NUMERICAL SOLUTION OF FRACTIONAL TELEGRAPH EQUATION *VIA* THE TAU METHOD

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This paper presents a computational technique based on the Tau method and Legendre polynomials for the solution of a class of time-fractional telegraph equations. An appropriate representation of the solution *via* the Legendre operational matrix of fractional derivative is used to reduces its numerical treatment to the solution of a set of linear algebraic equations. The fractional derivatives are described based on the Caputo sense. The method is easy to implement and yields very accurate results. Illustrative examples are included to demonstrate the validity and applicability of the proposed technique.

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Key words: fractional telegraph equations, Tau method, operational matrix, Caputo derivative, Legendre polynomials.

1. INTRODUCTION

In recent years, it has been found that derivatives of fractional (non-integer) order are very effective for the description of many phenomena in engineering and applied sciences such as diffusion process, rheology, damping laws, electric transmission, viscoelasticity and fluid mechanics, see, say [21] and the references therein. However, in the case of fractional order differentiation, the problem does not have a fully acceptable geometrical or physical interpretation [23]. Today, there are many works on fractional calculus [11, 21]. Most fractional differential equations do not have closed form solutions, so approximation and numerical techniques such as finite difference method [10, 24–26], Adomian decomposition method [27], variational iteration method [14, 28], homotopy analysis method [12, 13], pseudo-spectral method [17], Sinc-Legendre collocation method [35], Tau method [34], Fourier method [9], wavelet method [18, 37], and other methods [1, 2, 3, 6, 15, 16, 19, 22, 30, 32, 33], must be used.

Telegraph equations are hyperbolic partial differential equations that are applicable in several fields such as wave propagation [36], random walk theory [4], signal analysis [20], etc (see [31] and the references therein). The time fractional telegraph equations have recently been considered by many authors. Liu and coworkers [8] derived the analytical solution of the nonhomogeneous time-fractional telegraph equation under three types of nonhomogeneous boundary conditions by the method of separation of variables. Authors of [29] studied the fundamental solutions to time-fractional telegraph equations of order 2α. For the special case $\alpha = 1/2$, they showed that the fundamental solution is the probability density of a telegraph process with Brownian time. Also authors of [5] considered the fractional telegraph equation with partial fractional derivatives of rational order $\alpha = m/n$ with $m < n$. They proved that the fundamental solution to the Cauchy problem for the time-fractional telegraph equation can be expressed as the density of the composition of two processes, one depending on m and the other depending on n.

In this paper, we present a direct computational technique for the one-dimensional time-fractional telegraph equation of the form [19, 8]:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} + \lambda\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \mu\frac{\partial^{2}u(x,t)}{\partial x^{2}} + f(x,t), \quad 0 < x < \ell, \quad 0 < t \leq \tau, \tag{1.1}$$

subject to initial conditions

$$u(x,0) = \phi_1(x), \quad u_t(x,0) = \phi_2(x), \quad 0 \leq x \leq \ell, \tag{1.2}$$

and boundary conditions

$$u(0,t) + \theta_1 u_x(0,t) = \psi_1(t), \qquad 0 \leq t \leq \tau, \tag{1.3}$$

$$u(\ell,t) + \theta_2 u_x(\ell,t) = \psi_2(t), \qquad 0 \leq t \leq \tau, \tag{1.4}$$

where $f, \phi_1, \phi_2, \psi_1$ and ψ_2 are sufficiently smooth prescribed functions, the rate λ is an arbitrary nonnegative constant and μ is an arbitrary positive constant. Also $1/2 < \alpha \leq 1$ and the time-fractional derivative is defined as the Caputo fractional derivatives.

Definition 1. The Caputo fractional derivative of order $\alpha > 0$ is defined as [21]

$$D^{\alpha}f(t) = \frac{\mathrm{d}^{\alpha}f(t)}{\mathrm{d}t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(x)}{(t-x)^{\alpha+1-n}}\mathrm{d}x, & n-1 < \alpha < n, \quad n \in \mathbb{N}, \\ \frac{\mathrm{d}^n}{\mathrm{d}t^n}f(t), & \alpha = n \in \mathbb{N}. \end{cases} \tag{1.5}$$

As said in [8], when $\lambda = 0$, Eq. (1.1) is the fractional counterpart of the nonhomogeneous wave equation. In fact, without the forcing term $f(x,t)$, and with $\lambda = 0$, Eq. (1.1) is known as the fractional diffusion-wave equation.

The main idea of this work is to apply the Legendre polynomials and operational matrix of fractional derivative together with the Tau method to discretize Eq. (1.1). As a result, a linear system of algebraic equations is generated. Tau method consists of expanding the required approximate solution as the elements of a complete set of orthogonal polynomials [7]. It is worth to point out that, the method based on using the operational matrix for solving fractional-order differential equations is computer oriented.

The organization of the rest of this article is as follows. In Section 2, we explain the basic formulation of shifted Legendre polynomials required for our subsequent development. In Section 3 we illustrate how Legendre Tau method may be used to replace problem (1.1)–(1.4) by an explicit system of linear algebraic equations. In Section 4, we present some numerical examples to demonstrate the effectiveness of the proposed method.

2. PROPERTIES OF SHIFTED LEGENDRE POLYNOMIALS

Legendre polynomials are a well known family of orthogonal polynomials that have many applications. They are defined on the interval $[-1,1]$ and are recursively related by:

$$L_0(z)=1,\quad L_1(z)=z,\quad (i+1)L_{i+1}(z)=(2i+1)zL_i(z)-iL_{i-1}(z), i=1,2,...$$

For practical use of Legendre polynomial on the interval of interest $x\in[0,\ell]$ it is necessary to shift the defining domain by the following variable substitution:

$$z=\frac{2x-\ell}{\ell},\qquad 0\le x\le \ell.$$

The shifted Legendre polynomials $L_i^\ell(x)=L_i((2x-\ell)/\ell)$ are then obtained as:

$$(i+1)L_{i+1}^\ell(x)=(2i+1)\left(\frac{2x}{\ell}-1\right)L_i^\ell(x)-iL_{i-1}^\ell(x),\qquad i=1,2,\ldots \tag{2.1}$$

where $L_0^\ell(x)=1$ and $L_1^\ell(x)=(2x-\ell)/\ell$. They have the following orthogonality relation:

$$\int_0^\ell L_i^\ell(x)L_j^\ell(x)\mathrm{d}x=\begin{cases}\frac{\ell}{2i+1} & \text{for } i=j,\\ 0 & \text{for } i\ne j.\end{cases} \tag{2.2}$$

The shifted Legendre polynomials have the following analytic form:

$$L_i^\ell(x)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!x^k}{(i-k)!(k!)^2\ell^k}. \tag{2.3}$$

Note that $L_i^\ell(0)=(-1)^i$ and $L_i^\ell(\ell)=1$. A function $y(x)$, square integrable in $[0,\ell]$, may be expressed in terms of shifted Legendre polynomials as

$$y(x)=\sum_{j=0}^{\infty}c_jL_j^\ell(x).$$

In practice, only the first $(m+1)$-terms shifted Legendre polynomials are considered. Then we have

$$y_m(x) = \sum_{j=0}^{m} c_j L_j^{\ell}(x) = C^T \Phi_{m,\ell}(x),$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\Phi_{m,\ell}(x)$ are given by

$$C = [c_0, ..., c_m]^T,$$

$$\Phi_{m,\ell}(x) = [L_0^{\ell}(x), L_1^{\ell}(x), \cdots, L_m^{\ell}(x)]^T. \tag{2.4}$$

The coefficients c_j are chosen to minimize the mean integral square error

$$\varepsilon = \int_0^{\ell} \left(y(x) - C^T \Phi_{m,\ell}(x)\right)^2 \mathrm{d}x,$$

and are given by

$$c_j = \frac{(2j+1)}{\ell} \int_0^{\ell} y(x) L_j^{\ell}(x) \mathrm{d}x, \qquad j = 1, 2, \ldots$$

Similarly we approximate a function $u(x,t)$ of two independent variables defined for $0 \le x \le \ell$ and $0 \le t \le \tau$ by double shifted Legendre polynomials as:

$$u_{m,n}(x,t) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} L_i^{\tau}(t) L_j^{\ell}(x) = \Phi_{n,\tau}^T(t) \mathbf{A} \Phi_{m,\ell}(x), \tag{2.5}$$

where the shifted Legendre vector $\Phi_{n,\tau}(t)$ is defined similarly to Eq. (2.4). Also the shifted Legendre coefficient matrix A is given by

$$\mathbf{A} = \begin{pmatrix} a_{0,0} & \cdots & a_{0,m} \\ \vdots & \vdots & \vdots \\ a_{n,0} & \cdots & a_{n,m} \end{pmatrix}, \tag{2.6}$$

where

$$a_{ij} = \left(\frac{2i+1}{\tau}\right) \left(\frac{2j+1}{\ell}\right) \int_0^{\tau} \int_0^{\ell} u(x,t) L_i^{\tau}(t) L_j^{\ell}(x) \mathrm{d}x \mathrm{d}t,$$
$$i = 0, 1, \ldots, n, \quad j = 0, 1, \ldots, m.$$

The derivative of the vector $\Phi_{m,\ell}(x)$ can be expressed by [7]

$$\frac{d\Phi_{m,\ell}(x)}{dx} = \mathbf{D} \Phi_{m,\ell}(x), \tag{2.7}$$

where $\mathbf{D}^{(1)}$ is the $(m+1) \times (m+1)$ operational matrix of derivative given by

$$\mathbf{D}^{(1)} = (d_{ij}) = \begin{cases} \frac{2(2j+1)}{l}, & \text{for } j = i-k, \quad \begin{cases} k = 1, 3, ..., m, & \text{if } m \text{ odd}, \\ k = 1, 3, ..., m-1, & \text{if } m \text{ even}, \end{cases} \\ 0, & \text{otherwise}. \end{cases}$$

Using Eq. (2.7), it is clear that

$$\frac{\mathrm{d}^k\Phi_{m,\ell}(x)}{\mathrm{d}x^k} = (\mathbf{D}^{(1)})^k\Phi_{m,\ell}(x), \qquad k \in \mathbb{N}, \tag{2.8}$$

here the super script, in $\mathbf{D}^{(1)}$, denotes matrix powers. Thus,

$$\mathbf{D}^{(k)} = (\mathbf{D}^{(1)})^k, \qquad k = 1, 2, \ldots \tag{2.9}$$

In the following theorem, which plays an important role in this paper, the authors of [34, 33] generalized the operational matrix of derivative of shifted Legendre polynomials to fractional derivative.

THEOREM 1. *Let $\Phi_{m,\ell}(x)$ be shifted Legendre vector defined in (2.4) and also suppose $\alpha > 0$ then*

$$D^{\alpha}\Phi_{m,\ell}(x) \simeq \boldsymbol{D}^{(\alpha)}_{m,\ell}\Phi_{m,\ell}(x), \tag{2.10}$$

where $\boldsymbol{D}^{(\alpha)}_{m,\ell}$ is the $(m+1)\times(m+1)$ operational matrix of fractional derivative of order α in Caputo sense and is defined as follows:

$$\boldsymbol{D}^{(\alpha)}_{m,\ell} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,m,k} \end{pmatrix}, \tag{2.11}$$

where $\theta_{i,j,k}$ is given by

$$\theta_{i,j,k} = \frac{2j+1}{h^{k+1}}\sum_{\ell=0}^{j}\frac{(-1)^{i+j+k+\ell}(i+k)!(\ell+j)!}{(i-k)!k!\Gamma(k-\alpha+1)(j-\ell)!(\ell!)^2(k+\ell-\alpha+1)}.$$

We use the ceiling function $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α. Also, note that in $\mathbf{D}^{(\alpha)}_{m,\ell}$, the first $\lceil\alpha\rceil$ rows, are all zero and if $\alpha = k \in \mathbb{N}$, then Theorem 1 gives the same result as Eq. (2.9).

Proof. The proof of this Theorem is available in [34]. $\square$

3. IMPLEMENTATION OF THE METHOD

To solve problem (1.1)–(1.4), we approximate $f(x,t)$ by shifted Legendre series as:

$$f_{m,n}(x,t) = \Phi_{n,\tau}^T(t)\mathbf{F}\Phi_{m,\ell}(x), \tag{3.1}$$

where $\mathbf{F}$ is known $(n+1)\times(m+1)$ matrix and is obtained similarly to Eq. (2.6). Also we approximate $u(x,t)$ by shifted Legendre polynomials as:

$$u_{m,n}(x,t) = \Phi_{n,\tau}^T(t)\mathbf{A}\Phi_{m,\ell}(x), \tag{3.2}$$

where $\mathbf{A}$ is unknown $(n+1)\times(m+1)$ matrix. Using Eqs. (2.10) and (3.2) we have

$$\frac{\partial^\alpha}{\partial t^\alpha}u_{m,n}(x,t) = \left(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\Phi_{n,\tau}^T(t)\right)\mathbf{A}\Phi_{m,\ell}(x) \simeq \Phi_{n,\tau}^T(t)(\mathbf{D}_{n,\tau}^{(\alpha)})^T\mathbf{A}\Phi_{m,\ell}(x), \tag{3.3}$$

$$\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}u_{m,n}(x,t) \simeq \Phi_{n,\tau}^T(t)(\mathbf{D}_{n,\tau}^{(2\alpha)})^T\mathbf{A}\Phi_{m,\ell}(x), \tag{3.4}$$

$$\frac{\partial^2}{\partial x^2}u_{m,n}(x,t) = \Phi_{n,\tau}^T(t)\mathbf{A}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Phi_{m,\ell}(x)\right) \simeq \Phi_{n,\tau}^T(t)\mathbf{A}\mathbf{D}_{m,\ell}^{(2)}\Phi_{m,\ell}(x), \tag{3.5}$$

where $\mathbf{D}_{n,\tau}^{(\alpha)}$ is defined similarly to Eq. (2.11). Employing Eqs. (3.1), (3.2), (3.3), (3.4) and (3.5) the residual $R_{m,n}(x,t)$ for Eq. (1.1) can be written as

$$\begin{aligned} R_{m,n}(x,t) &= \Phi_{n,\tau}^T(t)\{(\mathbf{D}_{n,\tau}^{(2\alpha)})^T\mathbf{A} + \lambda(\mathbf{D}_{n,\tau}^{(\alpha)})^T\mathbf{A} - \mu\mathbf{A}\mathbf{D}_{m,\ell}^{(2)} - \mathbf{F}\}\Phi_{m,\ell} \\ &= \Phi_{n,\tau}^T(t)\mathbf{E}\Phi_{m,\ell}(x), \end{aligned} \tag{3.6}$$

where

$$\mathbf{E} = (\mathbf{D}_{n,\tau}^{(2\alpha)})^T\mathbf{A} + \lambda(\mathbf{D}_{n,\tau}^{(\alpha)})^T\mathbf{A} - \mu\mathbf{A}\mathbf{D}_{m,\ell}^{(2)} - \mathbf{F}.$$

As in a typical Tau method [7] we generate $(n-1)\times(m-1)$ linear algebraic equations using the following algebraic equations

$$\mathbf{E}_{i,j} = 0, \qquad i = 0,1,\ldots,n-2 \qquad j = 0,1,\ldots,m-2. \tag{3.7}$$

Also substituting Eq. (3.2) in Eq.(1.2) yields

$$\Phi_{n,\tau}^T(0)\mathbf{A}\Phi_{m,\ell}(x) = \phi_1(x), \tag{3.8}$$

$$\Phi_{n,\tau}^T(0)(\mathbf{D}_{n,\tau}^{(1)})^T\mathbf{A}\Phi_{m,\ell}(x) = \phi_2(x), \tag{3.9}$$

Eqs. (3.8) and (3.9) are collocated at $(m-1)$ points. For suitable points we use the shifted Legendre roots $x_i, i = 1,2,...,m-1$ of $L_{m-1}^{\ell}(x)$. Furthermore, applying Eq. (3.2) in Eqs. (1.3) and (1.4) we obtain

$$\Phi_{n,\tau}(t)^T(\mathbf{A} + \theta_1\mathbf{A}\mathbf{D}_{m,\ell}^{(1)})\Phi_{m,\ell}(0) = \psi_1(t), \tag{3.10}$$

$$\Phi_{n,\tau}(t)^T(\mathbf{A} + \theta_2 \mathbf{A}\mathbf{D}^{(1)}_{m,\ell})\Phi_{m,\ell}(\ell) = \psi_2(t), \tag{3.11}$$

respectively. Eqs. (3.10) and (3.11) are collocated at $(n+1)$ points. For suitable points we use the shifted Legendre roots t_j, $j = 1, 2, ..., n+1$ of $L^{\tau}_{n+1}(t)$. The number of unknown coefficients $a_{i,j}$ is equal to $(n+1) \times (m+1)$ can be obtained from Eqs. (3.7)–(3.11). Consequently $u_{n,m}(x,t)$ given in Eq. (3.2) can be calculated. It is worth to mention here that, throughout this paper, we use the Maple's `fsolve` command to find unknown coefficients $a_{i,j}$ from Eqs. (3.7)–(3.11).

4. ILLUSTRATIVE EXAMPLES

This section is devoted to computational results. We applied the method presented in this paper and solved two examples. We will report the accuracy and efficiency of the new method based on maximum absolute error $e_{m,n}$ defined as:

$$e_{m,n} = \max\{|u(x,t) - u_{m,n}(x,t)|, \quad 0 \le x \le \ell, \quad 0 < t \le \tau\}.$$

Example 1. Consider the fractional telegraph equation [19],

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),$$
$$0.5 < \alpha \le 1, \quad 0 < x < 1, \quad 0 < t \le 1,$$

subject to initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0, \quad 0 < x < 1,$$

and boundary conditions

$$u(0,t) + u_x(0,t) = 2t^2, \quad 0 < t \le 1,$$
$$u(1,t) - \frac{1}{2}u_x(1,t) = \frac{et^2}{2}, \quad 0 < t \le 1,$$

whose exact solution is $u(x,t) = t^2 e^x$. We solved the problem, by applying the technique described in section 3. To explore the dependence of errors on the discretization parameter m, n, in Figure 1 we represent $e_{m,n}$ with $n = 3, \alpha = 0.64$ and for different values of m. Also in Figure 2 we plot $e_{m,n}$ with $m = n$ and $\alpha = 0.64$ as a function of m. According to Figures 1, and 2 we find that the presented method provides accurate results even for small m, n and we see the errors decrease rapidly as m and n increase.

In [19], the reproducing kernel theorem is used to solve this problem. For the purpose of comparison in Table 1, we compare the maximum absolute error of our method with $m = n$, together with the method of [19]. From Table 1, we see that our method is clearly reliable if compared with the method introduced in [19].

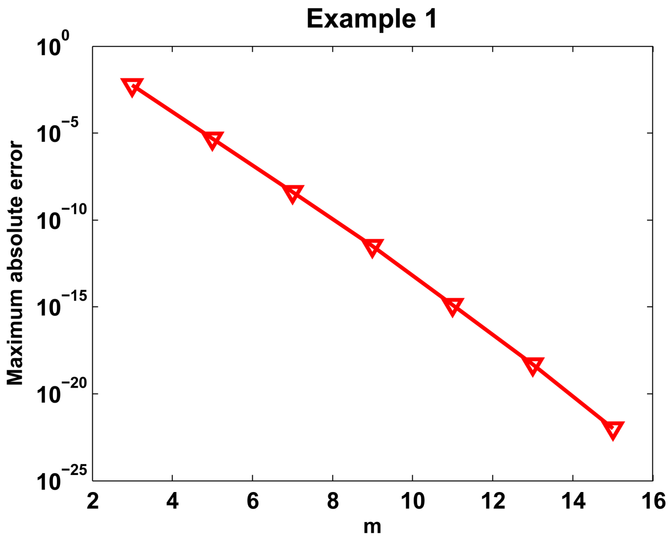

Fig. 1 – Comparison of $e_{m,n}$ with $n = 3$ and for different values of m.

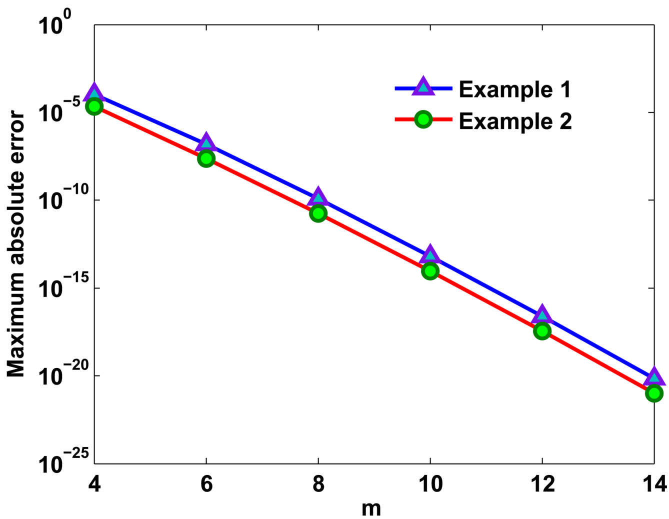

Fig. 2 – Comparison of $e_{m,n}$ with $m = n$ and for different values of m.

Table 1

The maximum absolute error for Example 1

	method in [19]			our method
n	$\alpha = 0.64$	$\alpha = 0.8$	$\alpha = 0.96$	$\alpha = 0.64, 0.8, 0.96$
4	1.86E-03	2.68E-03	2.33E-03	1.40E-04
8	2.49E-04	5.87E-04	4.99E-04	1.20E-10
12	7.77E-05	7.60E-05	2.10E-04	2.50E-17

Example 2. As the second example, we consider the following problem,

$$\frac{\partial^{\frac{4}{3}} u(x,t)}{\partial t^{\frac{4}{3}}} + \frac{\partial^{\frac{2}{3}} u(x,t)}{\partial t^{\frac{2}{3}}} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 < x < 1, \quad 0 < t \leq 1,$$

where

$$f(x,t) = 6\sin(x+1)\left(\frac{t^{\frac{5}{3}}}{\Gamma\left(\frac{8}{3}\right)} + \frac{t^{\frac{7}{3}}}{\Gamma\left(\frac{10}{3}\right)}\right) + \sin(x+1)(t^3+1),$$

and with the initial conditions

$$u(x,0) = \sin(x+1), \quad u_t(x,0) = 0, \quad 0 < x < 1,$$

and the boundary conditions

$$u(0,t) = \sin(1)(t^3+1), \qquad u(1,t) + 3u_x(1,t) = (t^3+1)(\sin(2) + 3\cos(2)).$$

The exact solution to this problem is $u(x,t) = (t^3+1)\sin(x+1)$ which can be verified by direct fractional differentiation of the given solution and substituting in the fractional differential equation. In Figure 2 we plot $e_{m,n}$ with $m = n$ and for different values of m and also in Figure 3 we represent $e_{m,n}$ with $n = 3$ and for different values of m. Furthermore the absolute error $|u(x,1) - u_{n,m}(x,1)|$ for $n = m = 4, 6, 8$ and 10 are shown in Table 2 and the absolute error function $|u(x,t) - u_{13,13}(x,t)|$ obtained by the present method is shown in Figure 4. As the previous example, it is seen from Table 2 and Figures 2, 3 and 4 that we can achieve a very good approximation with the exact solution using a few terms of the shifted Legendre polynomials.

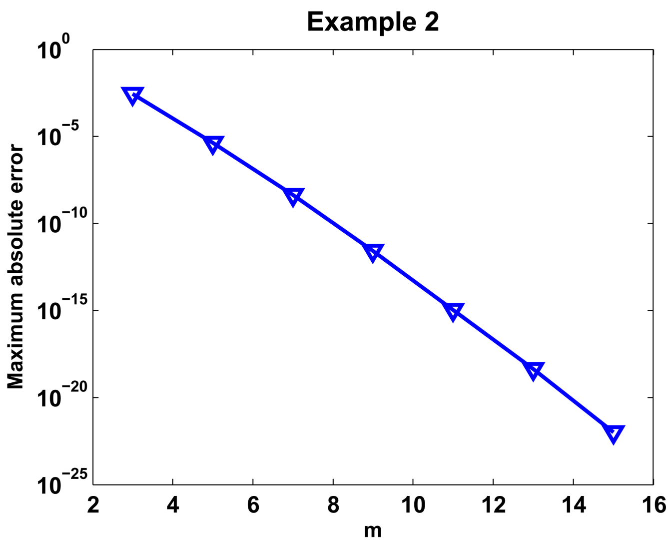

Fig. 3 – Comparison of $e_{m,n}$ with $n = 3$ and for different values of m.

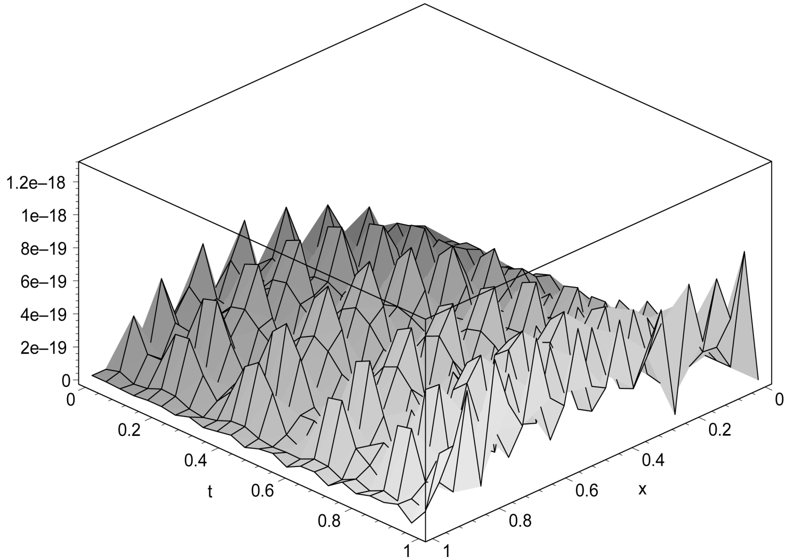

Fig. 4 – Plot of error function, $|u(x,t) - u_{13,13}(x,t)|$, from Example 2.

Table 2

The absolute errors for $u(x,1)$ from Example 2

x	$m = 4$	$m = 6$	$m = 8$	$m = 10$
0.1	1.52E-05	2.18E-08	2.23E-13	7.83E-15
0.2	2.16E-05	4.38E-09	1.56E-11	6.06E-15
0.3	1.56E-05	2.36E-08	1.01E-11	2.54E-15
0.4	2.49E-06	1.23E-08	1.50E-11	9.27E-15
0.5	9.93E-06	1.13E-08	8.32E-12	4.04E-15
0.6	1.56E-05	1.96E-08	1.39E-11	5.56E-15
0.7	1.27E-05	5.63E-09	5.65E-12	6.72E-15
0.8	4.02E-06	1.05E-08	9.09E-12	1.59E-15
0.9	4.54E-06	6.22E-09	6.17E-12	1.48E-15

5. CONCLUSION

In the present work, we proposed a numerical scheme, based on the shifted Legendre Tau method, to solve the time-fractional telegraph equation. Using the operational matrix of fractional derivative the problem can be reduced to a set of linear algebraic equations. The solution obtained using the suggested method shows that this approach can solve the problem effectively and it needs less CPU time. Two examples are given and the numerical results demonstrate the reliability and efficiency of the method for solving this type of problem.

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