SOME RESULTS ON $C(X)$ WITH SET OPEN TOPOLOGY

ABDERRAHMANE BOUCHAIR and SMAIL KELAIAIA

Communicated by Dan Crişan

Let X and Y be topological spaces, and λ and β be two nonempty families of compact subsets of X and Y, respectively. Let $C_\lambda(X)$ denote the set of all continuous real valued functions on X endowed with the set open topology defined by λ. Let $\Phi : X \to Y$ be a continuous function and $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ be its induced function. In this paper, the continuity and openness of Φ^* are studied. Also the weak α-favorability and Baireness of $C_\lambda(X)$ are characterized.

AMS 2010 Subject Classification: 54C35, 91A44.

Key words: function space, set open topology, induced function, topological game.

1. INTRODUCTION

For a topological space X, let $C(X)$ be the set of all continuous real-valued functions on X and λ a nonempty family of subsets of X. The set open topology on $C(X)$ has a subbase consisting of the sets $[A, V] = \{f \in C(X) : f(A) \subseteq V\}$, where $A \in \lambda$ and V is an open subset of $\mathbb{R}$, and the function space $C(X)$ endowed with this topology is denoted by $C_\lambda(X)$. The set open topology is a generalization of the compact-open topology and the topology of pointwise convergence. This topology was first introduced by Arens and Dugundji in [1] and studied over the last years by many authors, see [11, 13, 16–18].

A useful tool normally used for studying function spaces is the concept of induced function. For topological spaces X and Y, every continuous function $\Phi : X \to Y$ induces a function $\Phi^* : C(Y) \to C(X)$, called the induced function of Φ, defined by $\Phi^*(f) = f \circ \Phi$ for each $f \in C(Y)$. In ([13], Theorem 2.1), McCoy and Ntantu stated the following result: Let $\Phi : X \to Y$ be a continuous function, and let λ and β be families of compact subsets of X and Y, respectively. Then $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ is continuous if and only if β approximates $\Phi(\lambda)$. Also $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ is open onto its image if and only if $\Phi(\lambda)$ approximates $\beta \cap \overline{\Phi(X)}$. We will give an example (see Section 4) to show that the necessary conditions in this theorem, in general, are not true. However, as we will prove, they are correct in the presence of additional assumptions.

Completeness properties in a topological space range from complete metrizability to the Baire space property. (Complete) metrizability and Čech completeness of $C_\lambda(X)$ were characterized in [15] and necessary conditions for Baireness of $C_\lambda(X)$ are also given. Another completeness property that a topological space may have, stronger to the Baire property, is the property of being a weakly α-favorable space. This property is defined by means of a topological game. In the case of $C(X)$ endowed with the compact-open topology (the corresponding space is denoted by $C_k(X)$), weak α-favorability and Baireness of $C_k(X)$ were investigated in [6, 12, 14]. In particular, Ma in his paper ([12], Theorem 1.2) showed that, for a locally compact space X, $C_k(X)$ is weakly α-favorable if and only if X is paracompact. It remains an unsolved problem to obtain a complete characterization of α-favorability of $C_\lambda(X)$. In the present paper, we will be concerned to characterize the weak α-favorability and Baireness of $C(X)$ when it is equipped with the set-open topology.

We end this introduction with an outline of the paper. In Section 2, we give some basic definitions and properties of the families λ needed in the next sections. In Section 3, we introduce several infinite topological games played on a topological space X and present some results in this frame. In Section 4, we give a new proof of the result of McCoy-Ntantu mentioned above concerning the continuity and openness of Φ^*. We show that, if λ is a family of compact subsets of X and β is an admissible family of compact subsets of Y, then the continuity of $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ implies that β approximates $\Phi(\lambda)$. We also show that, if λ and β are admissible families of compact subsets of X and Y respectively, then $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ is open onto its image if and only if $\Phi(\lambda)$ approximates $\beta \cap \overline{\Phi(X)}$. The last section of this paper is reserved to weak-α-favorability and Baireness of $C_\lambda(X)$. The main result of this section is Theorem 11 which states that, if λ is an admissible family of compact subsets of X such that each point of X admits a member of λ as a neighborhood, then $C_\lambda(X)$ is weakly α-favorable if and only if X is paracompact. We then use this result to obtain a characterization for Baireness of $C_\lambda(X)$ when X is a paracompact q-space, extending by this a result obtained by McCoy and Ntantu in the framework of the compact open topology ([15], Corollary 5.3.4).

2. DEFINITIONS AND NOTATION

Let λ and β be two nonempty families of subsets of X, we recall the following definitions which will be used throughout the paper.

We say that λ *refines* or that it is a refinement of β if every member of λ is contained in some member of β. If λ refines β, then the family β is called an λ-*cover* of X.

We say that β *approximates* λ or that λ can be approximated by β provided that for every $A \in \lambda$ and every open U in X such that $A \subseteq U$, there exist $B_1, ..., B_n \in \beta$ with $A \subseteq \bigcup_{i=1}^{n} B_i \subseteq U$.

A family λ is said to be *admissible* if for every $A \in \lambda$ and every finite sequence $U_1, U_2, ..., U_n$ of open subsets of X such that $A \subseteq U_1 \cup ... \cup U_n$, there exists a finite sequence $A_1, ..., A_m$ of members of λ which refines $U_1, U_2, ..., U_n$ and whose union contains A. This notion was introduced by McCoy and Ntantu [13] in the characterization of countability properties of function spaces with set open topologies.

In view of the results obtained in [10, 13], the property being admissible for a family of subsets of X proves to be useful in the study of certain topological properties of $C_\lambda(X)$.

It would be useful to mention that if Z is a subset of X, then the set $C(Z)$ may be endowed with the set-open topology generated by the subbase $\{[A \cap Z, V] : A \in \lambda$ and V is an open subset of $\mathbb{R}\}$, and it will be denoted by $C_{\lambda \cap Z}(Z)$. In addition, if λ is an admissible family of subsets in X, then the family $\lambda \cap Z = \{A \cap Z : A \in \lambda\}$ in Z, endowed with its subspace topology, is also an admissible family. Note that, if λ is replaced by the family of all finite unions of its elements then the topology of $C_\lambda(X)$ does not change; therefore we can always assume that the family λ is closed under finite unions, *i.e.*, if $A, B \in \lambda$, then $A \cup B \in \lambda$.

Throughout this paper all spaces are assumed to be completely regular and Hausdorff, $C(X)$ is the set of all continuous real-valued functions on a topological space X, and λ is always a non-empty family of subsets of X. The symbols $\emptyset$ and $\mathbb{N}$ will stand for the empty set and the positive integers, respectively. We denote by $\mathbb{R}$ the real numbers with the usual topology. The complement and the closure of a subset A in X is denoted by A^c and $\overline{A}$, respectively. If $A \subseteq X$, the restriction of a function $f \in C(X)$ to the set A is denoted by $f_{|A}$. Notations not defined in this paper can be found in [4].

3. **TOPOLOGICAL GAMES**

In this section, we will give several infinite topological games played on a topological space X between two players I and II. Let λ be a nonempty family of subsets of X including the empty set.

The first game, called the Banach-Mazur game $BM(X)$ [3, 8, 14], is defined as follows: Player I starts the game by choosing a non-empty open subset U_0 of X, then player II chooses a non-empty open subset $V_0 \subseteq U_0$. At the nth step $(n \geq 1)$, player I selects a non-empty open subset $U_n \subseteq V_{n-1}$

then II answers by choosing a non-empty open subset $V_n \subseteq U_n \subseteq V_{n-1}$. We say that II wins the game if $\cap\{U_n : n \in \mathbb{N}\} \neq \emptyset$, otherwise player I wins. The Banach-Mazur game is also called Choquet game in [9], but we use here the former terminology.

The second game, denoted by $\Gamma^1_\lambda(X)$, is played on X as follows: Player I begins by choosing some $A_0 \in \lambda$, then player II takes $B_0 \in \lambda$. At the nth play ($n \geq 1$), player I chooses $A_n \in \lambda$ such that $A_n \cap (B_1 \cup ... \cup B_{n-1}) = \emptyset$, then II takes $B_n \in \lambda$. Player II wins the play if $\{A_n : n \in \mathbb{N}\}$ is a discrete family in X, otherwise player I wins. Recall that a family $\mathcal{A}$ of subsets of X is discrete if each point of X has a neighborhood which meets at most one member of $\mathcal{A}$. It should be mentioned that if λ is the family of all compact subsets of X then $\Gamma^1_\lambda(X)$ is the McCoy-Ntantu's game introduced in [14] in order to study completeness properties of $C(X)$ with compact-open topology.

The last game $G^*(X)$ introduced by Gruenhage in [5] is played as follows: players I and II (called K and L, respectively, in [5]) take turns in choosing compact subsets of X. At the nth play, player I chooses a compact subset K_n of X, and player II responds by choosing a compact subset L_n of X such that $L_n \cap K_n = \emptyset$. Player I wins the game if $\{L_n : n \in \mathbb{N}\}$ is a locally finite family in X.

A strategy in these games for player I is a rule which tells him what sets to select given all the previous sets chosen by the opponent. A winning strategy for a player is a strategy such that he/she wins all plays of the game according to this strategy.
Recall that a topological space is a Baire space if the intersection of any sequence of dense open subsets of X is dense. It is well known that a topological space X is a Baire space if and only if player I has no winning strategy in the game $BM(X)$. This result was first discovered by Oxtoby [19] (for the proof see, for example, [7], Theorem 3.16 or [20], Theorems 1 and 2).

Definition 1. A topological space X is weakly α-favorable (also called Choquet space in [9]) if player II has a winning strategy in the Banach-Mazur game $BM(X)$.

It is clear that weakly α -favorable spaces are Baire spaces. However the converse is not true in general (see, for example, [12], Section 5).
It is also well known [21] that the property of being weakly α -favorable is productive, *i.e.*, $\prod_{j \in J} X_j$ is a weakly α-favorable space if X_j so is for each $j \in J$. The following result, which will be used later, gives us a necessary condition for weak α-favorability of $C_\lambda(X)$.

Proposition 1 ([2]). *Let X be a topological space, and let λ be an admissible family of compact subsets of X. If $C_\lambda(X)$ is weakly α-favorable, then* *player II has a winning strategy in the game* $\Gamma^1_\lambda(X)$.

Gruenhage [5] proved the following characterization of paracompactness for the class of locally compact spaces.

THEOREM 1. *Let X be a locally compact space. Then X is paracompact if and only if player I has a winning strategy in $G^*(X)$.*

The following lemma will be used in the proof of Proposition 2.

LEMMA 1. *Let λ be an admissible family of compact subsets of X. If $A, K \in \lambda$, then there exists $C \in \lambda$ such that $A \setminus K \subseteq C$ and $C \cap K = \emptyset$.*

Proof. If $A \subseteq K$, we take $C = \emptyset$, otherwise let $x \in A \setminus K$. The family $\{\{x\}^c, K^c\}$ is an open cover of A. By admissibility of λ, there exists a finite sequence $A_1, ..., A_m$ of elements of λ which refines $\{\{x\}^c, K^c\}$ and whose union contains A. We set $C = \cup\{A_i : A_i \subseteq K^c\}$. It is clear that $A \setminus K \subseteq C$ and $C \cap K = \emptyset$. □

PROPOSITION 2. *Let X be a topological space, and let λ be an admissible family of compact subsets of X. If the family of all compact subsets of X refines λ, then player II has a winning strategy in the game $\Gamma^1_\lambda(X)$ implies player I has a winning strategy in $G^*(X)$.*

Proof. Let τ be a winning strategy for player II in $\Gamma^1_\lambda(X)$. Define a strategy σ for player I in the game $G^*(X)$ as follows. Suppose first that I chooses $K_0 = \emptyset$ and player II answers by a compact L_0 in the game $G^*(X)$. Since the compact sets refine λ, then there exists an element A_0 of λ which contains L_0. As τ is a winning strategy for player II in $\Gamma^1_\lambda(X)$, then whatever is the play of player I in this game, player II must win if he moves according to τ. So let A_0 be I's first move in the game $\Gamma^1_\lambda(X)$. Define $K_1 = \sigma(L_0) = \tau(A_0)$. For the nth step ($n \geq 1$), suppose that we have the compact sets K_n, L_n such that $K_n \cap L_n = \emptyset$ and $A_0, \tau(A_0), ..., A_n, \tau(A_0, ..., A_n) \in \lambda$ have been defined, representing a partial play of the game $\Gamma^1_\lambda(X)$, so that each $A_i \in \lambda$ contains L_i with $A_i \cap K_i = \emptyset$ and $K_i = \tau(A_0, ..., A_{i-1}) \cup K_{i-1}$. We set $K_{n+1} = \tau(A_0, ..., A_n) \cup K_n$ and define $\sigma(L_0, ..., L_n) = K_{n+1}$. Let L_{n+1} be II's next choice. Player I selects $A_{n+1} \in \lambda$ which contains L_{n+1} and, according to Lemma 1, such that $A_{n+1} \cap K_{n+1} = \emptyset$. Then player II answers by choosing an element $\tau(A_0, ..., A_{n+1})$ of λ. Since τ is a winning strategy, then the family $\{A_n : n \in \mathbb{N}\}$ is discrete in X and hence, $\{L_n : n \in \mathbb{N}\}$ is a locally finite family. Thus, $L_0, K_0, ..., L_n, K_n, ...$ is a winning play for player I in the game $G^*(X)$. Hence, σ is a winning strategy for player I in $G^*(X)$. □

4. INDUCED FUNCTIONS ON THE FUNCTION SPACE

Let X and Y be topological spaces. Every continuous function $\Phi : X \to Y$ induces a function $\Phi^* : C(Y) \to C(X)$, called the induced function, defined by $\Phi^*(f) = f \circ \Phi$ for each $f \in C(Y)$. This concept is a useful tool usually used for studying function spaces. In this section we study the continuity and openness of Φ^* when $C(X)$ and $C(Y)$ are endowed with set open topologies defined by two families λ and β of X and Y, respectively. If λ is a family of subsets of X and if $\Phi : X \to Y$ is a function, we denote $\Phi(\lambda) = \{\Phi(A) : A \in \lambda\}$.

In [13], McCoy and Ntantu anounced the following result.
Let $\Phi : X \to Y$ be a continuous function, and let λ and β be families of compact subsets of X and Y, respectively.

(1) Then $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ is continuous if and only if β approximates $\Phi(\lambda)$.

(2) Also $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ is open onto its image if and only if $\Phi(\lambda)$ approximates $\beta \cap \overline{\Phi(X)}$.

The necessary conditions in this result, in general, are not true as is shown by the example given below. However, we will give a new proof of this result under additional assumptions.

Example 1. Suppose $X = Y = \mathbb{R}$ and λ the family of all finite subsets of $\mathbb{R}$, the two intervals $[0, \frac{1}{2}]$ and $[0, 1]$, and the set $[\frac{1}{2}, 1] \cup \{2\}$. Take $\beta = \lambda \setminus [0, 1]$. Then β does not approximate λ, because there is no finite sequence $B_1, ..., B_n \in \beta$ such that the relation $[0, 1] \subseteq B_1 \cup B_2 \cup ... \cup B_n \subseteq]-\frac{1}{2}, \frac{3}{2}[$ holds. On the other hand, we have $C_\lambda(X) = C_\beta(X)$, see ([16], Example 2.6). Let $\Phi = id$. We have then $\Phi^* : C_\beta(X) \to C_\lambda(X)$ is continuous without that β approximates $\Phi(\lambda) = \lambda$, and $\Phi^* : C_\lambda(X) \to C_\beta(X)$ is open without that $\Phi(\beta) = \beta$ approximates $\lambda \cap \overline{\Phi(X)} = \lambda$.

We state the following theorem (Theorem 1.4.2 in [10]) which gives a necessary and sufficient criterion for the set open topology of $C_\beta(X)$ to be finer than the set-open topology of $C_\lambda(X)$. This result will be used in the proof of Theorem 3.

THEOREM 2 ([10]). *Let λ and β be two families of compact subsets of X, such that β is admissible. Then the set-open topology of $C_\beta(X)$ is finer than the set-open topology of $C_\lambda(X)$ if and only if β approximates λ.*

THEOREM 3. *Let λ and β be two families of compact subsets of X and Y respectively, and let $\Phi : X \to Y$ be a continuous mapping. If β approximates $\Phi(\lambda)$ then $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ is continuous. If, in addition, β is admissible the converse is true.*

Proof. The proof of the first part is given in ([13], Theorem 2.1). For the proof of the converse case, from Theorem 2, it is enough to prove that the topology of $C_\beta(Y)$ is finer than the topology of $C_{\Phi(\lambda)}(Y)$. To do this, let $A \in \lambda$, V be open in $\mathbb{R}$ and $f \in [\Phi(A), V]$. We have $\Phi^*(f) \in [A, V]$. Since Φ^* is continuous, there exist $B_1, ..., B_n \in \beta$ and $V_1, ..., V_n$ open subsets in $\mathbb{R}$ such that

$$f \in \bigcap_{i=1}^{n} [B_i, V_i] \subseteq (\Phi^*)^{-1}([A, V]) \subseteq [\Phi(A), V].$$

This means that the topology of $C_\beta(Y)$ is finer than the topology of $C_{\Phi(\lambda)}(Y)$. So β approximates $\Phi(\lambda)$. □

For the study of the openness of Φ^* onto its image, we begin with the particular case when X is a subspace of Y and Φ is the inclusion mapping. We need the following lemma given in [10].

LEMMA 2. *Let Y be a topological space, X a subspace of Y, β a family of compact subsets of Y such that $B \cap X = B \cap \overline{X}$ for each $B \in \beta$ and let $g \in C(X)$ be a function extendable to a continuous function over Y. Let $B_1, ..., B_n \in \beta$ and $V_1, ..., V_n$ be bounded open intervals in $\mathbb{R}$ such that $g\left(B_i \cap \overline{X}\right) \subseteq V_i$ for each $i = 1, ..., n$. Then, there exists $g' \in C(Y)$ an extension of g such that $g'(B_i) \subseteq V_i$ for each $i = 1, ..., n$.*

Recall that the standard base of the set open topology consists of all sets of the form $\cap_{i=1}^{n}[A_i, V_i]$, where $A_i \in \lambda$ and V_i is open in $\mathbb{R}$ for each $1 \leq i \leq n$. It is shown in ([10], Theorem 1.2.2) that, if λ is an admissible family of compact subsets of X, then the topology of $C_\lambda(X)$ does not change if the V_i are not arbitrary but belong to the collection of all open bounded intervals of $\mathbb{R}$. This allows us, for some families λ, to have only sets of the form $[A, V]$, where $A \in \lambda$ and V is an open bounded interval in $\mathbb{R}$, in the study of $C_\lambda(X)$.

THEOREM 4. *Let Y be a topological space, X a subspace of Y, β an admissible family of compact subsets of Y and λ a family of compact subsets of X. Let $i : X \to Y$ be the inclusion mapping. If λ approximates $\beta \cap \overline{X}$ then $i^* : C_\beta(Y) \to C_\lambda(X)$ is open onto its image.*

Proof. Suppose that λ approximates $\beta \cap \overline{X}$. Let us first show that $B \cap X = B \cap \overline{X}$ for every $B \in \beta$. Assume the contrary, then there is $B \in \beta$ such that $B \cap (\overline{X} \setminus X) \neq \emptyset$. Take a point x in $B \cap (\overline{X} \setminus X)$. Then x does not belong to any $A \in \lambda$. Thus, $B \cap \overline{X}$ can be in no finite union of elements of λ. This means that λ does not approximate $\beta \cap \overline{X}$, a contradiction. Hence, $B \cap X = B \cap \overline{X}$ for every $B \in \beta$. Now, let $\cap_{i=1}^{n}[B_i, V_i]$ be a basic open subset of $C_\beta(Y)$ and $f \in i^*(\cap_{i=1}^{n}[B_i, V_i])$. Let $f' \in C(Y)$ be an extension of f over Y such that

$f' \in \cap_{i=1}^{n}[B_i, V_i]$. Since λ approximates $\beta \cap \overline{X}$, there exists, for each $i = 1, ..., n$, a finite subfamily λ_i of the family λ such that:

$$B_i \cap \overline{X} \subseteq \bigcup \{A : A \in \lambda_i\} \subseteq f^{-1}(V_i).$$

Then

$$f \in \left(\bigcap_{i=1}^{n} \bigcap_{A \in \lambda_i} [A, V_i] \right) \bigcap i^* (C(Y)) = W.$$

We claim that $W \subseteq i^* (\cap_{i=1}^{n} [B_i, V_i])$. To justify our claim, let $g \in W$. Because $g(\bigcup \{A : A \in \lambda_i\}) \subseteq V_i$, then $g(B_i \cap \overline{X}) \subseteq V_i$ for every $i = 1, ..., n$. Also we have $B \cap X = B \cap \overline{X}$ for every $B \in \beta$, as it is shown above. Then, by Lemma 2, there exists a function $g' \in C(Y)$ which agrees with g on X and belongs to $\cap_{i=1}^{n}[B_i, V_i]$. We have then $g \in i^* (\cap_{i=1}^{n} [B_i, V_i])$. Hence, $W \subseteq i^* (\cap_{i=1}^{n} [B_i, V_i])$. It follows that i^* is open onto its image. $\square$

For a family λ of subsets of X we consider the family

$$\lambda' = \{A'/A' \text{compact of } X \text{ and } \exists A \in \lambda : A' \subseteq A\}.$$

The following lemmas are needed in the proof of the next theorem.

Lemma 3. *Let λ be a family of compact subsets of X. Then the following statements are true:*
(1) *The family λ' is always admissible.*
(2) *λ is admissible if and only if λ approximates λ'.*

Proof. (1) Let $A' \in \lambda'$ and let $\{U_1, ..., U_n\}$ be an open cover of A'. There exists a finite sequence $K_1, ..., K_m$ of compact subsets of X which refines $\{U_1, ..., U_n\}$ and whose union contains A'. Let $A \in \lambda$ such that $A' \subseteq A$. We have $A' \subseteq (\bigcup_{i=1}^{m} K_i) \cap A = \bigcup_{i=1}^{m} (K_i \cap A)$. Put $A'_i = K_i \cap A$ for $1 \le i \le m$. The family $A'_1, ..., A'_m$ of members of λ' covers A' and refines $U_1, ..., U_n$. Hence, λ' is admissible.

(2) Suppose that λ is admissible. We will show that λ approximates λ'. Let $A' \in \lambda'$ and U be an open subset of X such that $A' \subseteq U$. Let $A \in \lambda$ with $A' \subseteq A$. If $A \subseteq U$ the proof is over, otherwise suppose that $A \setminus U \neq \emptyset$. Then $\{U, A'^c\}$ is an open cover of A. By admissibility of λ, there are $A_1, ..., A_n$ members of λ which refine $\{U, A'^c\}$ and whose union contains A. Put $\mathcal{I} = \{i : A_i \subseteq U\}$. Then, we have $A' \subseteq \bigcup_{i \in \mathcal{I}} A_i \subseteq U$. This means that λ approximates λ'.

Conversely, suppose that λ approximates λ' and let us show that λ is admissible. Let $A \in \lambda$ and $\{U_1, ..., U_n\}$ be a family of open subsets of X which covers A. There exists a finite sequence $K_1, ..., K_m$ of compact subsets of X which refines $U_1, ..., U_n$ and whose union contains A. Put $A'_i = K_i \cap A$ for

$1 \leq i \leq m$. Then the family $\{A'_1, ..., A'_m\} \subseteq \lambda'$ covers A and refines $\{U_1, ..., U_n\}$. For each $j = 1, ..., m$, let us choose some i_j such that $A'_j \subseteq U_{i_j}$. By our assumption there is, for every $j = 1, ..., m$, a finite family $\mathcal{A}_j = \{A^j_1, ..., A^j_{m_j}\}$ of elements of λ such that $A'_j \subseteq \bigcup_{k=1}^{m_j} A^j_k \subseteq U_{i_j}$. We set $\mathcal{A} = \bigcup_{j=1}^{m} \mathcal{A}_j$, this is a finite family of elements of λ which covers A and refines $U_1, ..., U_n$. This completes the proof of the lemma. □

Lemma 4. *Let Y be a topological space, X a subspace of Y, and λ and β be two families of compact subsets of X and Y, respectively. Let $i : X \to Y$ be the inclusion mapping. If $i^* : C_\beta(Y) \to C_\lambda(X)$ is open onto its image then $B \cap X = B \cap \overline{X}$ for every $B \in \beta$.*

Proof. Suppose that, on the contrary, there is $B \in \beta$ such that $B \cap (\overline{X} \setminus X) \neq \emptyset$. Let $x \in B \cap (\overline{X} \setminus X)$, $V =]\frac{1}{3}, \frac{2}{3}[$, and $f \in i^*([B, V])$. We observe that $i^*(g) = g_{|X}$ for any $g \in C(Y)$. Then we have $f = f'_{|X}$ for some $f' \in [B, V]$. Since i^* is open onto its image, there exist $A_1, ..., A_n \in \lambda$ and $V_1, ..., V_n$ open subsets in $\mathbb{R}$ such that

$$f \in (\bigcap_{i=1}^{n} [A_i, V_i]) \cap i^*(C(Y)) \subseteq i^*([B, V]).$$

Since $x \in B \cap (\overline{X} \setminus X)$, we have $x \notin \cup_{i=1}^{n} A_i$. Complete regularity of Y gives us a function $h \in C(Y)$ with $h(x) = 0$ and $h(\cup_{i=1}^{n} A_i) = \{1\}$. Consider the function $h_1 = f'.h$. We have then $h_1(\cup_{i=1}^{n} A_i) \subseteq V$ and $h_1(x) = 0$. So $h_{1|X} \in (\bigcap_{i=1}^{n} [A_i, V_i]) \cap i^*(C(Y))$ and h_1 does not belong to $[B, V]$ as well as any continuous extension of $h_{1|X}$ over Y. This gives a contradiction. Hence, we must have $B \cap X = B \cap \overline{X}$ for every $B \in \beta$. □

The following result is the converse of Theorem 4.

Theorem 5. *Let Y be a topological space, X a subspace of Y, β a family of compact subsets of Y and λ an admissible family of compact subsets of X. Let $i : X \to Y$ be the inclusion mapping. If $i^* : C_\beta(Y) \to C_\lambda(X)$ is open onto its image then λ approximates $\beta \cap \overline{X}$.*

Proof. Note that, by Lemma 4, we have $B \cap X = B \cap \overline{X}$ for every $B \in \beta$. Let $B \in \beta$, G an open subset of X such that $B \cap \overline{X} = B \cap X \subseteq G$, and let G_1 be open in Y with $G_1 \cap X = G$. Now the subset $G_2 = G_1 \cup (Y \backslash \overline{X})$, which is open in Y, contains B and verifies $G_2 \cap X = G_1 \cap X = G$. Let $f : Y \to [0, 1]$ be a continuous function such that $f(B) = \{1\}$ and $f(Y \backslash G_2) = \{0\}$. Let us put $V =]\frac{1}{2}, \frac{3}{2}[$. We have $f^{-1}(V) \subseteq G_2$. Then $f^{-1}_{|X}(V) = f^{-1}(V) \cap X \subseteq G$.

Consider in $C_\beta(Y)$ the subbasic open subset $[B, V]$. We have then $f \in [B, V]$. Thus, $i^*(f) = f_{|X}$ belongs to $i^*([B, V])$ which is open in $i^*(C(Y))$, by our assumption. Therefore, there exist $A_1, ..., A_n$ members of λ and bounded open intervals $V_1, ..., V_n$ in $\mathbb{R}$ such that

$$f_{|X} \in \left(\cap_{i=1}^{n} [A_i, V_i]\right) \cap i^* \left(C\left(Y\right)\right) \subseteq i^* \left([B, V]\right).$$

This implies that $B \cap X \subseteq \cup_{i=1}^{n} A_i$. Indeed, suppose that there exists an $x_0 \in (B \cap X) \setminus \cup_{i=1}^{n} A_i$. Since Y is a completely regular space, we can find a continuous function $g : Y \longrightarrow [0, 1]$ such that $g(x_0) = 0$ and $g(\cup_{i=1}^{n} A_i) = \{1\}$. Then the function $h = f_{|X}.g_{|X}$ does not belong to $i^* \left([B, V]\right),$ because $h(x_0) = 0 \notin V$. But $h \in \left(\cap_{i=1}^{n} [A_i, V_i]\right) \cap i^* \left(C\left(Y\right)\right)$, which is a contradiction.
Now, put $A = A^1, A^c = A^0$ and $\mathcal{I} = \{1, ..., n\}$ and let

$$\triangle = \left\{(\delta_i)_{i\in\mathcal{I}} \in \{0, 1\}^n \setminus \{(0, ..., 0)\} : \left(\cap_{i\in\mathcal{I}} A_i^{\delta_i}\right) \cap \left(B \cap \overline{X}\right) \neq \phi\right\}.$$

It is clear that

$$B \cap \overline{X} = B \cap X \subseteq \cup \left\{\cap_{i=1}^{n} A_i^{\delta_i} \;/\; (\delta_i) \in \triangle\right\}.$$

We claim that $\cap_{\delta_i=1} V_i \subseteq V$ for each $(\delta_i) \in \triangle$ fixed. To prove our claim, arguing by contradiction. Let $t \in \cap_{\delta_i=1} V_i \setminus V$ for some $(\delta_i) \in \triangle$. We can assume $t > 1$ (the case $t < 1$ will be treated similarly). Pick $x_0 \in \left(\cap_{i\in\mathcal{I}} A_i^{\delta_i}\right) \cap \left(B \cap \overline{X}\right)$, then $x_0 \notin (\cup_{\delta_i=0} A_i)$ and $f(x_0) = 1 \in \cap_{\delta_i=1} V_i$. Let $\epsilon > 0$ with

$$\epsilon < \min(\inf\{\mid t - s \mid: s \in (\cap_{\delta_i=1} V_i)^c\}, \inf\{\mid 1 - s \mid: s \in (\cap_{\delta_i=1} V_i)^c\}).$$

By continuity of f, regularity of the space X, and the fact that $x_0 \notin (\cup_{\delta_i=0} A_i)$, we can take an open neighborhood U of x_0 such that $U \cap (\cup_{\delta_i=0} A_i) = \emptyset$ and $\mid f(y) - 1 \mid < \epsilon$ for any $y \in U$. Consider a continuous function $g : Y \to [0, t - 1]$ such that $g(x_0) = t - 1$ and $g(U^c) = \{0\}$, and put $h = f_{|X} + g_{|X}$. We shall now verify that $h \in \cap_{i=1}^{n} [A_i, V_i]$. If $A_i \cap U \neq \emptyset$, with $1 \leq i \leq n$, then $\delta_i = 1$ and for each $y \in A_i \cap U$, we have $h(y) = f(y) + g(y) \in]1 - \epsilon, t + \epsilon[\subseteq V_i$. If $y \in A_i \setminus U$, we have $h(y) = f(y) \in V_i$. Therefore $h \in \left(\cap_{i=1}^{n} [A_i, V_i]\right) \cap i^* \left(C\left(Y\right)\right)$. But $h \notin i^*([B, V])$, because $h(x_0) = f(x_0) + g(x_0) = t \notin V$. A contradiction. Hence, we have $\cap_{\delta_i=1} V_i \subseteq V$ for each $(\delta_i) \in \triangle$. So we obtain that

$$B \cap X \subseteq \cup \{\cap_{\delta_i=1} A_i \;/\; (\delta_i) \in \triangle\} \subseteq f_{|X}^{-1}(V).$$

Moreover, the admissibility of λ gives, by Lemma 3, that for each $(\delta_i) \in \triangle$, there exist $\lambda_{(\delta_i)}$ finite $\subseteq \lambda$ such that

$$\cap_{\delta_i=1} A_i \subseteq \cup \left\{A : A \in \lambda_{(\delta_i)}\right\} \subseteq f_{|X}^{-1}(V).$$

Hence,

$$B \cap \overline{X} = B \cap X \subseteq \bigcup_{(\delta_i)\in\triangle} \Big(\bigcup_{A\in\lambda_{(\delta_i)}} A\Big) \subseteq f_{|X}^{-1}(V) \subseteq G.$$

Thus, we have that λ approximates $\beta \cap \overline{X}$. $\square$

Theorems 4 and 5 give us the following.

THEOREM 6. *Let Y be a topological space, X a subspace of Y, and λ and β be two admissible families of compact subsets of X and Y, respectively. Let $i : X \to Y$ be the inclusion mapping. Then $i^* : C_\beta(Y) \to C_\lambda(X)$ is open onto its image if and only if λ approximates $\beta \cap \overline{X}$.*

Now we generalize Theorem 6 to the case where X and Y are arbitrary topological spaces and $\Phi : X \to Y$ is an arbitrary continuous function. To do this, let j be the mapping from X to $\Phi(X)$ defined by $j(x) = \Phi(x)$. Let $i : \Phi(X) \to Y$ be the inclusion mapping. We have $\Phi = i \circ j$, and so $\Phi^* = j^* \circ i^*$. We observe that j is continuous. If λ is a family of compact subsets of X then, from ([15], Theorem 2.2.7), $j^* : C_{\Phi(\lambda)}(\Phi(X)) \to C_\lambda(X)$ is an homeomorphism onto its image.

THEOREM 7. *Let $\Phi : X \to Y$ be a continuous function, λ a family of compact subsets of X and β an admissible family of compact subsets of Y. If $\Phi(\lambda)$ approximates $\beta \cap \overline{\Phi(X)}$ then $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ is open onto its image.*

Proof. Let W be an open subset of $C_\beta(Y)$. We have $\Phi^*(W) = j^*(i^*(W))$. From Theorem 4, the mapping i^* is open onto its image. Then there exists an open subset W_1 of $C_{\Phi(\lambda)}(\Phi(X))$ such that $\Phi^*(W) = j^*(W_1 \cap i^*(C(Y)))$, and hence,

$$\Phi^*(W) = j^*(W_1) \cap j^*(i^*(C(Y))) = W_2 \cap j^*(C(\Phi(X))) \cap \Phi^*(C(Y)),$$

where W_2 is an open subset of $C_\lambda(X)$, from the fact that j^* is an homeomorphism onto its image. Since $\Phi^*(C(Y)) \subseteq j^*(C(\Phi(X)))$, we obtain that $\Phi^*(W) = W_2 \cap \Phi^*(C(Y))$, and so, Φ^* is open onto its image. $\square$

THEOREM 8. *Let $\Phi : X \to Y$ be a continuous function, λ an admissible family of compact subsets of X and β a family of compact subsets of Y. If $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ is open onto its image then $\Phi(\lambda)$ approximates $\beta \cap \overline{\Phi(X)}$.*

Proof. Note that the admissibility of λ implies the admissibility of $\Phi(\lambda)$. Note also that j^*, defined as above, restricted to $i^*(\Phi(Y))$ is an homeomorphism from $i^*(\Phi(Y))$ to $j^*(i^*(\Phi(Y))) = \Phi^*(C(Y))$. Now, let W be an open subset of $C_\beta(Y)$. We have $i^*(W) = (j^*)^{-1}(\Phi^*(W))$. Since $\Phi^*(W)$ is open in $\Phi^*(C(Y))$, we obtain that $i^*(W)$ is open in $i^*(C(Y)) = (j^*)^{-1}(\Phi^*(C(Y)))$. Hence, i^* is open onto its image. It follows from Theorem 5 that $\Phi(\lambda)$ approximates $\beta \cap \overline{\Phi(X)}$. $\square$

THEOREM 9. *Let X and Y be topological spaces, λ and β be two admissible families of compact subsets of X and Y respectively, and $\Phi : X \to Y$ a* *continuous mapping. Then* $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ *is open onto its image if and only if* $\Phi(\lambda)$ *approximates* $\beta \cap \overline{\Phi(X)}$.

The following corollary is an immediate consequence of Theorems 3 and 9.

Corollary 1. *Let* X *and* Y *be topological spaces,* λ *and* β *be two admissible families of compact subsets of* X *and* Y *respectively, and* $\Phi : X \to Y$ *a continuous mapping such that* $\Phi(X)$ *is dense in* Y*. Then* $\Phi^* : C_\beta(Y) \to C_\lambda(X)$ *is an embedding if and only if each of* $\Phi(\lambda)$ *and* β *approximates the other.*

5. COMPLETENESS PROPERTIES OF $C_\lambda(X)$

In this section, we are going to characterize the weak-α-favorability and Baireness of $C(X)$ equipped with the set-open topology. Let X be a topological space and let λ be a family of compact subsets of X. Following [15] and [10], the space X is called a λ-*space* provided that every subset of X is closed whenever its intersection with any member of λ is closed. The space X is called λ-*hemicompact* if λ contains a countable λ-cover of X.

Proposition 3. *If each point of* X *admits a member of* λ *as a neighborhood, then* X *is a* λ*-space.*

Proof. Let A be a subset of X such that the intersection of A with any member of λ is closed. Suppose that A is not closed, let $x \in \overline{A} \setminus A$ and $U \in \lambda$ be a neighborhood of x. For every neighborhood V of x, we have $(V \cap U) \cap A \neq \emptyset$. Then $x \in \overline{U \cap A} \setminus U \cap A$, and hence, $U \cap A$ is not closed which is a contradiction. □

Before giving the next theorem which is due to McCoy and Ntantu, let d be the usual metric on $\mathbb{R}$ bounded by 1, *i.e.*, $d(x, y) = \min\{1, |x - y|\}$. Then $C_d(X)$ will denote $C(X)$ with the topology generated by the metric $d^*(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$, which is a complete metric. It would be useful to mention that if λ is an admissible family and $A \in \lambda$, then $C_{\lambda\cap A}(A) = C_d(A)$ (see [13], Lemma 2.4).

Theorem 10. *Let* X *be a topological space, and let* λ *be an admissible family of compact subsets of* X *such that* $\cup\{A : A \in \lambda\} = X$*. If* X *is a* λ*-hemicompact* λ*-space, then* $C_\lambda(X)$ *is completely metrizable (and hence, weakly* α*-favorable).*

Proof. Let $\{A_n : n \in \mathbb{N}\} \subseteq \lambda$ be an λ-cover of X. For each A_n, consider the product space $A'_n = A_n \times \{n\}$ which is naturally homeomorphic to A_n. Let Y be the topological sum of the A'_n's, and let $p : Y \to X$ be the natural projection. Since X is a λ-space, then p is a quotient map. Let $p^* : C_\lambda(X) \to C_\beta(Y)$ be the induced map, where $\beta = \{(A \times \{n\}) \cap A'_n \neq \emptyset : A \in \lambda, n \in \mathbb{N}\}$. It easy to verify that the family β is admissible. The map p^* is one-to-one because p is onto. In addition, each of λ and $p(\beta)$ approximates the other, this follows immediately from the fact that $\lambda \subseteq p(\beta) \subseteq \lambda^{'}$ and applying Lemma 3. It follows from Theorems 3 and 7 that p^* is continuous and open onto its image; thus, is an embedding onto its image.

Now we want to prove the following claim: $p^*(C_\lambda(X))$ is closed in $C_\beta(Y)$.

Let $g \in C_\beta(Y) \setminus p^*(C_\lambda(X))$. Let us show that there exists $y, y' \in Y$ such that $g(y) \neq g(y')$ and $p(y) = p(y')$. Assume the contrary. Then $p(y) = p(y')$ implies $g(y) = g(y')$ for all $y, y' \in Y$. Consider the function $h : X \to \mathbb{R}$ defined by $h(x) = g(y)$, where y is an arbitrary element in $p^{-1}(x)$. This is well defined since g is constant on $p^{-1}(x)$ for all $x \in X$. We have then $g = h \circ p$. Let W be an open subset of $\mathbb{R}$, then $p^{-1}(h^{-1}(W)) = g^{-1}(W)$. By continuity of g and quotientness of h, it results that $h^{-1}(W)$ is open in X and hence, h is continuous. Since $g = p^*(h)$, then $g \in p^*(C_\lambda(X))$ which is a contradiction.

Now, let $y, y' \in Y$ such that $g(y) \neq g(y')$ and $p(y) = p(y')$. Let U and U' be disjoint open neighborhoods of $g(y)$ and $g(y')$ in $\mathbb{R}$, respectively. Then $g^{-1}(U)$ and $g^{-1}(U')$ are open disjoint subsets of Y. Since β covers Y, there are B, B' members of β such that $y \in B \cap g^{-1}(U)$ and $y' \in B' \cap g^{-1}(U')$. By admissibility of β, there exist $B_1, B'_1 \in \beta$ such that $y \in B_1 \subseteq g^{-1}(U)$ and $y' \in B'_1 \subseteq g^{-1}(U')$. It is clear then that $[B_1, U_1] \cap [B'_1, U'_1]$ is a neighborhood of g in $C_\beta(Y)$ with $p^*(C_\lambda(X)) \cap ([B_1, U_1] \cap [B'_1, U'_1]) = \emptyset$. This proves the claim. Finally, since $C_\beta(Y)$ is homeomorphic to the product $\prod\{C_{\beta \cap A'_n}(A'_n) : n \in \mathbb{N}\}$ of completely metrizable spaces, then $p^*(C_\lambda(X))$ is completely metrizable and hence, $C_\lambda(X)$ is completely metrizable. $\square$

The forthcoming theorem is the main result of this section.

THEOREM 11. *Let X be a topological space, λ an admissible family of compacts subsets of X. If each point of X admits a member of λ as a neighborhood, then $C_\lambda(X)$ is weakly α-favorable if and only if X is paracompact.*

Proof. Suppose that $C_\lambda(X)$ is weakly α-favorable. By Proposition 1, player II has a winning strategy in the game $\Gamma^1_\lambda(X)$. Then, by Proposition 2, player I has a winning strategy in the game $G^*(X)$, which in turn is equivalent to X being paracompact from Theorem 1.

Conversely, suppose that X is paracompact. First, we prove that X can be written as a topological sum of λ_j-hemicompact λ_j-spaces, where λ_j are subfamilies of λ that will be defined during the proof. For each $x \in X$, let Ω_x be an open subset of X, and $K_x \in \lambda$ such that $x \in \Omega_x \subseteq K_x$. then $\{\Omega_x : x \in X\}$ is an open cover for X. Since X is paracompact, there is $\mathcal{U} = \{U_i : i \in \mathcal{I}\}$ an open locally finite refinement of $\{\Omega_x : x \in X\}$. We claim that every member of $\mathcal{U}$ meets only finitely many members of $\mathcal{U}$. Indeed, let $U_0 \in \mathcal{U}$ and K_{x_0} a member of λ with $U_0 \subseteq K_{x_0}$. Every point of K_{x_0} has an open neighborhood meeting only finitely many members of $\mathcal{U}$. Because K_{x_0} is compact, then it is covered by a finite number of these neighborhoods. Then K_{x_0}, in particular U_0, intersects only finitely many members of $\mathcal{U}$. Now, define on the set of indices $\mathcal{I}$ an equivalence relation as follows: two elements i and j of $\mathcal{I}$ are equivalent if and only if there exists a finite sequence of indices, $i_0 = i, i_1, ..., i_n = j$, such that $U_{i_k} \cap U_{i_{k+1}} \neq \emptyset$ for $k = 0, ..., n-1$. Let $\{\mathcal{I}_j : j \in \mathcal{J}\}$ be the family of equivalence classes under this relation. Each $\mathcal{I}_j$ is countable. Put $X_j = \bigcup_{i \in \mathcal{I}_j} U_i$. Since two arbitrary sets X_{j_1}, X_{j_2} either coincide or are disjoint, then all sets X_j are open and closed. Put $\lambda_j = \{A, A \in \lambda : A \subseteq X_j\}$, then we have $\lambda = \cup\{\lambda_j : j \in \mathcal{J}\}$. Thanks to disjointness of the spaces X_j, every family λ_j is admissible and each point of X_j has a neighborhood from λ_j. Hence, each space X_j is λ_j-space by Proposition 3, and is also λ_j-hemicompact . In fact, for every $i \in \mathcal{I}_j$ there is $K_i \in \lambda$ such that $U_i \subseteq K_i$. For $n \in \mathbb{N}$, let $A_n = \cup_{i \leq n} K_i$. It easy to see that $\{A_n : n \in \mathbb{N}\}$ is an λ_j-cover of X_j.

To complete the proof of the theorem, according to Theorem 10, X_j is weakly α-favorable space for every $j \in \mathcal{J}$. Since $C_\lambda(X)$ is homeomorphic to the product space $\prod\{C_{\lambda_j}(X_j) : j \in \mathcal{J}\}$ and a product of weakly α-favorable spaces is weakly α-favorable, then $C_\lambda(X)$ is weakly α-favorable. □

Corollary 2. *Let X be a topological space, and let λ be an admissible family of compact subsets of X verifying the property that any point of X admits a member of λ as a neighborhood. Then the following are equivalent.*
(1) *$C_\lambda(X)$ is completely metrizable;*
(2) *$C_\lambda(X)$ is weakly α-favorable;*
(3) *X is paracompact;*
(4) *X is λ-hemicompact.*

Proof. (1) ⇒ (2) is immediate.
(2) ⇒ (3) follows from Theorem 11.
(4) ⇒ (1) follows from Theorem 10.
(3) ⇒ (4) If X is a paracompact space, then it is a topological sum of spaces X_j, $j \in \mathcal{J}$, as it is shown above. Each space X_j is λ_j-hemicompact, where λ is the disjoint union of the families λ_j. Clearly X is λ-hemicompact. □

The problem of finding a property $\mathcal{P}$ of X such that $C_\lambda(X)$ is a Baire space if and only if X has $\mathcal{P}$ appears very difficult. Now we apply the above results to obtain a characterization for Baireness of $C_\lambda(X)$ in the special case when X is paracompact q-space. The following theorem, in which the proof is essentially the same to the one in ([11], Theorem 7), gives a necessary condition for $C_\lambda(X)$ to be a Baire space when X is a q-space. Before stating this result we recall that a space X is a q-space if for each point $x \in X$ there is a sequence $\{U_n : n \in \mathbb{N}\}$ of open neighborhoods of x such that whenever $x_n \in U_n$ for each n, the set $\{x_n : n \in \mathbb{N}\}$ has a cluster point.

Theorem 12. *Let X be a q-space, and let λ be an admissible family of compact subsets of X such that $\cup\{K : K \in \lambda\} = X$. If $C_\lambda(X)$ is a Baire space, then each point of X has a neighborhood from λ.*

Proof. It is proved in ([11], Theorem 5) that if $C_\lambda(X)$ is a Baire space, then each compact subset of X is contained in a member of λ. Therefore it is enough to show that each point of X has a compact neighborhood. Let $x \in X$, $\{U_n : n \in \mathbb{N}\}$ be a sequence of open neighborhoods of x, with $U_{n+1} \subseteq U_n$, such that whenever $x_n \in U_n$ for each n, the sequence $\{x_n : n \in \mathbb{N}\}$ has a cluster point, and suppose that x has no compact neighborhood. For each n, let $A_n = \overline{U_n} \setminus U_{n+1}$. Suppose that each A_n is compact and let $\{x_n : n \in \mathbb{N}\}$ be a sequence in $\overline{U_0}$. If $\{x_n : n \in \mathbb{N}\}$ is contained in the union of finitely many A_n, then it will have a cluster point. Otherwise there will exist a subsequence $\{x_{n_i} : i \in \mathbb{N}\}$ such that $x_{n_i} \in U_i$, for every $i \in \mathbb{N}$. This subsequence converges to x. Thus, the sequence $\{x_n : n \in \mathbb{N}\}$ would have a cluster point and then it follows that $\overline{U_0}$ is pseudocompact. Since $C_\lambda(X)$ is a Baire space then, from ([11], Theorem 5), $\overline{U_0}$ is compact. This gives a contradition, so there is $n_0 \in \mathbb{N}$ such that A_{n_0} is not compact. With the same reasoning we obtain by recurrence a subsequence $\{A_{n_k} : k \in \mathbb{N}\}$ of the sequence $\{A_n : n \in \mathbb{N}\}$ such that A_{n_k} is not compact for each $k \in \mathbb{N}$. Now for each k, we set $G_k = \cup\{[A,]k, k+1[] : A \in \lambda \text{ with } A \cap A_{n_k} \neq \emptyset\}$ which is open and dense in $C_\lambda(X)$. Since $C_\lambda(X)$ is a Baire space, there exists an $g \in \cap\{G_k : k \in \mathbb{N}\}$. Then for each k, there exists $a_k \in A_{n_k}$ such that $g(a_k) \in]k, k+1[$. The sequence $\{a_k : k \in \mathbb{N}\}$ converges to x, but $\{g(a_k) : k \in \mathbb{N}\}$ diverges which contradicts the fact that g is continuous. Hence, x admits a compact neighborhood. This finishes the proof of the theorem. □

The next result is an immediate consequence of Theorem 11 coupled with Theorem 12.

Corollary 3. *Let X be a paracompact q-space, and let λ be an admissible family of compact subsets of X such that $\cup\{K : K \in \lambda\} = X$. Then $C_\lambda(X)$ is a Baire space if and only if each point of X has a neighborhood from λ.*

Acknowledgments. Both authors would like to thank the referee for valuable comments and for careful reading of the manuscript.

REFERENCES

[1] R. Arens and J. Dugundji, *Topologies for function spaces.* Pacific J. Math. **1** (1951), 5–31.

[2] A. Bouchair and S. Kelaiaia, *Applications de jeux topologiques à l'étude de $C(X)$ muni d'une topologie set-open.* Révue Sciences & Technologie, Univ. Mentouri Constantine **20** (2003), 17–20.

[3] G. Choquet, *Lectures in analysis.* Benjamin New York Amsterdam, 1969.

[4] R. Engelking, *General Topology.* Polish Scientific Publishing, 1977.

[5] G. Gruenhage, *Games, covering properties and Eberlein compacts.* Topology Appl. **23** (1986), 291–297.

[6] G. Gruenhage and D.K. Ma, *Baireness of $C_k(X)$ for a locally compact X.* Topology App. **80** (1997), 131–139.

[7] R.C. Haworth and R.A. McCoy, *Baire spaces.* Dissertationes Math. (Rozprawy Mat.) **141** (1977), 1–77.

[8] L. Holá and L. Zsilinszky, *Completeness properties of the generalized compact-open topology on partial functions with closed domains.* Topology Appl. **110** (2001), 303–321.

[9] A.S. Kechris, *Classical descriptive set theory.* Springer, New York, 1995.

[10] S. Kelaiaia, *Propriétés de certaines topologies set-open sur $C(X)$.* Thèse de Doctorat de l'Université de Rouen, soutenu le 30 Juin 1995.

[11] S. Kelaiaia, *On a completeness property of $C(X)$.* Int. J. Appl. Math. **6** (2001), 287–291.

[12] D.K. Ma, *The Cantor tree, the λ-property and Baire function spaces.* Proc. Amer. Math. Soc. **119** (1993), 903–913.

[13] R.A. McCoy and I. Ntantu, *Countability properties of function spaces with set-open topology.* Topology proc. **10** (1985), 329–345.

[14] R.A. McCoy and I. Ntantu, *Completeness properties of function spaces.* Topology Appl. **2** (1986), 191–206.

[15] R.A. McCoy and I. Ntantu, *Topological properties of spaces of continuous functions.* Lecture Notes in Math. **1315** (1988), Springer Verlag Germany.

[16] S.E. Nokhrin, *Some properties of set-open topologies.* J. Math. Sci. **144** (2007), 4123–4151.

[17] S.E. Nokhrin and A.V. Osipov, *On the coincidence of set open and uniform topologies.* Proc. Steklov Inst. Math. Suppl. **3** (2009), 184–191.

[18] A.V. Osipov, *Topological algebraic properties of function spaces with set open topologies.* Topology Appl. **159** (2012), 800–805.

[19] J.C. Oxtoby, *The Banach-Mazur game and Banach Category Theorem, contributions to the theory of games.* Ann. of Math. Stud. *III*, **39** (1957).

[20] J. Saint-Raymond, *Jeux topologiques et espaces de Namioka.* Proc. Amer. Math. Soc. **87** (1983), 499–504.

[21] H.E. White, *Topological spaces that are α-favorable for player with perfect information.* Proc. Amer. Math. Soc. **50** (1975), 477–482.

Received 12 February 2013

University of Jijel,
LMAM Laboratory,
Department of Mathematics,
Jijel 18000, Algeria
abdebouchair@yahoo.com

University of Annaba,
LMA Laboratory,
Department of Mathematics,
Annaba 23000, Algeria
kelaiaiasmail@yahoo.fr