ON THE CHARACTERIZATION OF SIMPLE K_4-GROUP $L_2(3^m)$ IN THE CLASS OF FINITE CENTERLESS GROUPS

S. HEYDARI, N. AHANJIDEH and B. ASADIAN

Communicated by Ştefan Papadima

In this paper, we show that the simple K_4-group $L_2(3^m)$ is characterizable by the number of its Sylow subgroups in the class of finite centerless groups.

AMS 2010 Subject Classification: 20D06, 20D15.

Key words: Sylow subgroups of a finite group, simple K_n-groups.

1. INTRODUCTION

A finite group G is said to be characterizable by the number of its Sylow subgroups, if G is uniquely (up to isomorphism) determined by the number of its Sylow subgroups. In 1992, Bi showed that $L_2(q)$ uniquely (up to isomorphism) determined by the orders of its Sylow normalizers (see [1]). In [5], the authors concentrated on the number of Sylow subgroups of finite groups in the class of the finite centerless groups and they showed that some finite simple groups can be determined uniquely (up to isomorphism) by number of their Sylow subgroups in the class of the finite centerless groups.

For a finite group G, let $\pi(G)$ be the set of prime divisors of $|G|$. For the natural number n, a finite group S is named a simple K_n-group, when S is a simple group with $|\pi(S)| = n$. In this paper, we prove that:

THEOREM 1.1. *The finite simple K_4-group $L_2(3^m)$ is characterizable by the number of its Sylow subgroups in the class of finite centerless groups.*

2. MAIN RESULTS

Throughout this paper, we use the following notation: Let a and n be integers greater than 1. A primitive prime divisor of $a^n - 1$ is a prime l such that $l \mid (a^n - 1)$ but $l \nmid (a^i - 1)$ for $1 \leq i < n$. If q is odd, we put $\exp_2(q) = 1$ if $q \equiv 1 \pmod 4$, and $\exp_2(q) = 2$ otherwise. Put $Z_n(a) = \{l : l \text{ is a primitive prime divisor of } a^n - 1\}$. For a finite group G and the prime number p, let $n_p(G)$ denote the number of p-Sylow subgroups of G. Also, the set of all p-Sylow subgroups of G is denoted by $\mathrm{Syl}_p(G)$, and the p-part of G, denoted by $|G|_p$, is the order of any $P \in \mathrm{Syl}_p(G)$. In the following lemmas, we will quote some useful facts which will be used during the proof of the main theorem.

LEMMA 2.1 ([4]). *Let $a, n > 1$ be natural numbers. Then $Z_n(a) \neq \{\}$, unless $a = 2,\ n = 6$ or $n = 2$ and $a = 2^w - 1$ for some natural number w.*

The following lemma is a known fact that we will use it without reference during the proof of the main theorem. The number of Sylow subgroups of other finite simple groups can be obtained using [6]:

LEMMA 2.2. *Let $r \in \pi(L_2(q))$. Then:*

(i) *if $r \mid q$, then $n_r(L_2(q)) = (q+1)$;*

(ii) *if $r \in Z_2(q) - \{2\}$, then $n_r(L_2(q)) = q(q-1)/2$;*

(iii) *if $r \in Z_1(q) - \{2\}$, then $n_r(L_2(q)) = q(q+1)/2$.*

LEMMA 2.3. (i) [8] *If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $PSU_3(3)$ or $PSU_4(2)$;*

(ii) [9] *let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

(1) A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;

(2) $L_2(r)$, where r is a prime and satisfies $r^2-1 = 2^a.3^b.v^c$ with $a, b, c \geq 1$ and a prime $v > 3$;

(3) $L_2(2^m)$, where $m \geq 5$ satisfies $2^m - 1 = u$ and $2^m + 1 = 3t^b$, where u and t are primes, $t > 3$ and $b \geq 1$;

(4) $L_2(3^m)$, where $m \geq 2$ satisfies $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, where u and t are odd primes, and $b, c \geq 1$.

LEMMA 2.4 ([10]). *Let G be a finite group and M be a normal subgroup of G. Then for every prime p, $n_p(M)n_p(G/M)$ divides $n_p(G)$.*

LEMMA 2.5 ([7]). *Let G be a finite solvable group and $|G| = m.n$, where $m = p_1^{\alpha_1}...p_r^{\alpha_r}$ and $(m, n) = 1$. Let $\pi = \{p_1, ..., p_r\}$ and h_m be the number of π-Hall subgroups of G. Then $h_m = q_1^{\beta_1}...q_s^{\beta_s}$ and for all $i \in \{1, .., s\}$: $q_i^{\beta_i} \equiv 1 \pmod{p_j}$, for some p_j.*

Proof of Theorem 1.1. Let G be a finite centerless group such that for every prime number p, $n_p(G) = n_p(L_2(3^m))$. Since $L_2(3^m)$ is a simple K_4-group, $m \geq 3$ and m satisfies either $3^m-1 = 2u^c$ and $3^m+1 = 4t$ or $3^m-1 = 2u$ and $3^m + 1 = 4t^b$, where u and t are odd primes and $b, c \geq 1$. We show that $\pi(G) = \pi(L_2(3^m))$. If not, then there exists $p \in \pi(G) \setminus \pi(L_2(3^m))$, so $n_p(G) = n_p(L_2(3^m)) = 1$ and for every $q \in \pi(G)$, $p \nmid n_q(G)$. This shows that $p \in \pi(Z(G))$, which is a contradiction. Thus, $\pi(G) = \pi(L_2(3^m))$. Let $1 = N_0 \trianglelefteq N_1 \trianglelefteq \dots \trianglelefteq N_{n-1} \trianglelefteq N_n = G$ be a chief series of G. We claim that G is not solvable. If not, then since $n_t(G) = n_t(L_2(3^m)) = 3^m.u^c$, where $u^c = (3^m - 1)/2$, we deduce that by Lemma 2.5, $t \mid 3^m - 1$ or $t \mid u^c - 1$, which shows that $t \mid u^c - 1 = 3(3^{m-1} - 1)/2$. This forces t to divide $3^{m-1} - 1$, which is impossible. This shows that G is not solvable and hence, there exists $1 \leq i \leq n$ such that N_i/N_{i-1} is not solvable. Put $N := N_{i-1}$ and $H := N_i$. Since H/N is a normal minimal subgroup of G/N, $H/N \cong P_1 \times \dots \times P_w$, where for every $1 \leq i \leq w$, P_i is a simple group and $P_i \cong P_j$, for every $1 \leq i, j \leq w$. Fix j such that $1 \leq j \leq w$. As stated before, $|\pi(G)| = 4$ and it is evident that P_j is neither a K_1-group nor a K_2-group. Therefore, P_j is a simple K_3-group or a simple K_4-group. Let P_j be a simple K_3-group. Then P_j is isomorphic to one of the groups mentioned in Lemma 2.3(i) and also, since $\pi(P_j) \subset \pi(G) = \{2, 3, t, u\}$ and $|\pi(P_j)| = 3$, we deduce that either $t \in \pi(P_j)$ or $u \in \pi(P_j)$. Thus, comparing the maximal prime divisor of the orders of the groups mentioned in Lemma 2.3(i) with t and u leads us to conclude that t or $u \in \{5, 7, 13, 17\}$. In the following, we consider these cases:

I. Let $t \in \pi(H/N)$. Then we have the following possibilities for t:
a. $t = 5$. Then $5|3^2 + 1$ and $5|3^m + 1$. So, $5|\gcd(3^2 + 1, 3^m + 1)$, which implies that $2|m$. Thus, $m = 2$, because m is prime. Obviously, this is a contradiction with our assumption.

b. $t = 7$. Then since $7|3^3 + 1$ and $7|3^m + 1$, by the same argument as that of in (a), we can see that $m = 3$. On the other hand, among the groups mentioned in Lemma 2.3(i), the only groups which their orders are divisible by 7 are $L_2(7)$ and $PSU_3(3)$ and hence, either $P_j \cong L_2(7)$ or $P_j \cong PSU_3(3)$. It follows from Lemma 2.4 and [3] that either $8 = n_7(L_2(7)) = n_7(P_j)$ divides $n_7(G) = n_7(L_2(27)) = 3^3.13$ or $2^5.3^2 = n_7(P_j)$ divides $n_7(G) = n_7(L_2(27)) = 3^3.13$, which is impossible.

c. $t = 13$. Then since $13|3^3 - 1$ and $13|3^m + 1$, $13|\gcd(3^3 - 1, 3^m + 1) = 2$, which is impossible. The same reasoning rules out the case when $P_j \cong L_2(8)$.
d. $t = 17$. Then since $17|3^{16} - 1$ and $17|3^m + 1$, we obtain $8|m$, which is a contradiction with the fact that m is prime.

II. Let $u \in \pi(H/N)$. Then we have the following possibilities for u:

a. $u = 5$. Then since $5|3^m - 1$ and $5 \in Z_4(3)$, we obtain $4|m$, which is a contradiction.

b. $u = 7$. Then since $7|3^m - 1$ and $7 \in Z_6(3)$, we obtain $6|m$, which is a contradiction.

c. $u = 13$. Then since $13|3^m - 1$ and $13 \in Z_3(3)$, we get $m = 3$. Now, since $13 \in \pi(P_j)$ and P_j is a simple K_3-group, we deduce that by Lemma 2.3(i), $P_j \cong L_3(3)$. Thus, by Lemma 2.4, $2^4.3^2 = n_{13}(P_j)$ divides $n_{13}(L_2(27)) = 2.3^3.7$, which is impossible.

d. $u = 17$. Then since $17|3^m - 1$ and $17 \in Z_{16}(3)$, we deduce that $16|m$, which is a contradiction.

It follows that P_j is not a simple K_3-group and hence, P_j is a simple K_4-group. Recall that every simple K_4-group is isomorphic to one of the groups mentioned in Lemma 2.3(ii). Thus, we have the following cases:

Case 1. P_j is isomorphic to one of the groups mentioned in Lemma 2.3(ii)(1). Then comparing the prime divisors of the order of the groups mentioned in Lemma 2.3(ii)(1) and the prime divisors of the order of $L_2(3^m)$ leads us to see that $P_j \in \{G_2(3), {}^3D_4(2)\}$ (up to isomorphism). Then:

• if $P_j \cong G_2(3)$ and t is the maximal prime divisor of $|L_2(3^m)|$, then, since $|P_j| = 2^6.3^6.7.13$ and $\pi(P_j) = \pi(L_2(3^m))$, we have $t = 13$. We claim that $3^m + 1 = 4t$. If not, then Lemma 2.3(ii) forces $3^m + 1 = 4t^b$ and $3^m - 1 = 2u$, where $b \geq 2$ and hence, $u > t$, which is a contradiction with our assumption. Thus, $3^m + 1 = 4t$ and hence, $3^m + 1 = 52$, which is impossible;

• if $P_j \cong G_2(3)$ and u is the maximal prime divisor of $|L_2(3^m)|$, then the same reasoning as that of in the above forces $u = 13$, $c = 1$ and $m = 3$. Let $x \in G_2(3)$ such that $O(x) = 13$. Then $\langle x\rangle \in \mathrm{Syl}_{13}(G_2(3))$. Also, [3] shows that $|C_{G_2(3)}(x)| = 13$ and N-C theorem shows that $N_{G_2(3)}(\langle x\rangle)/C_{G_2(3)}(\langle x\rangle) \lesssim \mathrm{Aut}(\langle x\rangle) \cong \mathbb{Z}_{12}$. Therefore, $|N_{G_2(3)}(\langle x\rangle)|$ divides 12.13. This forces $2^4.3^5.7$ to divide $n_{13}(P_j) = n_{13}(G_2(13))$. But by Lemma 2.4, $n_{13}(P_j)$ divides $n_{13}(G) = n_{13}(L_2(27)) = 2.3^3.7$. This shows that $2^4.3^5.7$ divides $2.3^3.7$, which is impossible.

The same reasoning rules out the case where $P_j \cong {}^3D_4(2)$.

Case 2. $P_j \cong L_2(r)$, where r is a prime and satisfies $r^2 - 1 = 2^a.3^b.v^c$ with $a, b, c \geq 1$ and a prime $v > 3$. Then since r is the maximal prime divisor of $|L_2(r)|$ and $\pi(H/N) = \pi(G)$, we deduce that either $t = r$ or $u = r$. Thus, we have the following subcases:

Subcase i. $t = r$. Then since $r = t$ is the maximal prime divisor of $|L_2(3^m)|$, we deduce that $t = (3^m + 1)/4$ and $v = u$. Thus, by Lemma 2.4, $t(t - 1)/2 = r(r - 1)/2 = n_u(L_2(r))$ divides $n_u(L_2(3^m)) = 3^m(3^m + 1)/2$, so $(3^m - 3)t/8$ divides $2.t.3^m$. This implies that $m = 3$ and hence, $r = t = 7$. Thus, $\pi(P_j) = \pi(L_2(7)) = \{2, 3, 7\}$, a contradiction;

Subcase ii. $u = r$. Then since $r = u$ is the maximal prime divisor of

$|L_2(3^m)|$, we deduce that $u = (3^m - 1)/2$ and $v = t$. Thus, by Lemma 2.4, $u(u-1)/2 = r(r-1)/2 = n_t(L_2(r))$ divides $n_t(L_2(3^m)) = 3^m(3^m-1)/2$, so $u(3^m - 3)/4$ divides $u.3^m$. This implies that $m = 2$, which is a contradiction;

Case 3. $P_j \cong L_2(2^{m'})$, where $m' \geq 5$ satisfies $2^{m'} - 1 = u'$ and $2^{m'} + 1 = 3t'^{b'}$, where u' and t' are primes, $t' > 3$ and $b' \geq 1$. Then u' is the maximal prime divisor of $|P_j|$ and hence, as stated before, we have the following subcases:

Subcase i. $t = u'$ and hence, $t = (3^m + 1)/4$ and $t' = u$. Thus, by Lemma 2.4, $2^{m'}(2^{m'}+1)/2 = n_{u'}(L_2(2^{m'}))$ divides $n_t(L_2(3^m)) = 3^m(3^m-1)/2$, so $3.2^{m'}.t'^{b'}$ divides $3^m.u^c$, which is impossible;

Subcase ii. $u = u'$ and hence, $u = (3^m - 1)/2$ and $t' = t$. Thus, by Lemma 2.4, $2^{m'}(2^{m'}+1)/2 = n_{u'}(L_2(2^{m'}))$ divides $n_u(L_2(3^m)) = 3^m(3^m+1)/2$, so $3.2^{m'}.t'^{b'}$ divides $2.3^m.t^b$. It follows that $m' \leq 1$, a contradiction;

Case 4. $P_j \cong L_2(3^{m'})$, where $m' \geq 3$ satisfies $3^{m'}+1 = 4t'$, $3^{m'}-1 = 2u'^{c'}$ or $3^{m'} + 1 = 4t'^{b'}$, $3^{m'} - 1 = 2u'$, where u' and t' are odd primes, and $b', c' \geq 1$. Then comparing the maximal prime divisor of $|L_2(3^m)|$ with the maximal prime divisor of $|L_2(3^{m'})|$ leads us to see that $m = m'$ and hence, $P_j \cong L_2(3^m)$. It follows from Lemma 2.4 that $n_t(H/N) = (3^m(3^m - 1)/2)^w$ divides $n_t(G) = 3^m(3^m - 1)/2$, so $w = 1$ and consequently, $H/N \cong L_2(3^m)$. Set $\overline{G} := G/N$ and $\overline{H} := H/N$, which is isomorphic to $L_2(3^m)$. Since $Z(\bar{H}) = 1$, we have

$$L_2(3^m) \cong \overline{H} \cong \frac{\overline{H}C_{\overline{G}}(\overline{H})}{C_{\overline{G}}(\overline{H})} \leq \frac{\overline{G}}{C_{\overline{G}}(\overline{H})} = \frac{N_{\overline{G}}(\overline{H})}{C_{\overline{G}}(\overline{H})} \lesssim \mathrm{Aut}(\overline{H}).$$

Let $C_{\bar{G}}(\bar{H}) = K/N$. Then $K \trianglelefteq G$ and $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Since by Lemma 2.4, for every $p \in \pi(G)$, $n_p(L_2(3^m))$ divides $n_p(G/K)$ and $n_p(G/K)$ divides $n_p(G) = n_p(L_2(3^m))$, we deduce that $n_p(G/K) = n_p(L_2(3^m)) = n_p(G)$ and hence, Lemma 2.4 leads us to see that $n_p(K) = 1$, for every $p \in \pi(G)$. Thus, K is nilpotent. We claim that $K = 1$. Let Q be a q-Sylow subgroup of K. Then since K is nilpotent, Q is normal in G. For $p \in \pi(G)$ and $P \in \mathrm{Syl}_p(G)$, set $\widetilde{P} = PK/K$ and $\widetilde{G} = G/K$. It is easy to check that $\widetilde{P} \in \mathrm{Syl}_p(\widetilde{G})$ and $N_{\widetilde{G}}(\widetilde{P}) = N_G(P)K/K$. But as mentioned before, $n_p(G/K) = n_p(G)$ and hence, $|N_G(P)K| = |N_G(P)|$. This shows that $K \leq N_G(P)$. Thus, Q normalizes P and so, if $p \neq q$, then P and Q centralizes each other. Let $C = C_G(Q)$. Then for every $p \in \pi(G) \setminus \{q\}$, C contains the p-Sylow subgroups of G and hence, $|G : C|$ is a power of q. Now let S be a q-Sylow subgroup of G. Then $G = CS$. Also if $Q \neq 1$, then "$Q \trianglelefteq G$" guarantees that $C_Q(S) \neq 1$. But $C_Q(S) \leq Z(G) = 1$. This forces Q to be trivial. Since q is an arbitrary element of $\pi(G)$, we have $K = 1$. Thus, $L_2(3^m) \trianglelefteq G \cong \bar{G}/C_{\bar{G}}(\bar{H}) \lesssim \mathrm{Aut}(L_2(3^m)) \cong PGL_2(3^m).\mathbb{Z}_m$. Thus, we conclude that either $G \cong PGL_2(3^m).\mathbb{Z}_n$ or $G \cong L_2(3^m).\mathbb{Z}_n$, where $n \mid m$. If $G \cong PGL_2(3^m).\mathbb{Z}_n$, then by Lemma 2.4, $n_2(PGL_2(3^m))$ divides $n_2(G) =$

$n_2(L_2(3^m))$ and [2, Lemma 3] shows that $|PGL_2(3^m)|/|PGL_2(3^m)|_2$ divides $|L_2(3^m)|/(3.|L_2(3^m)|_2) = |PGL_2(3^m)|/(3.|PGL_2(3^m)|_2)$, which is impossible. Thus, $G \cong L_2(3^m).\mathbb{Z}_n$. Then since $n \mid m$ and m is a prime number, we get that either $n = 1$ or $n = m$. We claim that $n = 1$. If not, $n = m \in \pi(G) = \pi(L_2(3^m)) = \{2, 3, u, t\}$. But $t \in Z_{2m}(3)$ and $u \in Z_m(3)$, so Fermat's little theorem shows that $m \mid u - 1$ and $2m \mid t - 1$ and hence, $t, u \nmid m$, consequently, $m \neq u$ and $m \neq t$. Thus, $n = m \in \{2, 3\}$. If $n = m = 2$, then $\pi(G) = \pi(L_2(9)) = \{2, 3, 5\}$, which is a contradiction. This forces $n = m = 3$. Thus, $364 = n_3(L_2(27).\mathbb{Z}_3) = n_3(G) = n_3(L_2(27)) = 28$, which is impossible. This shows that $n = 1$. So $G \cong L_2(3^m)$, which implies that the simple K_4-group $L_2(^m)$ is uniquely (up to isomorphism) determined by the number of its Sylow subgroups, as desired. □

REFERENCES

[1] J. Bi, *A characterization of $L_2(q)$.* J. Liaoning Univ. Nat. Sci. **19** (1992), 1–4.

[2] R. Carter and P. Fong, *The Sylow 2-subgroups of the finite classical groups.* J. Algebra **1** (1964), 139–151.

[3] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups.* Clarendon, Oxford University Press, 1985.

[4] W. Feit, *On large Zsigmondy primes.* Proc. Amer. Math. Soc. **102** (1988), 29–33.

[5] A. Khalili Asboei, S.S. Salehi Amiri, A. Iranmanesh and A. Tehranian, *A characterization of some linear groups by the number of Sylow subgroups.* Australian J. Basic Appl. Sci. **5** (2011), 831–834.

[6] P. Kleidman and M. Liebeck, *The subgroup structure of finite classical groups.* London Mathematical Society Lecture Note Series **129**. Cambridge University Press, 1990.

[7] M. Hall, *The Theory of Groups.* Macmillan, New York, 1959.

[8] M. Herzog, *On finite simple groups of order divisible by three primes only.* J. Algebra **120** (1968), 383–388.

[9] W. Shi, *On simple K_4-groups.* Chinese Science Bull. **36** (1991), 1281–1283.

[10] J. Zhang, *Sylow numbers of finite groups.* J. Algebra **176** (1995), 111–123.

Received 14 March 2013

Shahre-kord University,
Faculty of Mathematical Sciences,
Department of pure Mathematics,
P.O. Box 115,
Shahre-kord, Iran
heydarisomaye@yahoo.com

ahanjideh.neda@sci.sku.ac.ir

asadian.bahare@gmail.com