ON SOME FRACTIONAL DIRICHLET PROBLEMS IN BOUNDED DOMAINS

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Communicated by Daniel Tătaru

We prove the existence of positive continuous solutions to the nonlinear fractional problem

$$\left(-\Delta_{|D}\right)^{\frac{\alpha}{2}} u + \lambda f(., u) = 0,$$

in a bounded $C^{1,1}$-domain D in $\mathbb{R}^n$ $(n \geq 2)$, subject to some Dirichlet conditions, where $0 < \alpha < 2$ and λ is a positive number. The function f is nonnegative continuous monotone with respect to the second variable and satisfying some adequate hypotheses related to the Kato class. Our approach is based on Schauder's fixed point Theorem.

AMS 2010 Subject Classification: 34A08, 34B27, 35B09.

Key words: fractional nonlinear problem, bounday value problems, Green's function, positive solutions.

1. INTRODUCTION

Let D be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ $(n \geq 2)$ and $0 < \alpha < 2$. In this paper, we deal with the existence of positive continuous solutions for the following nonlinear fractional problem

$$(1) \quad \begin{cases} \left(-\Delta_{|D}\right)^{\frac{\alpha}{2}} u + \lambda f(., u) = 0 & \text{in } D, \text{ (in the sense of distributions)} \\ u > 0 & \text{in } D, \\ \lim\limits_{x \to z \in \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = \varphi(z), & \end{cases}$$

where λ is a positive number, φ is a fixed positive continuous function on ∂D. Here the fractional power $\left(-\Delta_{|D}\right)^{\frac{\alpha}{2}}$ of the negative Dirichlet Laplacian in D, is the infinitesimal generator of the subordinate killed Brownian motion process Z_α^D. For more description of the process Z_α^D we refer to [12, 13, 16, 17].

The nonnegative function $M_\alpha^D 1$ is defined by the formula

$$(2) \quad M_\alpha^D 1(x) = \frac{1 - \frac{\alpha}{2}}{\Gamma(\frac{\alpha}{2})} \int_0^\infty t^{-2+\frac{\alpha}{2}} (1 - P_t^D 1(x)) \mathrm{d}t,$$

where $(P_t^D)_{t>0}$ is the semi-group corresponding to the killed Brownian motion upon exiting D.

We recall that from ([12], Theorem 3.1), the function $M_\alpha^D 1$ is harmonic with respect to Z_α^D and by ([17], Remark 3.3), there exists a constant $c>0$ such that

$$\frac{1}{c}\,(\delta(x))^{\alpha-2} \leq M_\alpha^D 1(x) \leq c\,(\delta(x))^{\alpha-2}\,, \text{ for all } x\in D, \tag{3}$$

where $\delta(x)$ denotes the Euclidian distance from x to the boundary of D.

When $\alpha=2$ and the nonlinearity f is negative, there exist a lot of work related to the problem (1); see for example, the papers of Alves, Carriao and Faria [1], Bandle [3], Bandle and M. Marcus [4], de Figueiredo, Girardi and Matzeu [9], Cîrstea, Ghergu and Rădulescu [5], Cîrstea and Rădulescu [6], Dumont, Dupaigne, Goubet and V. Rădulescu [7], Ghergu and Rădulescu [10, 11], Lair and Wood [14], Rădulescu [15], Zhang [18] and references therein. In all these papers, the main tools used are Galerkin method, sub-supersolution method and variational techniques. In a recent article [8], the authors studied the problem (1) for $\lambda=1$, $\varphi\equiv 0$ and f is a non-trivial nonnegative measurable function in $D\times(0,\infty)$ which is continuous and nonincreasing with respect to the second variable and satisfying some appropriate assumptions. Using a fixed point theorem, they have proved (see [8], Theorem 3) that the problem (1) has a positive continuous solution u in D.

In this paper, we aim to give two existence results for (1) as f is nondecreasing or nonincreasing with respect to the second variable and satisfying some appropriate conditions related to the Kato class $K_\alpha(D)$ (see Definition 1.1 below).

Throughout this paper, we denote by $M_\alpha^D\varphi$ (see [12]) the unique positive continuous solution of

$$\begin{cases} \left(-\Delta_{|D}\right)^{\frac{\alpha}{2}} u = 0 & \text{in } D, \text{ (in the sense of distributions)} \\ \lim\limits_{x\to z\in\partial D} \frac{u(x)}{M_\alpha^D 1(x)} = \varphi(z). \end{cases} \tag{4}$$

We also denote by $G_\alpha^D(x,y)$ the Green function of Z_α^D.

To state our first existence result, we assume that $f: D\times[0,\infty)\to[0,\infty)$ is Borel measurable function satisfying

($\mathbf{H}_1$) f is continuous and nondecreasing with respect to the second variable.

($\mathbf{H}_2$) The function $y\to \frac{1}{M_\alpha^D\varphi(y)} f(y, M_\alpha^D\varphi(y))$ belongs to the class $K_\alpha(D)$, defined below

Definition 1.1. A Borel measurable function q in D belongs to the Kato class $K_\alpha(D)$ if

$$\lim_{r\to 0}(\sup_{x\in D}\int_{(|x-y|\leq r)\cap D} \frac{\delta(y)}{\delta(x)} G_\alpha^D(x,y)|q(y)|\mathrm{d}y) = 0.$$

As a typical example of functions in $K_\alpha(D)$, we cite (see [8])

$$q(x) = (\delta(x))^{-\nu}, \text{ for } \nu < \alpha. \tag{5}$$

Our first existence result is the following

THEOREM 1.2. *Assume that (H_1)–(H_2) are satisfied. Then there exists a constant $\lambda_0 > 0$ such that for each $\lambda \in [0, \lambda_0)$, problem (1) has a positive continuous solution u such that*

$$(1 - \frac{\lambda}{\lambda_0}) M_\alpha^D \varphi \le u \le M_\alpha^D \varphi \quad \text{in } D.$$

To state our second existence result, we consider the special nonlinearity $f(x,u) = p(x)\, g(u)$ and we fix ϕ a positive continuous functions on ∂D. Put $h_0 = M_\alpha^D \phi$ and assume that

($\mathbf{H}_3$) The function $g : (0, \infty) \to [0, \infty)$ is continuous and nonincreasing.

($\mathbf{H}_4$) The function $p_0 := p\frac{g(h_0)}{h_0}$ belongs to the class $K_\alpha(D)$.

Using the Schauder fixed point theorem, we prove the following:

THEOREM 1.3. *Under the assumptions (H_3) and (H_4), there exists a constant $c > 1$ such that if $\varphi \ge c\phi$ on ∂D, then problem*

$$\begin{cases} \left(-\Delta_{|D}\right)^{\frac{\alpha}{2}} u + p(x)\, g(u) = 0 & \text{in } D, \ (\text{in the sense of distributions}) \\ u > 0 & \text{in } D, \\ \lim\limits_{x \to z \in \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = \varphi(z), & \end{cases} \tag{6}$$

has a positive continuous solution u satisfying for each $x \in D$

$$h_0(x) \le u(x) \le M_\alpha^D \varphi(x).$$

This result follows from the one of Athreya [2], who considered the following problem

$$(*) \begin{cases} \Delta u = g(u), & \text{in } \Omega \\ u = \varphi \text{ on } \partial\Omega, \end{cases}$$

where Ω is a simply connected bounded C^2-domain and $g(u) \le \max(1, u^{-\beta})$, for $0 \underset{\sim}{<} \beta < 1$. Then he proved that there exists a constant $c > 1$ such that if $\varphi \ge c\widetilde{h_0}$ on $\partial\Omega$, where $\widetilde{h_0}$ is a fixed positive harmonic function in Ω, problem $(*)$ has a positive continuous solution u such that $u \ge \widetilde{h_0}$.

Our paper is organized as follows. In Section 2, we collect some properties of functions belonging to the Kato class $K_\alpha(D)$, which are useful to establish our main result. In Section 3, we prove Theorems 1.2 and 1.3.

As usual, let $C_0(D)$ be the set of continuous functions in D vanishing continuously on ∂D. Note that $C_0(D)$ is a Banach space with respect to the uniform norm $\|u\|_\infty = \sup_{x\in D} |u(x)|$. When two positive functions ρ and ψ are defined on a set S, we write $\rho \approx \psi$ if the two sided inequality $\frac{1}{c}\psi \leq \rho \leq c\psi$ holds on S. Finally, we define the potential kernel G_α^D for a nonnegative Borel measurable functions ψ in D, by

$$G_\alpha^D \psi(x) = \int_D G_\alpha^D(x,y)\psi(y)\mathrm{d}y, \text{ for } x \in D.$$

2. ON THE KATO CLASS $K_\alpha(D)$

First, we recall the following sharp estimates on the Green function $G_\alpha^D(x,y)$.

PROPOSITION 2.1 (see [16]). *For x, y in D, we have*

$$G_\alpha^D(x,y) \approx \frac{1}{|x-y|^{n-\alpha}} \min\left(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right). \tag{7}$$

Next, we collect some properties of functions belonging to the Kato class $K_\alpha(D)$.

PROPOSITION 2.2 (see [8]). *Let q be a function in $K_\alpha(D)$, then we have*
(i)

$$a_\alpha(q) := \sup_{x,y\in D} \int_D \frac{G_D^\alpha(x,z)G_D^\alpha(z,y)}{G_D^\alpha(x,y)} q(y)\,\mathrm{d}y < \infty. \tag{8}$$

(ii) *Let h be a positive excessive function on D with respect to Z_α^D. Then there exists a constant $C_0 > 0$ such that*

$$\int_D G_\alpha^D(x,y)h(y)|q(y)|\mathrm{d}y \leq a_\alpha(q)\, h(x). \tag{9}$$

Furthermore, for each $x_0 \in \overline{D}$, we have

$$\lim_{r\to 0}(\sup_{x\in D}\frac{1}{h(x)}\int_{B(x_0,r)\cap D} G_\alpha^D(x,y)h(y)|q(y)|\mathrm{d}y) = 0 \tag{10}$$

(iii) *The function $x \to (\delta(x))^{\alpha-1} q(x)$ is in $L^1(D)$.*

The next Lemma is crucial in the proof of Theorems 1.2 and 1.3.

LEMMA 2.3. *Let q be a nonnegative function in $K_\alpha(D)$, then the family of functions*

$$\Lambda_q = \{\frac{1}{M_\alpha^D\varphi(x)}\int_D G_\alpha^D(x,y)M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y,\ |\rho| \leq q\}$$

is uniformly bounded and equicontinuous in $\overline{D}$. *Consequently* Λ_q *is relatively compact in* $C_0(D)$.

Proof. Taking $h \equiv M_\alpha^D \varphi$ in (9), we deduce that for ρ such that $|\rho| \leq q$ and $x \in D$, we have

(11)
$$\left|\int_D \frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y\right| \leq \int_D \frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y)q(y)\mathrm{d}y \leq a_\alpha(q) < \infty.$$

So the family Λ_q is uniformly bounded.

Next, we aim at proving that the family Λ_q is equicontinuous in $\overline{D}$.

Let $x_0 \in \overline{D}$ and $\varepsilon > 0$. By (10), there exists $r > 0$ such that

$$\sup_{z\in D} \frac{1}{M_\alpha^D\varphi(z)} \int_{B(x_0,2r)\cap D} G_\alpha^D(z,y) M_\alpha^D\varphi(y)q(y)\mathrm{d}y \leq \frac{\varepsilon}{2}.$$

If $x_0 \in D$ and $x, x' \in B(x_0, r) \cap D$, then for ρ such that $|\rho| \leq q$, we have

$$\begin{aligned}
&\left|\int_D \frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y - \int_D \frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')} M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y\right| \\
&\leq \int_D \left|\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} - \frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}\right| M_\alpha^D\varphi(y)q(y)\mathrm{d}y \\
&\leq 2\sup_{z\in D} \int_{B(x_0,2r)\cap D} \frac{1}{M_\alpha^D\varphi(z)} G_\alpha^D(z,y) M_\alpha^D\varphi(y)q(y)\mathrm{d}y \\
&\quad + \int_{(|x_0-y|\geq 2r)\cap D} \left|\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} - \frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}\right| M_\alpha^D\varphi(y)q(y)\mathrm{d}y \\
&\leq \varepsilon + \int_{(|x_0-y|\geq 2r)\cap D} \left|\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} - \frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}\right| M_\alpha^D\varphi(y)q(y)\mathrm{d}y.
\end{aligned}$$

On the other hand, using (7) and the fact that $M_\alpha^D\varphi(z) \approx (\delta(z))^{\alpha-2}$, for every $y \in B^c(x_0, 2r) \cap D$ and $x, x' \in B(x_0, r) \cap D$, we have

$$\left|\frac{1}{M_\alpha^D\varphi(x)} G_\alpha^D(x,y) - \frac{1}{M_\alpha^D\varphi(x')} G_\alpha^D(x',y)\right| M_\alpha^D\varphi(y) \leq C\,(\delta(y))^{\alpha-1}.$$

Now since $x \to \frac{1}{M_\alpha^D\varphi(x)} G_\alpha^D(x,y)$ is continuous outside the diagonal and $q \in K_\alpha(D)$, we deduce by the dominated convergence theorem and Proposition 2.2 (iii), that

$$\int_{(|x_0-y|\geq 2r)\cap D} \left|\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} - \frac{G_\alpha^D(x',y)}{M_\alpha^D\varphi(x')}\right| M_\alpha^D\varphi(y)q(y)\mathrm{d}y \to 0 \text{ as } |x-x'| \to 0.$$

If $x_0 \in \partial D$ and $x \in B(x_0, r) \cap D$, then we have

$$\left|\int_D \frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y)\rho(y)\mathrm{d}y\right| \leq \frac{\varepsilon}{2} + \int_{(|x_0-y|\geq 2r)\cap D} \frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y)q(y)\mathrm{d}y.$$

Now, since $\frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} \to 0$ as $|x-x_0| \to 0$, for $|x_0-y| \geq 2r$, then by the same argument as above, we get

$$\int_{(|x_0-y|\geq 2r)\cap D} \frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} M_\alpha^D\varphi(y)q(y)\mathrm{d}y \to 0 \text{ as } |x-x_0| \to 0.$$

So the family Λ_q is equicontinuous in $\overline{D}$.

Therefore by Ascoli's theorem, the family Λ_q becomes relatively compact in $C_0(D)$. □

3. PROOFS OF THEOREMS 1.2 AND 1.3

3.1. PROOF OF THEOREM 1.2

Put

$$\lambda_0 := \inf_{x\in D} \frac{M_\alpha^D\varphi(x)}{G_\alpha^D(f(.,M_\alpha^D\varphi))(x)}. \tag{12}$$

Using (H_2) and (9) we deduce that $\lambda_0 > 0$.

Let $\lambda \in [0,\lambda_0)$ and Λ be the nonempty closed bounded convex set given by

$$\Lambda = \{v \in C(\overline{D}) : \ (1-\frac{\lambda}{\lambda_0}) \leq v \leq 1\}.$$

We define the operator T on Λ by

$$Tv(x) = 1 - \frac{\lambda}{M_\alpha^D\varphi(x)} \int_D G_\alpha^D(x,y) f\left(y, v(y)M_\alpha^D\varphi(y)\right)\mathrm{d}y. \tag{13}$$

We claim that the family $T\Lambda$ is relatively compact in $C(\overline{D})$.

Indeed, using (H_1), (H_2) and Lemma 2.3 with $q(y) = \frac{1}{M_\alpha^D\varphi(y)} f(y, M_\alpha^D\varphi(y))$, we deduce that the family

$$\{\frac{1}{M_\alpha^D\varphi(x)} \int_D G_\alpha^D(x,y) f\left(y, v(y)M_\alpha^D\varphi(y)\right)\mathrm{d}y, \quad v \in \Lambda\}, \tag{14}$$

is relatively compact in $C_0(D)$. Hence, the set $T\Lambda$ is relatively compact in $C(\overline{D})$.

On the other hand, since f is a nonnegative function, it is clear from (13), (H_1) and (12) that $T\Lambda \subset \Lambda$. Next, we prove the continuity of the operator T in Λ in the supremum norm.

Let $(v_k)_k$ be a sequence in Λ which converges uniformly to a function v in Λ. Then we have

$$|Tv_k(x) - Tv(x)| \leq \lambda \int_D \frac{G_\alpha^D(x,y)}{M_\alpha^D\varphi(x)} \left| f\left(y, v(y)M_\alpha^D\varphi(y)\right) - f\left(y, v_k(y)M_\alpha^D\varphi(y)\right)\right| \mathrm{d}y.$$

From the monotonicity of f, we have

$$\left|f\left(y,v(y)M_\alpha^D\varphi(y)\right)-f\left(y,v_k(y)M_\alpha^D\varphi(y)\right)\right|\leq 2M_\alpha^D\varphi(y)q(y).$$

So we conclude by the continuity of f with respect to the second variable, Proposition 2.2 and again the dominated convergence theorem, that

$$\forall x\in\overline{D},\ Tv_k(x)\to Tv(x)\text{ as }k\to\infty.$$

Using the fact that $T\Lambda$ becomes relatively compact in $C(\overline{D})$, we obtain the uniform convergence, namely

$$\|Tv_k-Tv\|_\infty\to 0\ \text{ as }k\to\infty.$$

Thus, we have proved that T is a compact operator mapping from Λ to itself. Hence, by the Schauder's fixed point theorem, there exists $v\in\Lambda$ such that

$$v(x)=1-\frac{\lambda}{M_\alpha^D\varphi(x)}\int_D G_\alpha^D(x,y)f\left(y,v(y)M_\alpha^D\varphi(y)\right)\mathrm{d}y. \tag{15}$$

Let $u(x)=v(x)M_\alpha^D\varphi(x)$. Then u is a positive continuous function, satisfying for each $x\in D$

$$u(x)=M_\alpha^D\varphi(x)-\lambda\int_D G_\alpha^D(x,y)f\left(y,u(y)\right)\mathrm{d}y. \tag{16}$$

In addition, since for each $x\in D$, $f\left(y,u(y)\right)\leq C\left(\delta(y)\right)^{\alpha-2}q(y)$, we deduce by Proposition 2.2 (iii) that the map $y\to f\left(y,u(y)\right)\in L^1_{loc}(D)$ and by (16), that $G_\alpha^D f\left(.,u\right)\in L^1_{loc}(D)$. Hence, applying $\left(-\Delta_{|D}\right)^{\frac{\alpha}{2}}$ on both sides of (16), we conclude by ([13], p. 230) that u is the required solution. □

Example 3.1. Let $\sigma\geq 0$ and $r+(1-\sigma)(\alpha-2)<\alpha$. Let p be a positive Borel measurable functions such that

$$p(x)\leq C\left(\delta(x)\right)^{-r},\quad\text{for all }x\in D.$$

Let φ be positive continuous functions on ∂D. Therefore by Theorem 1.2, there exists a constant $\lambda_0>0$ such that for each $\lambda\in[0,\lambda_0)$, problem

$$\begin{cases}\left(-\Delta_{|D}\right)^{\frac{\alpha}{2}}u+\lambda p(x)u^\sigma=0 & \text{in } D,\ \text{(in the sense of distributions)}\\ u>0 & \text{in } D,\\ \lim\limits_{x\to z\in\partial D}\frac{u(x)}{M_\alpha^D 1(x)}=\varphi(z), & \end{cases}$$

has a positive continuous solution u such that

$$(1-\frac{\lambda}{\lambda_0})M_\alpha^D\varphi\leq u\leq M_\alpha^D\varphi\ \text{ in }D.$$

3.2. PROOF OF THEOREM 1.3

We recall that by (H_4), the function $p_0 := p\frac{g(h_0)}{h_0}$ belongs to $K_\alpha(D)$. Let $c := 1+a_\alpha(p_0)$. Observe that from Proposition 2.2 (i), we have $a_\alpha(p_0) < \infty$. Let φ be a positive continuous functions on ∂D such that $\varphi \geq c\phi$. It follows from the integral representation of $M_\alpha^D \varphi(x)$ (see [8] p. 265), that for each $x \in D$ we have

$$M_\alpha^D \varphi(x) \geq ch_0(x) = cM_\alpha^D \phi. \tag{17}$$

Let S be the nonempty closed convex set given by

$$S = \left\{ \omega \in C(\overline{D}) : \frac{h_0}{M_\alpha^D \varphi} \leq \omega \leq 1 \right\}.$$

We define the operator $\digamma$ on S by

$$\digamma(\omega) = 1 - \frac{1}{M_\alpha^D \varphi} G_\alpha^D \left(pg\left(\omega M_\alpha^D \varphi\right)\right). \tag{18}$$

We will prove that $\digamma$ has a fixed point. Since for $\omega \in S$, we have $\omega \geq \frac{h_0}{M_\alpha^D \varphi}$, then we deduce from hypotheses (H_3), (H_4) and (9) that

$$G_\alpha^D \left(pg\left(\omega M_\alpha^D \varphi\right)\right) \leq G_\alpha^D \left(pg\left(h_0\right)\right) = G_\alpha^D(p_0 h_0) \leq a_\alpha(p_0) h_0.$$

Using further (17), we deduce that

$$\digamma\omega \geq 1 - \frac{a_\alpha(p_0)h_0}{M_\alpha^D \varphi} \geq \frac{h_0}{M_\alpha^D \varphi}. \tag{19}$$

Next by similar argument as in the proof of Theorem 1.2, we prove that $\digamma$ is a compact operator mapping from S to itself. Hence, again by the Schauder's fixed point theorem, there exists $\omega \in S$ such that

$$\omega(x) = 1 - \frac{1}{M_\alpha^D \varphi(x)} \int_D G_\alpha^D(x,y) p(y) g\left(\omega(y) M_\alpha^D \varphi(y)\right) dy. \tag{20}$$

Let $u(x) = \omega(x) M_\alpha^D \varphi(x)$. Then u satisfies for each $x \in D$

$$u(x) = M_\alpha^D \varphi(x) - \int_D G_\alpha^D(x,y) p(y) g\left(u(y)\right) dy. \tag{21}$$

Finally, we verify that u is the required solution. □

Example 3.2. Let $\gamma > 0$ and ϕ be a positive continuous functions on ∂D. Put $h_0 = M_\alpha^D \phi$ and consider the problem

$$\begin{cases} \left(-\Delta_{|D}\right)^{\frac{\alpha}{2}} u + p(x) u^{-\gamma} = 0 & \text{in } D, \text{ (in the sense of distributions)} \\ u > 0 & \text{in } D, \\ \lim\limits_{x \to z \in \partial D} \frac{u(x)}{M_\alpha^D 1(x)} = \varphi(z), & \end{cases} \tag{22}$$

where φ is a positive continuous function on ∂D and p is a positive measurable function satisfying

$$p(x) \leq \frac{C}{(\delta(x))^r} \quad \text{with } r + (1+\gamma)(\alpha - 2) < \alpha,$$

where $C > 0$. Using the fact that $h_0(x) \approx (\delta(x))^{\alpha-2}$ and (5), we can verify that hypothesis (H_4) is satisfied. Then by Theorem 1.3, there exists a constant $c > 1$ such that if $\varphi \geq c\phi$ on ∂D, then problem (22) has a positive continuous solution u satisfying for each $x \in D$

$$h_0(x) \leq u(x) \leq M_\alpha^D \varphi(x).$$

Acknowledgments. The author wants to thank the referees for a careful reading of the paper. This project is supported by NSTIP strategic technologies program number (13-MAT1813-02) in the Kingdom Saudi Arabia.

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Received 7 February 2013

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