STABILITY ON GENERALIZED SASAKIAN SPACE FORMS

NAJMA ABDUL REHMAN

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In this paper, the stability of the identity map on a compact generalized Sasakian space form is studied.

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1. INTRODUCTION

The theory of harmonic maps combines both global and local aspects and borrows both from Riemannian geometry and analysis. There are a lot of interesting results about harmonic maps on complex manifolds (see [9, 15]). In the analogy to the complex case, in the last decade harmonic maps on almost contact metric manifolds were studied [1, 3, 5–7]. The identity map of a compact Riemannian manifold is a trivial example of a harmonic map but in this case, the theory of the second variation is much more complicated and interesting. For instance, the stability of the identity map on Einstein manifolds is related with the first eigenvalue of the Laplacian acting on functions [13]. In [14] and [11] the authors find classifications of compact simply connected irreducible Riemannian symmetric spaces for which the identity map is unstable.

By a well known result, the identity map on the euclidean sphere S^{2n+1} is unstable [13]. More generally, C. Gherghe, S. Ianus and A.M. Pastore have studied the stability of the identity map on compact Sasakian manifolds with constant φ-sectional curvature [7].

P. Alegre, D.E. Blair and A. Carriazo introduced the generalized Sasakian space forms a generalization of Sasakian space forms [1]. So it is natural to study the stability of the identity map on a compact domain of such a manifold. The paper is organized as follows: after recalling in Section 2 the necessary facts about harmonic maps between Riemannian manifolds, we give some definitions on almost contact manifolds and recall in Section 3 how generalized Sasakian space forms are defined. Finally in Section 4, we give some results on the stability of the identity map on a compact generalized Sasakian space form.

2. HARMONIC MAPS ON RIEMANNIAN MANIFOLDS

In this section, we recall some well known facts concerning harmonic maps (see [4] for more details). Let $\phi : (M, g) \longrightarrow (N, h)$ be a smooth map between two Riemannian manifolds of dimensions m and n respectively.

The energy density of ϕ is a smooth function $e(\phi): M \longrightarrow [0, \infty)$ given by

$$e(\phi)_p = \frac{1}{2} Tr_g(\phi^* h)(p) = \frac{1}{2} \sum_{i=1}^{m} h(\phi_{*p} u_i, \phi_{*p} u_i),$$

for $p \in M$ and any orthonormal basis $\{u_1, \ldots, u_m\}$ of T_pM. If M is a compact Riemannian manifold, the energy $E(\phi)$ of ϕ is the integral of its energy density:

$$E(\phi) = \int_M e(\phi) v_g,$$

where v_g is the volume measure associated with the metric g on M. A map $\phi \in C^\infty(M, N)$ is said to be harmonic if it is a critical point of E in the set of all smooth maps between (M, g) and (N, h) *i.e.* for any smooth variation $\phi_t \in C^\infty(M, N)$ of ϕ ($t \in (-\epsilon, \epsilon)$) with $\phi_0 = \phi$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\phi_t)\Big|_{t=0} = 0.$$

Now, let (M, g) be a compact Riemannian manifold and $\phi : (M, g) \longrightarrow (N, h)$ be a harmonic map. We take a smooth variation $\phi_{s,t}$ with parameters $s, t \in (-\epsilon, \epsilon)$ such that $\phi_{0,0} = \phi$. The corresponding variation vector fields are denoted by V and W. The Hessian H_ϕ of a harmonic map ϕ is defined by

$$H_\phi(V, W) = \frac{\partial^2}{\partial s \partial t} (E(\phi_{s,t}))\Big|_{(s,t)=(0,0)}.$$

The second variation formula of E is [10, 13]:

$$H_\phi(V, W) = \int_M h(J_\phi(V), W) v_g,$$

where J_ϕ is a second order self-adjoint elliptic operator acting on the space of variation vector fields along ϕ (which can be identified with $\Gamma(\phi^{-1}(TN))$) and is defined by

$$J_\phi(V) = -\sum_{i=1}^{m} (\widetilde{\nabla}_{u_i} \widetilde{\nabla}_{u_i} - \widetilde{\nabla}_{\nabla_{u_i} u_i}) V - \sum_{i=1}^{m} R^N(V, \mathrm{d}\phi(u_i)) \mathrm{d}\phi(u_i),$$

for any $V \in \Gamma(\phi^{-1}(TN))$ and any local orthonormal frame $\{u_1, \ldots, u_m\}$ on M. Here R^N is the curvature tensor of (N, h) and $\widetilde{\nabla}$ is the pull-back connection by ϕ of the Levi-Civita connection of N.

The index of a harmonic map $\phi : (M, g) \longrightarrow (N, h)$ is defined as the dimension of the largest subspace of $\Gamma(\phi^{-1}(TN))$ on which the Hessian H_ϕ is negative definite. A harmonic map ϕ is said to be weakly stable if the index of ϕ is zero and otherwise is said to be unstable.

The operator $\overline{\triangle}_\phi$ defined by

$$\overline{\triangle}_\phi V = -\sum_{i=1}^{m}(\widetilde{\nabla}_{u_i}\widetilde{\nabla}_{u_i} - \widetilde{\nabla}_{\nabla_{u_i}u_i})V, \qquad V \in \Gamma(\phi^{-1}(TN))$$

is called the rough Laplacian.

Due to the Hodge de Rham Kodaira theory, the spectrum of J_ϕ consists of a discrete set of an infinite number of eigenvalues with finite muiltiplicities and without accumulation points.

3. GENERALIZED SASAKIAN SPACE FORMS

In this section, we recall some definitions and basic formulas on almost contact metric manifolds and generalized Sasakian space forms (see [2] for more details).

An odd dimensional Riemannian manifold M^{2n+1} is said to be an almost contact manifold if there exist on M a $(1, 1)$-tensor field φ, a vector field ξ and a 1-form η such that

$$\varphi^2 = -I + \eta \otimes \xi \quad and \quad \eta(\xi) = 1.$$

In an almost contact manifold we also have $\varphi(\xi)$=0 and $\eta o \varphi = 0$.

On any almost contact manifold, we can define a compatible metric that is a metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M. In this case, the manifold will be called almost contact metric manifold. An almost contact metric manifold is said to be a contact metric manifold if $d\eta$=Ω, where Ω is the fundamental 2 form defined by Ω(X, Y)=g(X, φY) for X, Y $\in \Gamma(TM)$. In analogy with the integrability condition on almost complex manifolds, the almost contact metric structure of M is said to be normal if

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ, given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

A normal contact metric manifold is called a Sasakian manifold. A normal almost contact metric manifold is called a Kenmostsu manifold if $d\eta$=0 and

$$d\Omega(X,Y,Z) = \frac{2}{3}\sigma_{(X,Y,Z)}\{\eta(X)\Omega(Y,Z)\}, \quad X,Y,Z \in \Gamma(\phi^{-1}(TN)),$$

where σ denotes the cyclic sum (see [8]). A simply connected Riemannian manifold with constant sectional curvature c is called a real space form and its curvature tensor satisfies the equation

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}.$$

An almost contact metric manifold $M(\varphi,\ \xi,\ \eta,$ g) is called a generalized Sasakian space form if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor on M satisfies the identity:

$$\begin{aligned} R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + \\ +f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + +2g(X,\phi Y)\phi Z\} + \\ +f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned}$$

for any vector fields X, Y, Z on M. (See [1]).

In particular:
If $f_1 = (c+3)/4$ and $f_2 = f_3 = (c-1)/4$, then M is a Sasakian space form.
If $f_1 = (c-3)/4$ and $f_2 = f_3 = (c+1)/4$, then M is a Kenmostsu-space form (see [1, 12]).

Example 1. Classical examples of generalized Sasakian space forms are warped products between the real line and a complex space form. For instance the warped products $R \times_f \mathbb{C}^n$, $R \times_f \mathbb{CP}^n(4)$ and $R \times_f \mathbb{CH}^n(-4)$ are generalized Sasakian space forms with the functions
$f_1 = -\frac{f'^2}{f^2}$, $f_2 = 0$, $f_3 = -\frac{f'^2}{f^2} + \frac{f''}{f}$,
$f_1 = \frac{1-f'^2}{f^2}$, $f_2 = \frac{1}{f^2}$, $f_3 = \frac{1-f'^2}{f^2} + \frac{f''}{f}$,
$f_1 = \frac{-1-f'^2}{f^2}$, $f_2 = \frac{-1}{f^2}$, $f_3 = \frac{-1-f'^2}{f^2} + \frac{f''}{f}$,
respectively, where f is a positive smooth function (see [1]).

4. MAIN RESULTS

Let M be a compact generalized Sasakian space form $M(\varphi,\ \xi,\ \eta,$ g). We consider the identity map on such a manifold ($\phi = 1_M$). In this case, see [15], the second variation formula is

$$H_{1_M}(V,V) = \int_M h(\overline{\triangle}V,V)\upsilon_g - \sum_{i=1}^{2n+1}\int_M h(R(V,u_i)u_i,V)\upsilon_g,$$

where $V \in \Gamma(TM)$ and $\{u_1, ..., u_{2n+1}\}$ is a local orthonormal frame on TM.

Let $\{e_i, \varphi e_i, \xi\}$ be an orthonormal local φ-adapted frame. Then we have

$$R(e_i, V)e_i = f_1\{g(V, e_i)e_i - V\} + f_2\{-3g(V, \varphi e_i)\varphi e_i\} + f_3\{g(V, \xi)\xi\}. \tag{1}$$

$$R(\varphi e_i, V)\varphi e_i = f_1\{g(V, \varphi e_i)\varphi e_i - V\} + f_2\{-3g(V, e_i)e_i\} + f_3\{g(V, \xi)\xi\}. \tag{2}$$

$$R(\xi, V)\xi = f_1\{g(V, \xi)\xi - V\} + f_3\{V - g(V, \xi)\xi\}. \tag{3}$$

From the above three relations, we get

$$\sum_{i=1}^{2n+1} g(R(u_i, V)u_i, V) = (f_1 - 3f_2)\sum_{i=1}^{n}\{g(V, e_i)^2 + g(V, \varphi e_i)^2\} - \\ -[(2n+1)f_1 - f_3]g(V, V) + [(2n-1)f_3 + f_1]g(V, \xi)^2,$$

and thus,

$$\sum_{i=1}^{2n+1} g(R(u_i, V)u_i, V) = -[3f_2 + 2nf_1 - f_3]g(V, V) + \\ +[3f_2 + (2n-1)f_3]g(V, \xi)^2.$$

THEOREM 1. *Let M be a compact generalized Sasakian space form. If $(3f_2 + (2n-1)f_3) \geq 0$ and $(3f_2 + 2nf_1 - f_3) \leq 0$, then the identity map 1_M is weakly stable.*

Proof. It is not very difficult to prove that

$$\int_M h(\overline{\triangle}V, V)\upsilon_g = \int_M h(\widetilde{\nabla}V, \widetilde{\nabla}V)\upsilon_g, \qquad V \in \Gamma(TM).$$

Now the second varition formula becomes

$$H_{1_M}(V, V) = \int_M h(\widetilde{\nabla}V, \widetilde{\nabla}V) - \int_M (3f_2 + 2nf_1 - f_3)g(V, V)\upsilon_g + \\ + \int_M (3f_2 + (2n-1)f_3)g(V, \xi)^2\upsilon_g,$$

and thus, the identity map 1_M is weakly stable if $(3f_2 + 2nf_1 - f_3) \leq 0$ and $(3f_2 + (2n-1)f_3) \geq 0$. $\square$

COROLLARY 1. *Let Ω be a compact domain of a Kenmotsu space form M of constant φ-sectional curvature c. If $c \in (-1, \frac{3n-1}{n+1})$ then the identity map 1_M is weakly stable.*

Proof. For a Kenmotsu space form M, we have $f_1 = \frac{(c-3)}{4}$, $f_2 = f_3 = \frac{c+1}{4}$ (see [1]). And thus, $2(3f_2+2nf_1-f_3) = c(n+1)-3n+1$ and $2(3f_2+(2n-1)f_3) = (c+1)(n+1)$ for $c \in (-1, \frac{3n-1}{n+1})$. Then by Theorem 1, the identity 1_M map is weakly stable for $c \in (-1, \frac{3n-1}{n+1})$. $\square$

Example 2. Using Example 1, it is not difficult to see that for $f(t) = \exp(t)$ and $f(t) = \frac{1}{t^p}$ $(t > 0, \quad p > 0)$, the inequalities in Theorem 1 are satisfied and thus, the identity map of any compact domain of the warped product $R \times_f C^n$ is weakly stable.

Similarly, for $f(t) = e^t$, $t \in [\frac{1}{2}ln(\frac{n+1}{n}), \infty)$, the identity map of any compact domain of the warped product $\mathbb{R} \times_f \mathbb{CP}^n$ is weakly stable.

We recall now the Weitzenbock formula: Let E be a vector bundle over an m-dimensional Riemannian manifold M. Then for any 1-form $\omega \in A^1(E)$ we have

$$\triangle_1 \omega = \overline{\triangle}\omega - \rho(\omega),$$

where

$$\rho(\omega)(X) = \sum_{i=1}^{m} R(X, u_i)(\omega(u_i)) - \sum_{i=1}^{m} \omega(R(X, u_i)u_i),$$

for any $\Gamma(TM)$. Here $\triangle_1$ is the Laplacian of E-valued 1-forms and $\overline{\triangle}$ is rough Laplacian on $A^1(E)$. Then, using the Weitzenbock formula for $E = M \times R$, we have

$$\triangle_1(V) = \overline{\triangle}V + \sum_{i=1}^{2n+1} R(V, u_i)u_i.$$

Then the second variational formula becomes now

$$H_{1_M}(V, V) = \int_M h(\triangle_1 V, V)v_g - 2\sum_{i=1}^{2n+1} \int_M h(R(V, u_i)u_i, V)v_g.$$

THEOREM 2. *Let M be a compact domain of a generalized Sasakian space form such that* $(3f_2 + (2n-1)f_3) \leq 0$. *If the first eigenvalue* λ_1 *of the Laplacian* $\triangle_g$ *acting on* $C^\infty(M)$ *satisfies* $\lambda_1 < 2(3f_2 + 2nf_1 - f_3)$, *then the identity map* 1_M *is unstable.*

Proof. Let λ_1 be the first eigenvalue of the Laplacian $\triangle_g$ acting on $C^\infty(M)$ and f be a non constant eigenfunction of $\triangle_g$ such that $\triangle_g f = \lambda_1 f$. Let $f \in C^\infty(M)$ be taken as $\triangle_g f = \lambda_1(g)f$ and let $V = gradf \neq 0$. Then

$$\begin{aligned}\int_M h(\triangle_1 V, V)v_g &= \int_M < (\mathrm{d}\delta + \delta d)\mathrm{d}f, \mathrm{d}f > v_g = \int_M < \mathrm{d}\delta\mathrm{d}f, \mathrm{d}f > v_g \\ &= \lambda_1(g)\int_M < \mathrm{d}f, \mathrm{d}f > v_g \quad (\textit{since} \quad \delta\mathrm{d}f = \triangle_g f) \\ &= \lambda_1(g)\int_M < \mathrm{d}f, \mathrm{d}f > v_g.\end{aligned}$$

Then the second variational formula becomes

$$H_{1_M}(V,V) = \int_M (\lambda_1 - 2(3f_2 + 2nf_1 - f_3)g(V,V)\upsilon_g + \\ + \int_M 2(3f_2 + (2n-1)f_3)g(V,\xi)^2\upsilon_g.$$

If the first eigenvalue λ_1 of the Laplacian satisfies $\lambda_1 < 2(3f_2+2nf_1-f_3)$ then 1_M is unstable. □

Using Theorem 2, we have the folowing corollary (see [7]).

Corollary 2. *Let M be a compact Sasakian space form of constant φ-sectional curvature c such that $c \leq 1$. If the first eigenvalue λ_1 of the Laplacian satisfies $\lambda_1 < c(n+1) + 3n - 1$ then the identity map is unstable.*

Proof. Indeed for a Sasakian space form, $f_1 = \frac{(c+3)}{4}$, $f_2 = f_3 = \frac{(c-1)}{4}$, and thus, $(3f_2+2nf_1-f_3) = (n+1)c+3n-1$ and $3f_2+(2n-1)f_3 = (n+1)(c-1)$. Then if $c \leq 1$ and $\lambda_1 < c(n+1) + 3n - 1$, from Theorem 2 we have that 1_M map is unstable. □

Remark 1. If M is a Sasakian space form with $c = 1$, then M becomes a space with constant curvature 1, hence, isometric to the unit sphere. As we know the first eigenvalue of the Laplacian $\triangle_g$ for the euclidean (2n+1)-sphere is $\lambda_1 = (2n+1)$ and therefore, from Corollary 2 the identity $1_{S^{(2n+1)}}$ is an unstable map.

Example 3. We can apply Theorem 2 also for a compact domain of a generalized Sasakian space form which is not Sasakian. First of all, such examples do exist. For example, if $N(c)$ is a complex space form of real dimension $2n$, we consider a compact domain of $M = (-\frac{\pi}{2}, \frac{\pi}{2}) \times_f N(c)$ with the function $f(t) = \cos t$. Then M is a generalized Sasakian space form (which is not Sasakian space form) with the functions (see Theorem 4.8. in [1]):

$$f_1 = \frac{c - 4\sin^2 t}{4\cos^2 t}, \quad f_2 = \frac{c}{4\cos^2 t}, \quad f_3 = \frac{c - 4\sin^2 t}{4\cos^2 t} - 1.$$

It easy to see that

$$3f_2 + (2n-1)f_3 == \frac{1}{2}\left[c + 2 + n(c-4)\right]\sec^2 t$$

and thus, if we chose c such that $c \leq \frac{2(2n-1)}{n+1}$, then the first condition of Theorem 2 is satisfied. On the other hand it is not difficult to see that if $\frac{2(2n-1)}{n+2} < c$ then, if the first eigenvalue λ_1 satisfies the condition $\lambda_1 < 4n$, then the second condition of Theorem 2 is also satisfied and thus, the identity map is unstable.

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GC University,
"Abdus Salam" School of Mathematical Sciences,
68-B, New Muslim Town,
Lahore 54600,
Pakistan
najma_ar@hotmail.com