9) *and* (4.8)–(4.10) *hold. Then, Problem $\mathcal{P}_2^V$ has a unique solution, which satisfies*

$$\boldsymbol{u} \in C(\mathbb{R}_+; U), \qquad \boldsymbol{\sigma} \in C(\mathbb{R}_+; Q). \tag{4.17}$$

The proof of Theorem 4.2 can be found in [3]. It is based on arguments of history-dependent variational inequalities developed in [6].

We study now the dependence of the solution of Problem $\mathcal{P}_2^V$ with respect to perturbations of the data. To this end, we assume in what follows that (3.7)–(3.9), (3.13), (4.8)–(4.10) hold and we denote by $(\boldsymbol{u}, \boldsymbol{\sigma})$ the solution of Problem $\mathcal{P}_2^V$ obtained in Theorem 4.2. For each $\rho > 0$ let p_ρ, b_ρ, $\boldsymbol{f}_{0\rho}$, $\boldsymbol{f}_{2\rho}$, $\boldsymbol{u}_{0\rho}$ and $\boldsymbol{\sigma}_{0\rho}$ be perturbations of p, b, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$, $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$, respectively, which satisfy conditions (3.8), (3.9), (3.7), (4.10). We define the operators $\mathcal{S}_{1\rho} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; Q)$ and $\mathcal{S}_\rho : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$ by

$$\mathcal{S}_{1\rho}\boldsymbol{u}(t) = \int_0^t \mathcal{G}(\mathcal{S}_{1\rho}\boldsymbol{u}(s) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,\mathrm{d}s + \boldsymbol{\sigma}_{0\rho} - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_{0\rho}), \tag{4.18}$$

$$(\mathcal{S}_\rho\boldsymbol{u}(t), \boldsymbol{v})_V = (\mathcal{S}_{1\rho}\boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathcal{B}_\rho\boldsymbol{u}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V \tag{4.19}$$

and we consider the following variational problem.

Problem $\mathcal{P}_{2\rho}^V$. *Find a displacement field $\boldsymbol{u}_\rho : \mathbb{R}_+ \to V$ and a stress field $\boldsymbol{\sigma}_\rho : \mathbb{R}_+ \to Q$, such that*

$$\boldsymbol{\sigma}_\rho(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)) + \mathcal{S}_{1\rho}\boldsymbol{u}_\rho(t), \tag{4.20}$$

$$(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + (\mathcal{S}_\rho\boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \tag{4.21}$$

$$+(P_\rho \boldsymbol{u}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \geq (\boldsymbol{f}_\rho(t), \boldsymbol{v} - \boldsymbol{u}_\rho(t))_V \ \ \forall \boldsymbol{v} \in U.$$

It follows from Theorem 4.2 that, for each $\rho > 0$ Problem $\mathcal{P}^V_{2\rho}$ has a unique solution $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho)$ with the regularity $\boldsymbol{u}_\rho \in C(\mathbb{R}_+; U)$, $\boldsymbol{\sigma}_\rho \in C(\mathbb{R}_+; Q)$. Consider now the assumptions (3.23)–(3.26) and

$$\text{(4.22)} \qquad \boldsymbol{u}_{0\rho} \to \boldsymbol{u}_0 \quad \text{in } \ V, \quad \boldsymbol{\sigma}_{0\rho} \to \boldsymbol{\sigma}_0 \quad \text{in } \ Q \quad \text{as} \quad \rho \to 0.$$

We have the following convergence result.

THEOREM 4.3. *Under assumptions* (3.23)–(3.26) *and* (4.22) *the solution* $(\boldsymbol{u}_\rho, \boldsymbol{\sigma}_\rho)$ *of Problem* $\mathcal{P}^V_{2\rho}$ *converges to the solution* $(\boldsymbol{u}, \boldsymbol{\sigma})$ *of Problem* $\mathcal{P}^V_2$, *i.e.*

$$\text{(4.23)} \qquad \boldsymbol{u}_\rho \to \boldsymbol{u} \quad in \quad C(\mathbb{R}_+; V), \quad \boldsymbol{\sigma}_\rho \to \boldsymbol{\sigma} \quad in \quad C(\mathbb{R}_+; Q) \quad as \quad \rho \to 0.$$

Proof. Let $\rho > 0$, $n \in \mathbb{N}^*$ and let $t \in [0, n]$. We take $\boldsymbol{v} = \boldsymbol{u}(t)$ in (4.21) and $\boldsymbol{v} = \boldsymbol{u}_\rho(t)$ in (4.16) and add the resulting inequalities to obtain

$$\text{(4.24)} \qquad (\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q$$

$$\leq (P_\rho \boldsymbol{u}_\rho(t) - P\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V + (\boldsymbol{f}_\rho(t) - \boldsymbol{f}(t), \boldsymbol{u}_\rho(t) - \boldsymbol{u}(t))_V$$

$$+(\mathcal{S}_\rho \boldsymbol{u}_\rho(t) - \mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V.$$

We have

$$\text{(4.25)} \qquad (\mathcal{S}_\rho \boldsymbol{u}_\rho(t) - \mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V$$

$$= (\mathcal{S}_{1\rho} \boldsymbol{u}_\rho(t) - \mathcal{S}_1 \boldsymbol{u}(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{\varepsilon}(\boldsymbol{u}_\rho(t)))_Q + (\mathcal{B}_\rho \boldsymbol{u}_\rho(t) - \mathcal{B}\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V.$$

Using Lemma 4.1, (2.4) and (4.8) we have the following inequality

$$\text{(4.26)} \qquad \|\mathcal{S}_{1\rho} \boldsymbol{u}_\rho(t) - \mathcal{S}_1 \boldsymbol{u}(t)\|_Q \leq k_n \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \mathrm{d}s + \alpha_{0\rho} q_n,$$

where

$$\text{(4.27)} \qquad \alpha_{0\rho} = \|\boldsymbol{\sigma}_{0\rho} - \boldsymbol{\sigma}_0\|_Q + d\,\|\mathcal{F}\|_{\mathbf{Q}_\infty} \|\boldsymbol{u}_{0\rho} - \boldsymbol{u}_0\|_V$$

and q_n is a positive constant which depends on n and $\mathcal{G}$.

From (4.25), (3.36) and (4.26) we obtain

$$\text{(4.28)} \ (\mathcal{S}_\rho \boldsymbol{u}_\rho(t) - \mathcal{S}\boldsymbol{u}(t), \boldsymbol{u}(t) - \boldsymbol{u}_\rho(t))_V \leq \Big(k_n \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \mathrm{d}s + \alpha_{0\rho} q_n$$

$$+\theta_{\rho n} \int_0^t \|\boldsymbol{u}_\rho(s) - \boldsymbol{u}(s)\|_V \,\mathrm{d}s + \omega_{\rho n} \int_0^t \|\boldsymbol{u}(s)\|_V \,\mathrm{d}s\Big) \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V.$$

Next, we combine (4.24), (4.8) (c), (3.33), (3.38) and (4.28) to see that

$$\text{(4.29)} \qquad m_{\mathcal{F}} \|\boldsymbol{u}_\rho(t) - \boldsymbol{u}(t)\|_V \leq G(\rho)\big(c_0^2 \,\|\boldsymbol{u}(t)\|_V + c_0 \beta \,\mathrm{meas}\,(\Gamma_3)^{\frac{1}{2}}\big)$$

$$+(k_n+\theta_{\rho n})\int_0^t \|\boldsymbol{u}_\rho(s)-\boldsymbol{u}(s)\|_V \mathrm{d}s+\omega_{\rho n}\int_0^t \|\boldsymbol{u}(s)\|_V\,\mathrm{d}s+\alpha_{0\rho}q_n+\delta_{\rho n}.$$

Next, using (4.15), (4.20), (2.4), (4.8), (4.9) and (4.26) we deduce that

$$\|\boldsymbol{\sigma}_\rho(t)-\boldsymbol{\sigma}(t)\|_Q\le c\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V+k_n\int_0^t \|\boldsymbol{u}_\rho(s)-\boldsymbol{u}(s)\|_V \mathrm{d}s+\alpha_{0\rho}q_n, \tag{4.30}$$

where c is a positive generic constant and whose value may change from line to line.

Following, we use the notation $\xi_{n,u}:=\max\{\xi_{n,u},q_n\}$ and we add now inequalities (4.30) and (4.29) to obtain

$$\|\boldsymbol{\sigma}_\rho(t)-\boldsymbol{\sigma}(t)\|_Q+\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V\le c(G(\rho)+\alpha_{0\rho}+\omega_{\rho n}+\delta_{\rho n})\xi_{n,u} \tag{4.31}$$
$$+c(k_n+\theta_{\rho n})\int_0^t (\|\boldsymbol{u}_\rho(s)-\boldsymbol{u}(s)\|_V+\|\boldsymbol{\sigma}_\rho(s)-\boldsymbol{\sigma}(s)\|_Q)\mathrm{d}s.$$

Then, we use Gronwall inequality to see that

$$\|\boldsymbol{\sigma}_\rho(t)-\boldsymbol{\sigma}(t)\|_Q+\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V \tag{4.32}$$
$$\le c(G(\rho)+\alpha_{0\rho}+\omega_{\rho n}+\delta_{\rho n})\xi_{n,u}e^{c(\theta_{\rho n}+k_n)t}.$$

We pass to the upper bound as $t\in[0,n]$ in (4.32) and use (3.44) to obtain

$$\max_{t\in[0,n]}(\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V+\|\boldsymbol{\sigma}_\rho(t)-\boldsymbol{\sigma}(t)\|_Q)$$
$$\le c(G(\rho)+\alpha_{0\rho}+\omega_{\rho n}+\delta_{\rho n})\xi_{n,u}e^{cn(\zeta_n+k_n)}.$$

Finally, (4.22) yields

$$\alpha_{0\rho}\to 0 \quad \text{as} \quad \rho\to 0. \tag{4.33}$$

We use now (3.26) (b), (3.46) and (4.33) in the above inequality to obtain

$$\max_{t\in[0,n]}(\|\boldsymbol{u}_\rho(t)-\boldsymbol{u}(t)\|_V+\|\boldsymbol{\sigma}_\rho(t)-\boldsymbol{\sigma}(t)\|_Q)\to 0 \quad \text{as} \quad \rho\to 0. \tag{4.34}$$

Since the convergence (4.34) holds for each $n\in\mathbb{N}^*$ we deduce from (2.5) that (4.23) holds, which concludes the proof. □

In addition to the mathematical interest in the convergence results (3.27), (3.50), (4.23) it is of importance from a mechanical point of view, since it states that the weak solution of the problems (3.1)–(3.6) and (4.1)–(4.7) depends continuously on the normal compliance function, the surface memory function, the densities of body forces and surface tractions. Moreover, for the second problem the weak solution depends continuously on the initial data too.

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