ON NICELY AND SEPARATELY ω_1-$p^{\omega+n}$-PROJECTIVE ABELIAN p-GROUPS

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Communicated by Vasile Brînzănescu

We define the class of *nicely* ω_1-$p^{\omega+n}$-*projective* abelian p-groups as well as some other closely related concepts. A few recurring relationships between them, and the class of *strongly* ω_1-$p^{\omega+n}$-*projective* abelian p-groups with some of its modifications, are also established. These results continue our recent publication in Hacettepe J. Math. Stat. (2014) and also strengthen statements due to Keef in J. Algebra Numb. Theory Acad. (2010).

AMS 2010 Subject Classification: 20K10.

Key words: countable groups, direct sums of cyclic groups, $p^{\omega+n}$-projective groups, strongly ω_1-$p^{\omega+n}$-projective groups, separably ω_1-$p^{\omega+n}$-projective groups, separately ω_1-$p^{\omega+n}$-projective groups, nicely ω_1-$p^{\omega+n}$-projective groups.

1. INTRODUCTION AND TERMINOLOGY

Let all groups into consideration be p-torsion abelian groups, where p is a prime fixed for the duration of the paper, and let $n \geq 0$ be an arbitrary integer. Our notions and notations are standard and follow essentially those from [8]. These that are not stated there can be found in [9] and [2], respectively. For instance, imitating [13], a group G is $p^{\omega+n}$-*projective* if there exists a p^n-bounded subgroup A such that G/A is Σ-cyclic (*i.e.*, a direct sum of cyclic groups).

In his seminal work [9], Keef successfully generalized the classical concept of $p^{\omega+n}$-projective groups by defining the class of so-called ω_1-$p^{\omega+n}$-*projective groups*. Two of their important characterizations are the following:

THEOREM 1.1. *The group G is ω_1-$p^{\omega+n}$-projective if exactly one of the next conditions hold:*

(i) *There is a countable nice subgroup C such that ($p^{\omega+n}G \subseteq C \subseteq p^{\omega}G$ and) G/C is $p^{\omega+n}$-projective.*

(ii) *There is a p^n-bounded subgroup B with G/B as the direct sum of a countable group and a Σ-cyclic group.*

It is appropriate to point out that the subgroup B in clause (ii) need not be nice in G, so that the additional restriction on niceness of B could be interesting and worthy of investigation. This was done in [2] by calling such groups *strongly* ω_1-$p^{\omega+n}$-*projective.* In the spirit of [10, 11] and [12] certain comprehensive classifications of these groups were additionally established in [2], and in some more generalized form in [4] and [3], too.

The purpose here is to define a new subclass of groups of the class of ω_1-$p^{\omega+n}$-projectives, suggested in [2] and termed *nicely* ω_1-$p^{\omega+n}$-*projective groups*, which properly contain strongly ω_1-$p^{\omega+n}$-projectives. The paper is structured as follows. In the next section, we foremost add two crucial technical claims which are necessary for applicable purposes in the sequel. After that, we explore nicely ω_1-$p^{\omega+n}$-projective groups and a number of their several different characterizations are obtained. Later on, in a separate section, we study so-named *co-nicely* ω_1-$p^{\omega+n}$-*projective groups.* Next, we investigate the aforementioned strongly ω_1-$p^{\omega+n}$-projective groups as well as some relations with the above mentioned nicely ω_1-$p^{\omega+n}$-projective groups are given. Finally, we close the article with some explicitly stated left-open questions and problems.

2. TWO PRELIMINARY TECHNICALITIES

The following two lemmas possess a central position.

Lemma 2.1. *Suppose that α is an ordinal, and that G and F are groups where F is finite. Then the following formula is fulfilled:*

$$p^\alpha(G+F) = p^\alpha G + F \cap p^\alpha(G+F).$$

Proof. We will use a transfinite induction on α. First, if $\alpha - 1$ exists, we have

$$p^\alpha(G+F) = p(p^{\alpha-1}(G+F)) = p(p^{\alpha-1}G + F \cap p^{\alpha-1}(G+F)) =$$
$$p(p^{\alpha-1}G)+p(F\cap p^{\alpha-1}(G+F)) \subseteq p^\alpha G+F\cap p(p^{\alpha-1}(G+F)) = p^\alpha G+F\cap p^\alpha(G+F).$$

Since the reverse inclusion "$\supseteq$" is obvious, we obtain the desired equality.

If now $\alpha - 1$ does not exist, we have that $p^\alpha(G+F) = \cap_{\beta<\alpha}(p^\beta(G+F)) \subseteq \cap_{\beta<\alpha}(p^\beta G+F) = \cap_{\beta<\alpha}p^\beta G+F = p^\alpha G+F$. In fact, the second sign "=" follows like this: Given $x \in \cap_{\beta<\alpha}(p^\beta G+F)$, we write that $x = g_{\beta_1}+f_1 = \cdots = g_{\beta_s}+f_s = \cdots$ where $f_1, \cdots, f_s \in F$ are all the elements of F; $g_{\beta_1} \in p^{\beta_1}G, \cdots, g_{\beta_s} \in p^{\beta_s}G$ with $\beta_1 < \cdots < \beta_s < \cdots$.

Since F is finite, while the number of equalities is infinite due to the infinite cardinality of α, we infer that $g_{\beta_s} \in p^\beta G$ for any ordinal $\beta < \alpha$ which means that $g_{\beta_s} \in \cap_{\beta<\alpha}p^\beta G = p^\alpha G$. Thus $x \in \cap_{\beta<\alpha}p^\beta G + F = p^\alpha G + F$, as claimed. Furthermore, $p^\alpha(G+F) \subseteq (p^\alpha G+F)\cap p^\alpha(G+F) = p^\alpha G + F \cap p^\alpha(G+F)$ which is obviously equivalent to an equality. $\square$

LEMMA 2.2. *Let N be a nice subgroup of a group G. Then*
(i) *$N+R$ is nice in G for every finite subgroup $R \leq G$;*
(ii) *N is nice in $G+F$ for each finite group F.*

Proof. (i) For any limit ordinal γ, we deduce that $\cap_{\delta<\gamma}(N+R+p^\delta G) \subseteq R+\cap_{\delta<\gamma}(N+p^\delta G) = R+N+p^\gamma G$, as required. Indeed, the relation "$\subseteq$" follows like this: Given $x \in \cap_{\delta<\gamma}(N+R+p^\delta G)$, we write $x = a_1+r_1+g_1 = \cdots = a_s+r_s+g_s = \cdots = a_k+r_1+g_k = \cdots$, where $a_1, \cdots, a_k \in N$; $r_1, \cdots, r_k \in R$; $g_1 \in p^{\delta_1}G, \cdots, g_k \in p^{\delta_k}G$ with $\delta_1 < \cdots < \delta_k$. So $a_1+g_1 = \cdots = a_k+g_k = \cdots \in \cap_{\delta<\gamma}(N+p^\delta G)$ and hence $x \in R+\cap_{\delta<\gamma}(N+p^\delta G)$, as requested.

(ii) Since N is nice in G, we may write $\cap_{\delta<\gamma}[N+p^\delta G] = N+p^\gamma G$ for every limit ordinal γ. Furthermore, with Lemma 2.1 at hand, we subsequently deduce that

$$\cap_{\delta<\gamma}[N+p^\delta(G+F)] = \cap_{\delta<\gamma}[N+p^\delta G+F\cap p^\delta(G+F)] \subseteq$$

$$\cap_{\delta<\gamma}(N+p^\delta G)+F\cap p^\gamma(G+F) = N+p^\gamma G+F\cap p^\gamma(G+F) = N+p^\gamma(G+F).$$

In fact, the inclusion "$\subseteq$" follows thus: Given $x \in \cap_{\delta<\gamma}[N+p^\delta G+F\cap p^\delta(G+F)]$, we write $x = a_1+g_1+f_1 = \cdots = a_s+g_s+f_s = \cdots = a_k+g_k+f_1 = \cdots$, where $a_1, \cdots, a_k \in N$; $g_1 \in p^{\delta_1}G, \cdots, g_k \in p^{\delta_k}G$; $f_1 \in F\cap p^{\delta_1}(G+F), \cdots, f_k \in F\cap p^{\delta_k}(G+F)$ with $\delta_1 < \cdots < \delta_k$. Hence $a_1+g_1 = \cdots = a_k+g_k = \cdots \in \cap_{\delta<\gamma}(N+p^\delta G)$ and because the number of the f_i's $(1 \leq i \leq k)$ is finite whereas the number of equalities is not, we can deduce that $f_1 \in \cap_{\delta<\gamma}(F\cap p^\delta(G+F)) = F\cap p^\gamma(G+F)$, as needed. $\square$

3. NICELY ω_1-$p^{\omega+n}$-PROJECTIVE GROUPS

We begin here with

Definition 3.1. The group G is said to be *nicely ω_1-$p^{\omega+n}$-projective* if there exists a nice $p^{\omega+n}$-projective subgroup N of G such that G/N is countable (that is, $G = N+K$ for some countable $K \leq G$).

Notice that if the quotient G/N is finite, *i.e.*, so is K, then it is obvious that G must be necessarily $p^{\omega+n}$-projective.

Note that nicely ω_1-$p^{\omega+n}$-projective groups are themselves ω_1-$p^{\omega+n}$-projective; actually, Definition 3.1 without the limitation on niceness of N in G is precisely one of the characterizations of ω_1-$p^{\omega+n}$-projectives (see, *e.g.*, [9]). Likewise, it is easily checked that the direct sum of a countable group and a $p^{\omega+n}$-projective group is necessarily nicely ω_1-$p^{\omega+n}$-projective.

Under some length requirements, the above definition can be weakened. Specifically, the following is true:

Proposition 3.1. *Suppose G is a group such that $p^{\omega+n}G = \{0\}$. Then G is nicely ω_1-$p^{\omega+n}$-projective if and only if there exists a $p^{\omega+n}$-projective subgroup P of G such that G/P is separable countable (i.e., a countable Σ-cyclic group).*

Proof. The sufficiency being self-evident, we concentrate on the necessity. To that aim, write G/N is countable for some $p^{\omega+n}$-projective nice subgroup $N \subseteq G$. Since $p^{\omega+n}(G/N) = (p^{\omega+n}G + N)/N = \{0\}$, it follows in view of [8] that G/N is countable $p^{\omega+n}$-projective. Hence, there is a subgroup $P/N \leq G/N$ with $p^nP \subseteq N$ and $(G/N)/(P/N) \cong G/P$ is Σ-cyclic. Also, G/P is countable being an epimorphic image of the countable G/N. One seeing readily that p^nP is $p^{\omega+n}$-projective, whence so does P, as stated. $\square$

Remark 1. One may observe that inseparable $p^{\omega+n}$-bounded nicely ω_1-$p^{\omega+n}$-projectives need not be $p^{\omega+n}$-projective.

A helpful consequence is the following one:

Corollary 3.2. *Any subgroup of a $p^{\omega+n}$-bounded nicely ω_1-$p^{\omega+n}$-projective group is again nicely ω_1-$p^{\omega+n}$-projective.*

Proof. Appealing to Proposition 3.1 one may write that G/P is separable countable for some $p^{\omega+n}$-projective subgroup P of a group G whenever the latter is nicely ω_1-$p^{\omega+n}$-projective. Letting $A \leq G$ be its arbitrary subgroup, it is plainly seen that $A/(A \cap P) \cong (A + P)/P \subseteq G/P$ remains countable separable, and $A \cap P \subseteq P$ remains $p^{\omega+n}$-projective, as required. $\square$

Another reformulation of Definition 3.1 listed above is the following:

Theorem 3.3. *The group G is nicely ω_1-$p^{\omega+n}$-projective if and only if there exists a p^n-bounded subgroup X of G such that G/X is equal to the sum $K/X + N/X$ of a countable group K/X and a Σ-cyclic group N/X with N nice in G.*

Proof. "$\Rightarrow$". Write G/N is countable for some $p^{\omega+n}$-projective nice subgroup $N \subseteq G$. There is a p^n-bounded subgroup $X \leq N$ such that N/X is Σ-cyclic. Therefore, $G/N \cong (G/X)/(N/X)$ being countable implies that $G/X = N/X + K/X$ for some countable K/X with $K \leq G$.

"$\Leftarrow$". It is plainly seen that $(G/X)/(N/X) \cong G/N$ is countable, and N is $p^{\omega+n}$-projective and nice in G, as required. $\square$

Remark 2. First of all, notice that using the same idea as that in Corollary 4.1 from [5], if for some group A the quotient A/B is countable and B is a Σ-cyclic subgroup, then A is the direct sum of a countable group and a Σ-cyclic group. Thus, we may write $G/X = (C/X) \oplus (L/X)$ where C/X is countable and L/X is Σ-cyclic (whence L is $p^{\omega+n}$-projective). Moreover, if X is nice in G, then L is nice in G because L/X is nice in G/X being its direct summand (cf. [8]). The converse perhaps does not hold, that is, if L is nice in X, then X is probably not necessarily nice in G too.

However, it is not obvious whether or not we can take $(N/X)\cap(K/X)=\{0\}$.

Concerning now certain important properties of these groups, nicely ω_1-$p^{\omega+n}$-projectives are closed under taking Ulm subgroups and countable direct sums; for Ulm factors the problem is more complicated and will be treated below.

Proposition 3.4. *If G is a nicely ω_1-$p^{\omega+n}$-projective group, then so is $p^\alpha G$ for any ordinal α.*

Proof. Write G/P is countable for some nice $p^{\omega+n}$-projective subgroup P of G. Clearly, $P\cap p^\alpha G$ is nice in $p^\alpha G$ (see [8]), and is also $p^{\omega+n}$-projective being a subgroup of P. Moreover, $G/P\supseteq(p^\alpha G+P)/P\cong p^\alpha G/(p^\alpha G\cap P)$ is countable, as required, and this gives the claim. □

Proposition 3.5. *The countable direct sum of nicely ω_1-$p^{\omega+n}$-projective groups is again a nicely ω_1-$p^{\omega+n}$-projective group.*

Proof. Suppose $i\in I$ is an arbitrary index in some countable set I such that G_i is a nicely ω_1-$p^{\omega+n}$-projective group. We claim that the direct sum $\oplus_{i\in I}G_i$ is also a nicely ω_1-$p^{\omega+n}$-projective group. In fact, there exist corresponding nice subgroups P_i of G_i which are also $p^{\omega+n}$-projective and for which G_i/P_i are countable. It is well known that $P=\oplus_{i\in I}P_i$ is nice in $\oplus_{i\in I}G_i=G$ and $G/P\cong\oplus_{i\in I}(G_i/P_i)$ is countable because $|I|\leq\aleph_0$. Likewise, P is obviously $p^{\omega+n}$-projective, that gives the result. □

The last affirmation can be slightly extended for $p^{\omega+n}$-bounded nicely ω_1-$p^{\omega+n}$-projective groups, but for such groups of lengths beyond $\omega+n$ this cannot happen, because neither the structure of subgroups nor even of direct summands is known yet (compare with Proposition 5.9 below).

Proposition 3.6. *Let $G=\oplus_{i\in I}G_i$ be a group of length not exceeding $\omega+n$. Then G is nicely ω_1-$p^{\omega+n}$-projective if and only if any G_i is nicely ω_1-$p^{\omega+n}$-projective and there exists a countable subset $J\subseteq I$ with the property that G_i are $p^{\omega+n}$-projective for any $i\in I\setminus J$.*

Proof. To treat the necessity, observe that all G_i are nicely ω_1-$p^{\omega+n}$-projective from the utilization of Corollary 3.2. Moreover, G/Y is countable for some $p^{\omega+n}$-projective subgroup $Y\leq G$. Thus $G/Y=\sum_{i\in I}(G_i+Y)/Y$ is countable, whence I is either countable and we may choose $J=I$, or I is uncountable and $(G_i+Y)/Y\cong G_i/(G_i\cap Y)=\{0\}$ for almost all indices

i, that is, for all but a countable number of indexes i. Hence there exists a countable subset $J \subseteq I$ with $G_i = G_i \cap Y \subseteq Y$ for every $i \in J$; thus these G_i are $p^{\omega+n}$-projective being subgroups, as needed.

Conversely, to deal with the sufficiency, write $G = \oplus_{i\in I} G_i = (\oplus_{i\in J} G_i) \oplus (\oplus_{i\in I\setminus J} G_i)$, and set $X = (\oplus_{i\in J} X_i) \oplus (\oplus_{i\in I\setminus J} G_i)$ where for each $i \in J$ these X_i are nice in G, X_i are $p^{\omega+n}$-projective and G_i/X_i are countable. Consequently, X is nice in G and $G/X \cong \oplus_{i\in J}(G_i/X_i)$ is countable, as required. $\square$

A new useful restatement of Definition 3.1 is the following statement which is very similar to that of [2].

Proposition 3.7. *The group G is nicely ω_1-$p^{\omega+n}$-projective if and only if there is a nice $p^{\omega+n}$-projective group P of G such that both $p^\omega G/(p^\omega G \cap P)$ and $G/(p^\omega G + P)$ are countable.*

Proof. Observing that G/P is countable uniquely when so are $p^\omega(G/P) = (p^\omega G+P)/P \cong p^\omega G/(P\cap p^\omega G)$ and $(G/P)/p^\omega(G/P) = (G/P)/(P+p^\omega G)/P \cong G/(P + p^\omega G)$, we are done. $\square$

We will now extend (Corollary 4.1, [5]) to the following statement.

Lemma 3.8. *Suppose B is a Σ-cyclic subgroup of a group A such that A/B is countable. Then A is the direct sum of a countable group and a Σ-cyclic group.*

Proof. Write $A = B + C$ for some countable $C \leq A$. Decompose $B = B_1 \oplus B_2$ where $B_2 \supseteq B \cap C$ and it is countable as well. Thus $C + B_2$ is also countable, and one may see that $A = B_1 \oplus (B_2 + C)$. In fact, that A is generated by the sum $B_1 + B_2 + C$ is trivial. That the intersection $B_1 \cap (B_2 + C) = \{0\}$, we choose x in it. Hence $x = b_1 = b_2 + c$ where $b_1 \in B_1$, $b_2 \in B_2$ and $c \in C$. But $b_1 - b_2 = c \in B \cap C \subseteq B_2$, so that $b_1 \in B_1 \cap B_2 = \{0\}$, *i.e.*, $x = 0$ as needed. $\square$

The next assertion shows that, although different, nicely ω_1-$p^{\omega+n}$-projective groups are very close to strongly ω_1-$p^{\omega+n}$-projective groups. To that aim, we will say that a subgroup U of a group G is *almost nice* in G if U is a nice subgroup of some nice subgroup of G. Clearly, each nice subgroup is almost nice while the converse claim is no longer true because niceness is not a transitive property (see cf. [8]).

Proposition 3.9. *If G is a nicely ω_1-$p^{\omega+n}$-projective group, then there exists an almost nice p^n-bounded subgroup Y of G such that G/Y is a direct sum of a countable group and a Σ-cyclic group.*

Proof. Let us use the terms of Definition 3.1. Since there is a p^n-bounded subgroup Y of P such that P/Y is Σ-cyclic, we deduce that Y is nice in P. But P is nice in G and thus Y is almost nice in G. Moreover, the quotient $G/P \cong$

$(G/Y)/(P/Y)$ is countable. Hence the above Lemma 3.8 applies to get the desired decomposition. $\square$

Of some interest and importance is whether or not the converse implication of the last result is also valid. We have some doubts about its validity.

PROPOSITION 3.10. *Suppose that A is a group with a subgroup G such that A/G is finite. If G is nicely ω_1-$p^{\omega+n}$-projective, then A is nicely ω_1-$p^{\omega+n}$-projective.*

Proof. Suppose P is a $p^{\omega+n}$-projective nice subgroup of G such that G/P is countable. Note that Lemma 2.2 assures that P is nice in A, and moreover $(A/P)/(G/P) \cong A/G$ being countable guarantees that A/P is countable, as needed. $\square$

PROPOSITION 3.11. *Let F be a finite subgroup of a group A. If A is nicely ω_1-$p^{\omega+n}$-projective, then A/F is nicely ω_1-$p^{\omega+n}$-projective.*

Proof. Assume that A/X is countable for some nice $p^{\omega+n}$-projective subgroup X. Furthermore, $(A/X)/((X+F)/X) \cong A/(X+F) \cong (A/F)/((X+F)/F)$ is countable, and $(X+F)/F \cong X/(X\cap F)$ is $p^{\omega+n}$-projective in view of [1] because $X \cap F$ is finite. Besides, Lemma 2.2 gives that $X+F$ is nice in A whence $(X+F)/F$ is nice in A/F (cf. [8]). $\square$

As a useful consequence, we yield:

COROLLARY 3.12. *Suppose A is a nicely ω_1-$p^{\omega+n}$-projective group for which $p^\lambda A$ is finite. Then $A/p^\lambda A$ is nicely ω_1-$p^{\omega+n}$-projective.*

A question which directly arises is whether or not the converse holds, *i.e.*, if $A/p^\lambda A$ is nicely ω_1-$p^{\omega+n}$-projective and $p^\lambda A$ is finite, is then A nicely ω_1-$p^{\omega+n}$-projective as well?

Remark 3. The converse implication that A/F being nicely ω_1-$p^{\omega+n}$-projective implies the same for A perhaps does not hold in general; one reason for that is that if P/F is $p^{\omega+n}$-projective, then P need not be $p^{\omega+n}$-projective too – in fact, P is ω-$p^{\omega+n}$-projective.

In this aspect, is it true that ω-$p^{\omega+n}$-projectives are nicely ω_1-$p^{\omega+n}$-projective? Notice that in Example 2.3 of [9] was constructed an ω-$p^{\omega+n}$-projective group of length $\omega+n$ which is not strongly ω_1-$p^{\omega+n}$-projective ($= p^{\omega+n}$-projective).

4. CO-NICELY ω_1-$p^{\omega+n}$-PROJECTIVE GROUPS

We start here with

Definition 4.1. The group G is said to be *co-nicely ω_1-$p^{\omega+n}$-projective* if there exists a $p^{\omega+n}$-projective group P with a countable nice subgroup $S \leq P$ such that $G \cong P/S$.

Owing to [9] these groups are of necessity ω_1-$p^{\omega+n}$-projective. However, they possess some more attractive properties. We first list a few elementary pieces of them:

(P1) If G is co-nicely ω_1-$p^{\omega+n}$-projective, then $p^{\omega+n}G=\{0\}$.

Indeed, $p^{\omega+n}G\cong p^{\omega+n}(P/S)=(p^{\omega+n}G+S)/S=\{0\}$.

(P2) If G is a co-nicely ω_1-$p^{\omega+n}$-projective group with $p^{\omega}G=\{0\}$, then G is $p^{\omega+n}$-projective.

In fact, G being p^{ω}-bounded forces that $p^{\omega}P\subseteq S$. Consequently, $P/S\cong(P/p^{\omega}P)/(S/p^{\omega}P)$. Moreover, it is readily verified that $P/p^{\omega}P$ is also separable $p^{\omega+n}$-projective, while $S/p^{\omega}P$ is countable. Thus, applying [5], we deduce that $P/S\cong G$ is $p^{\omega+n}$-projective, as claimed.

Note that this fact also follows directly from [9].

(P3) For each $n\geq 0$ there exists an inseparable co-nicely ω_1-$p^{\omega+n}$-projective group which is not $p^{\omega+n}$-projective.

We continue with some other major properties concerning Ulm subgroups and Ulm factors.

Proposition 4.1. *If G is a co-nicely ω_1-$p^{\omega+n}$-projective group, then both $p^{\alpha}G$ and $G/p^{\alpha}G$ are co-nicely ω_1-$p^{\omega+n}$-projective for every ordinal $\alpha\leq\omega+n-1$.*

Proof. For the first part, we have that the following isomorphism sequence is valid:

$$p^{\alpha}G\cong p^{\alpha}(P/S)=(p^{\alpha}P+S)/S\cong p^{\alpha}P/(p^{\alpha}P\cap S).$$

Next, since $p^{\alpha}P\cap S$ is countable, and $p^{\alpha}P\cap S$ is nice in $p^{\alpha}P$, where the latter group is $p^{\omega+n}$-projective, we are finished.

For the second part, observe that the following isomorphism sequence is fulfilled:

$$G/p^{\alpha}G\cong(P/S)/p^{\alpha}(P/S)=(P/S)/(p^{\alpha}P+S)/S\cong$$
$$P/(p^{\alpha}P+S)\cong(P/p^{\alpha}P)/(p^{\alpha}P+S)/p^{\alpha}P.$$

But it is easily checked that $P/p^{\alpha}P$ remains $p^{\omega+n}$-projective as is P, and that the factor-group $(p^{\alpha}P+S)/p^{\alpha}P\cong S/(S\cap p^{\alpha}P)$ is countable and also nice in $P/p^{\alpha}P$, as required. $\square$

Problem 1. If $p^{\alpha}G$ and $G/p^{\alpha}G$ are both co-nicely ω_1-$p^{\omega+n}$-projective groups for some $\alpha\leq\omega+n-1$, then does it follow that so is G?

5. STRONGLY ω_1-$p^{\omega+n}$-PROJECTIVE GROUPS

These groups were originally defined and carefully explored in [2] – compare with Section 1. Recall once again that a group G is said to be *strongly ω_1-$p^{\omega+n}$-projective* if there is a p^n-bounded nice subgroup H such that G/H

is the direct sum of a countable group and a Σ-cyclic group. As it will be observed below, a valuable example of such type of groups is the direct sum of a countable group and a $p^{\omega+n}$-projective group.

Here, we obtain a new simple but useful reformulation of the original definition for strongly ω_1-$p^{\omega+n}$-projective groups, listed in the first section, which is the following statement:

THEOREM 5.1. *A group G is strongly ω_1-$p^{\omega+n}$-projective if and only if there exist a countable subgroup K and a p^n-bounded nice subgroup P such that $G/(K+P)$ is Σ-cyclic.*

Proof. "$\Rightarrow$". By definition, write $G/X = (A/X) \oplus (B/X)$ where X is nice in G with $p^nX = \{0\}$, and A/X is countable, whereas B/X is Σ-cyclic. Consequently, $(G/X)/(A/X) \cong G/A \cong B/X$ is Σ-cyclic, and $A = X + K$ for some countable subgroup $K \leq A$, as required.

"$\Leftarrow$". Assume that $G/(K+P)$ is Σ-cyclic for some countable $K \leq G$ and p^n-bounded nice $P \leq G$. Observing that $(G/P)/((K+P)/P) \cong G/(K+P)$ is Σ-cyclic, and $(K+P)/P \cong K/(K \cap P)$ is countable, we deduce by the classical Charles' lemma (see, *e.g.*, [3]) that G/P is the direct sum of a countable group and a Σ-cyclic group, as required. $\square$

The class of strongly ω_1-$p^{\omega+n}$-projectives, and especially the two different characterizations that are ([2], Theorem 4.13) and Theorem 5.1 alluded to above, generate under some extra conditions the classes of separably ω_1-$p^{\omega+n}$-projectives and separately ω_1-$p^{\omega+n}$-projectives. Our further work in this section is devoted to their mirror relationships.

And so, this leads us to the following definition (notice that it is apparent that $p^{\omega+n}$-projectives are strongly ω_1-$p^{\omega+n}$-projective by choosing $K = \{0\}$).

Definition 5.1. We shall say that a group G is *separately ω_1-$p^{\omega+n}$-projective* if there is a countable subgroup K and a p^n-bounded nice subgroup P with $K \cap P = \{0\}$ such that $G/(K \oplus P)$ is Σ-cyclic.

It is worthwhile noticing that the direct sum of a countable group and a $p^{\omega+n}$-projective group is separately ω_1-$p^{\omega+n}$-projective; in fact, write $A = M \oplus N$ where M is countable and N is $p^{\omega+n}$-projective. So there is a nice subgroup $S \leq N$ with $p^nS = \{0\}$ and N/S is Σ-cyclic. Therefore, S is nice in A and $A/(M \oplus S) = (M \oplus N)/(M \oplus S) \cong N/S$ is Σ-cyclic, as wanted.

In particular, one may derive that each simply presented group G with countable $p^{\omega+n}G$ is separately ω_1-$p^{\omega+n}$-projective.

A valuable necessary and sufficient condition for separately ω_1-$p^{\omega+n}$-projective groups is the following one:

Theorem 5.2. *The group G is separately ω_1-$p^{\omega+1}$-projective if and only if the following two conditions are fulfilled:*

(1) *$p^{\omega+1}G$ is countable;*

(2) *$G/p^{\omega+1}G$ is $p^{\omega+1}$-projective.*

Proof. The "and only if" part follows directly from [9].

As for the "if" part, with [7] at hand, we write that $G = K \oplus S$ where K is countable and S is $p^{\omega+1}$-projective. Thus there is $T \leq S[p]$ such that S/T is Σ-cyclic. Hence T is nice in S, and so in G. Furthermore, $G/(K \oplus T) = (K \oplus S)/(K \oplus T) \cong S/T$ is Σ-cyclic, as required. $\square$

As immediate consequences, we derive:

Corollary 5.3. *The group G is strongly ω_1-$p^{\omega+1}$-projective if and only if G is separately ω_1-$p^{\omega+1}$-projective.*

Proof. Follows from ([2], Proposition 4.7) and Theorem 5.2. $\square$

Now, we recollect another critical concept from [2]. We will say that the group G is *separably ω_1-$p^{\omega+n}$-projective* if there exists a p^n-bounded nice subgroup P such that $P \cap p^\omega G = \{0\}$ and $G/(p^\omega G \oplus P)$ is Σ-cyclic.

We can note that the direct sum of a countable group and a separable $p^{\omega+n}$-projective group is separably ω_1-$p^{\omega+n}$-projective; indeed, write $B = K \oplus L$ where K is countable and L is separable $p^{\omega+n}$-projective. Thus there exists $P \leq L$ such that $p^n P = \{0\}$ and L/P is Σ-cyclic. That is why P is nice in L and hence in B. Moreover, $p^\omega B \cap P = p^\omega K \cap P \subseteq K \cap L = \{0\}$ whence $B/(p^\omega B \oplus P) = (K \oplus L)/(p^\omega K \oplus P) \cong (K/p^\omega K) \oplus (L/P)$ is Σ-cyclic, as required.

It is also worthy of noticing that there exists a separately ω_1-$p^{\omega+1}$-projective group G which is not separably ω_1-$p^{\omega+1}$-projective if we take $p^\omega G$ to be uncountable. However, for countable $p^\omega G$, the problem seems to be more difficult than we anticipate.

Specifically, we will prove the following:

Proposition 5.4. *Suppose that G is a group whose $p^\omega G$ is countable. If G is separably ω_1-$p^{\omega+n}$-projective, then G is separately ω_1-$p^{\omega+n}$-projective.*

Proof. Just take $K = p^\omega G$. $\square$

It follows from (Proposition 4.9, [2]) that G is even separably n-simply presented (see also [3]).

A useful idea to prove the converse relation could be the following: write $G/(K \oplus P)$ is Σ-cyclic for some p^n-bounded nice subgroup $P \leq G$ and countable $K \leq G$. Hence $p^\omega G \subseteq K \oplus P$. The critical step is whether or not we may assume that $p^\omega G \subseteq K$, whence $P \cap p^\omega G = \{0\}$. If yes, we therefore have that

$$G/(p^\omega G \oplus P)/(K \oplus P)/(p^\omega G \oplus P) \cong G/(K \oplus P)$$

is Σ-cyclic, where the quotient $(K \oplus P)/(p^\omega G \oplus P) \cong K/p^\omega G$ is countable. Furthermore, the Charles' lemma assures that $G/(p^\omega G \oplus P)$ is the direct sum of a countable group and a Σ-cyclic group. However, $G/(p^\omega G \oplus P)$ is obviously checked to be separable (because $p^\omega G \oplus P$ is nice in G), and consequently it is Σ-cyclic, as wanted.

Remark 3. However, for $n = 1$ there is an equivalence according to ([2], Proposition 4.8) combined with Theorem 5.2 stated above. So, we can state (compare with Corollary 5.3).

Corollary 5.5. *Let G be a group for which $p^\omega G$ is countable. Then G is separably ω_1-$p^{\omega+1}$-projective if and only if G is separately ω_1-$p^{\omega+1}$-projective.*

We end this section with the following result.

Proposition 5.6. *If G is strongly ω_1-$p^{\omega+n}$-projective, then G is nicely ω_1-$p^{\omega+n}$-projective.*

Proof. In virtue of the corresponding definition, write $G/B = (A/B) \oplus (C/B)$ where A/B is Σ-cyclic and C/B is countable for some $B \leq A$ and $B \leq C$ with $p^n B = \{0\}$ and B nice in G. Thus A is $p^{\omega+n}$-projective and $(G/B)/(A/B) \cong G/A \cong C/B$ is countable. But A/B is nice in G/B, and hence A is nice in G, as required, because B is nice in G (see, *e.g.*, [8]). □

Remark 4. The converse implication is not valid since, as we have seen above, there exists a nicely ω_1-$p^{\omega+n}$-projective group of length $\omega + n$ which is not $p^{\omega+n}$-projective, in contrast with strongly ω_1-$p^{\omega+n}$-projective groups (see [2]).

We will now explore how strongly ω_1-$p^{\omega+n}$-projectives are situated concerning finite extensions. Specifically, the following holds:

Proposition 5.7. *Suppose A is a group with a subgroup G such that A/G is finite. If G is strongly ω_1-$p^{\omega+n}$-projective, then so is A.*

Proof. Appealing to Theorem 5.1, suppose P is a p^n-bounded nice subgroup of G and K is a countable subgroup of G with $G/(K+P)$ being Σ-cyclic. Writing $A = G + F$ for some finite $F \leq A$, we deduce that $A/(K + P) = [G/(K+P)]+[(F+K+P)/(K+P)]$ where the latter quotient is finite. Hence the sum is also Σ-cyclic, as required. Observing that with Lemma 2.2 in hand P is nice in A, we are done. □

A helpful consequence could be the following:

Corollary 5.8. *Suppose that $A = G \oplus H$ where H is countable. Then A is strongly ω_1-$p^{\omega+n}$-projective if and only if G is strongly ω_1-$p^{\omega+n}$-projective.*

Proof. In virtue of Corollary 4.19 from [2], the necessity is true.

As for the sufficiency, we observe that $p^{\omega+n}A = p^{\omega+n}G \oplus p^{\omega+n}H$ is countable because so are $p^{\omega+n}G$ and $p^{\omega+n}H$ as well as $A/p^{\omega+n}A \cong (G/p^{\omega+n}G) \oplus (H/p^{\omega+n}H)$ is $p^{\omega+n}$-projective since both $G/p^{\omega+n}G$ and $H/p^{\omega+n}H$ are so. We now apply the First Reduction Criterion from [2] to infer that A is strongly ω_1-$p^{\omega+n}$-projective, as stated. $\square$

The last construction may be extended to the following one (compare also with Proposition 2.4 from [9]):

PROPOSITION 5.9. *Suppose that $G = \oplus_{i\in I}G_i$. Then G is strongly ω_1-$p^{\omega+n}$-projective if and only if all G_i are strongly ω_1-$p^{\omega+n}$-projective and there is a countable subset $J \subseteq I$ such that G_i are $p^{\omega+n}$-projective for all $i \in I \setminus J$.*

Proof. "$\Rightarrow$". That G_i is strongly ω_1-$p^{\omega+n}$-projective for any $i \in I$ follows from Corollary 4.19 in [2], and hence each $p^{\omega+n}G_i$ is countable.

Since $p^{\omega+n}G = \oplus_{i\in I}p^{\omega+n}G_i$, it follows that either I is countable (whence we may take $J = I$, so that $I \setminus J = \emptyset$), or I is uncountable and $p^{\omega+n}G_i = \{0\}$ for almost all indices i (that is, for all but a countable number of them) and thus we set $J = \{i \mid p^{\omega+n}G_i \neq \{0\}\}$. That is why these G_i are $p^{\omega+n}$-projective, as claimed.

"$\Leftarrow$". We see that $p^{\omega+n}G = \oplus_{i\in I}p^{\omega+n}G_i = \oplus_{i\in J}p^{\omega+n}G_i$ is countable because so is every $p^{\omega+n}G_i$ $(i \in I)$. Moreover, $G/p^{\omega+n}G \cong \oplus_{i\in I}(G_i/p^{\omega+n}G_i)$ is $p^{\omega+n}$-projective since all factors $G_i/p^{\omega+n}G_i$ are so. Now the First Reduction Criterion from [2] works to get the claim. $\square$

As a direct consequence, we derive:

COROLLARY 5.10. *The countable direct sum of strongly ω_1-$p^{\omega+n}$-projective groups is a strongly ω_1-$p^{\omega+n}$-projective group.*

For some results of the type described in the next result the interested reader can see [1].

PROPOSITION 5.11. *Suppose F is a finite subgroup of a group A. If A is strongly ω_1-$p^{\omega+n}$-projective, then A/F is strongly ω_1-$p^{\omega+n}$-projective.*

Proof. Observe that $p^{\omega+n}(A/F) = (p^{\omega+n}A + F)/F \cong p^{\omega+n}A/(p^{\omega+n}A \cap F)$ is countable since as mentioned before $p^{\omega+n}A$ is so. Next, utilizing [1], $A/p^{\omega+n}A$ is $p^{\omega+n}$-projective and thus $(A/F)/p^{\omega+n}(A/F) \cong A/(p^{\omega+n}A + F) \cong (A/p^{\omega+n}A)/((p^{\omega+n}A+F)/p^{\omega+n}A)$ is $p^{\omega+n}$-projective too, because the quotient $(p^{\omega+n}A + F)/p^{\omega+n}A \cong F/(F \cap p^{\omega+n}A)$ is finite. We therefore apply [2] to get the claim. $\square$

Remark 5. Unfortunately, the converse is not true in general, that is, A/F being strongly ω_1-$p^{\omega+n}$-projective does not imply that so is A. In fact, let us assume that A/F is strongly ω_1-$p^{\omega+n}$-projective. Thus $p^{\omega+n}(A/F) = (p^{\omega+n}A+$

$F)/F \cong p^{\omega+n}A/(p^{\omega+n}A \cap F)$ should be countable, whence so is $p^{\omega+n}A$. Moreover, $(A/F)/p^{\omega+n}(A/F) \cong A/(p^{\omega+n}A+F) \cong (A/p^{\omega+n}A)/((p^{\omega+n}A+F)/p^{\omega+n}A)$. However, $(p^{\omega+n}A + F)/p^{\omega+n}A \cong F/(p^{\omega+n}A \cap F)$ is finite, and as observed above, $(A/F)/p^{\omega+n}(A/F)$ being $p^{\omega+n}$-projective does not yield that $A/p^{\omega+n}A$ is so. We therefore conclude with the help of the First Reduction Criterion from [2] that A need not be strongly ω_1-$p^{\omega+n}$-projective, as asserted.

On another vein, a group is called *$\omega + n$-totally $p^{\omega+n}$-projective* if each of its $p^{\omega+n}$-bounded subgroup is $p^{\omega+n}$-projective. These groups are shown in Proposition 3.1 from [9] to be ω_1-$p^{\omega+n}$-projective.

On the other hand, a group is said to be *ω-totally $p^{\omega+n}$-projective* if each of its p^{ω}-bounded subgroup is $p^{\omega+n}$-projective. These groups are proved in Corollary 3.4 of [9] to contain the ω_1-$p^{\omega+n}$-projective ones. However, for groups with countable first Ulm subgroup, these two group classes coincide (see [9], Theorem 3.6). In Question 1 from [9] it is asked whether or not these two classes of groups are absolutely equal in general.

Since strongly ω_1-$p^{\omega+n}$-projective groups are ω_1-$p^{\omega+n}$-projective, it is reasonably right to ask whether or not strongly ω_1-$p^{\omega+n}$-projectives are $\omega + n$-totally $p^{\omega+n}$-projective and/or vice versa, again provided the first Ulm subgroup is countable. However, the next construction illustrates that this is not the case, at least in one direction (see Example 2 of [11] too).

Example 5.12. For any fixed integer $n \geq 1$, there exists a strongly ω_1-$p^{\omega+n}$-projective group G with finite but not p^n-bounded Ulm subgroup $p^{\omega}G$, which is not $\omega + n$-totally $p^{\omega+n}$-projective.

Proof. Let A be a separable (proper) $p^{\omega+1}$-projective group whose socle $A[p]$ is not $\aleph_0$-coseparable (for its existence see [6] and [7]), and let H be a countable group whose $p^{\omega}H$ is finite and $p^{\omega+n}H \neq 0$. Supposing now that $G = A \oplus H$, we see that G is strongly ω_1-$p^{\omega+n}$-projective. In fact, since $p^{\omega+n}G = p^{\omega+n}H$ is countable and $G/p^{\omega+n}G \cong A \oplus (H/p^{\omega+n}H)$ is $p^{\omega+n}$-projective, it follows from [2] that G is so as claimed.

Furthermore, since G is neither a direct sum of countable groups (otherwise A must be Σ-cyclic, a contradiction with its choice) nor a $p^{\omega+n}$-projective group (otherwise, $p^{\omega+n}G = \{0\}$ will imply that $p^{\omega+n}H = \{0\}$, contrary to its construction), if it were $\omega + n$-totally $p^{\omega+n}$-projective, it would be proper. But this contradicts Theorem 3.1 of [7]. $\square$

Remark 6. The eventual existence of an $\omega + n$-totally $p^{\omega+n}$-projective group G, with countable $p^{\omega}G$ in addition, that is not strongly ω_1-$p^{\omega+n}$-projective, is unknown yet. Nevertheless, for $p^{\omega+n}$-bounded groups, the classes of strongly ω_1-$p^{\omega+n}$-projective groups and $\omega + n$-totally $p^{\omega+n}$-projective groups are exactly the $p^{\omega+n}$-projective groups. So, we can state:

Problem 2. If G is an $\omega+n$-totally $p^{\omega+n}$-projective group, is it true that the factor-group $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective? If so, all $\omega+n$-totally $p^{\omega+n}$-projectives will be strongly ω_1-$p^{\omega+n}$-projective.

We suspect the answer is "no" because of the following thoughts: Let G be a proper $\omega+n$-totally $p^{\omega+n}$-projective group. By the utilization of [7], under the assumption that $2^{\aleph_0} < 2^{\aleph_1}$, we derive that $p^{\omega}G$ is countable. Assuming that $G/p^{\omega+1}G$ is $p^{\omega+1}$-projective, again [7] applies to get the decomposition $G = K \oplus L$, where K is countable and L is separable $p^{\omega+1}$-projective. Thus any proper $\omega+1$-totally $p^{\omega+1}$-projective group must have such a decomposition. However, it seems to be with some doubts about its validity.

Problem 3. Does it follow that the property of being a strongly or nicely ω_1-$p^{\omega+n}$-projective group is closed under taking ω-bijections?

Note that by virtue of Remark 5 the answer seems to be "no" for strongly ω_1-$p^{\omega+n}$-projectives.

It was shown before that the direct sum of a countable group and a separable $p^{\omega+n}$-projective group is separable ω_1-$p^{\omega+n}$-projective as well as the direct sum of a countable group and a $p^{\omega+n}$-projective group is separately ω_1-$p^{\omega+n}$-projective. In this regard, we pose:

Problem 4. Describe the classes of

(a) nice subgroups of the direct sum of a countable group and a $p^{\omega+n}$-projective group;

(b) isotype subgroups of the direct sum of a countable group and a $p^{\omega+n}$-projective group;

(c) balanced subgroups of the direct sum of a countable group and a $p^{\omega+n}$-projective group.

Corrections: In [11] on p. 769, Example 2 there are two misprints. They are as follows: Firstly, on line 1, the phrase "inseparable first Ulm subgroup" should be "non-trivial first Ulm subgroup", and secondly, on line 3 of the proof of the same example, the phrase "is strongly n-totally projective" should be "is inseparable strongly n-totally projective".

Acknowledgments. The author would like to thank the referee for the suggestions made, which improve the presentation of the paper.

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Received 8 May 2013

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