# SUFFICIENCY AND DUALITY IN MINIMAX FRACTIONAL PROGRAMMING WITH GENERALIZED $(\Phi, \rho)$-INVEXITY

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In the present paper, we establish sufficient optimality conditions for a minimax fractional programming problem under the assumptions of $(\Phi, \rho)$-invexity. Weak, strong and strict converse duality theorems are also derived for a dual model of minimax fractional programming problem.

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*Key words:* fractional programming, minimax programming, $(\Phi, \rho)$-invexity, sufficiency, duality.

## 1. INTRODUCTION

Amongst various important applications, one important application of nonlinear programming is to maximize or minimize the ratio of two functions, commonly called fractional programming. The characteristics of fractional programming problems have been investigated widely [1, 6, 10] and [13]. In noneconomic situations, fractional programming problems arisen in information theory, stochastic programming, numerical analysis, approximation theory, cluster analysis, graph theory, multifacility location theory, decomposition of large-scale mathematical programming problems, goal programming and among others. Recently, some biologists have been studying fractional programming problems to improve the accuracy of melting temperature estimations (cf. Leber *et al.* [15]).

The necessary and sufficient conditions for generalized minimax programming were first developed by Schmitendorf [17]. Later, several authors considered these optimality and duality theorems for minimax fractional programming problems, one can consult [2, 11, 16] and [20].

Antczak [4] proved optimality conditions for a class of generalized fractional minimax programming problems involving $B$-$(p, r)$-invexity functions and established duality theorems for various duality models. Later on, Ahmad *et al.* [3] discussed sufficient optimality conditions and duality theorems for a nondifferentiable minimax fractional programming problem with $B$-$(p, r)$-invexity.

Zalmai and Zhang [18, 19] introduce a new class of generalized $(\mathcal{F}, b, \phi, \rho, \theta)$-univex functions and derived necessary optimality conditions, sufficient optimality conditions and duality theorems for minimax fractional programming problem and its different types of dual models.

In this paper, we are motivated by Zalmai and Zhang [18, 19], and Ferrara and Stefanescu [7] to discuss sufficient optimality conditions and duality theorems for a minimax fractional programming problem with $(\Phi, \rho)$-invex functions [5]. The remainder of the paper is organized as follows. In Section 2, we recall some definitions and notations from the literature. In Section 3, we are devoted to establish sufficient optimality conditions for a class of minimax fractional programming problem involving $(\Phi, \rho)$-invex functions. Moreover, weak, strong and strict converse duality theorems for a dual model of minimax fractional programming problem are discussed in Section 4. Finally, conclusion and further development that might take in this direction is given in Section 5.

## 2. NOTATION AND PRELIMINARIES

Throughout the paper, let $R^n$ be the $n$-dimensional Euclidean space and $R^n_+$ its non-negative orthant.

We consider now the following minmax fractional programming problem:

**(P)** Minimize $\max\limits_{y\in Y} \frac{f(x,y)+\|A(y)x\|_a}{g(x,y)-\|B(y)x\|_b}$

subject to

$$G_j(x) + \|C_j(x)\|_{c(j)} \leq 0, j \in \underline{q},\ H_k(x) = 0,\ k \in \underline{r},\ x \in X,$$

where $X$ is a nonempty open convex subset of $R^n$, $Y$ is a compact metrizable topological space, $f(.,y)$, $g(.,y)$, $y \in Y$, $G_j$, $j \in \underline{q} \equiv \{1, 2, ..., q\}$, and $H_k$, $k \in \underline{r} \equiv \{1, 2, ..., r\}$, are real valued functions defined on $X$. For each $y \in Y$ and $j \in \underline{q}$, $A(y), B(y)$ and $C_j$ are respectively $l \times n$, $m \times n$, and $n_j \times n$ matrices. $\|.\|_a$, $\|.\|_b$, and $\|.\|_{c(j)}$ are arbitrary norms on $R^l$, $R^m$, and $R^{n_j}$, respectively. It is assumed that for each $y \in Y$ and for all $x$ satisfying the constraints of (P), $f(x,y) + \|A(y)x\|_a \geq 0$ and $g(x,y) - \|B(y)x\|_b > 0$. We denote $J_+(v) = \{j \in \underline{q} : v_j > 0\}$ for fixed $v \in R^q_+$, $K_*(w) = \{k \in \underline{r} : w_k \neq 0\}$ for fixed $w \in R^r$, and $\gamma = \{\gamma^1, \gamma^2, ..., \gamma^q\}$.

*Remark* 2.1. The problem (P) considered here is a general prototype optimization model that contains a well-known nondifferentiable minimax fractional programming problem studied by Lai *et al.* [14] as a special case.

In the next definition, an element of the $(n+1)$-dimensional Euclidean space $R^{n+1}$ is represented as the ordered pair $(y, r)$ with $y \in R^n$ and $r \in R$. Let

$\rho$ be a real number and $\Phi$ a real-valued function defined on $X \times X \times R^{n+1}$ such that $\Phi(x, a, .)$ is convex on $R^{n+1}$ and $\Phi(x, a, (0, r)) \geq 0$ for every $(x, a) \in X \times X$ and $r \in R_{+}$. Let $\varphi : X \subseteq R^n \to R$ be a differentiable function and $a \in X$.

*Definition* 2.1 ([5]). The function $\varphi$ is said to be (strictly) $(\Phi, \rho)$-invex at $a \in X$, if for all $x \in X$, we have

$$\varphi(x) - \varphi(a)(>) \geq \Phi(x, a, (\nabla\varphi(a), \rho).$$

In the sequel of the paper, we need the following result from Zalmai and Zhang [18].

THEOREM 2.1 (Necessary conditions). *Let $x^*$ be an optimal solution to* (P) *and assume that the functions $f(., y)$ ,$g(., y)$, $y \in Y$, $G_j$, $j \in \underline{q}$, and $H_k$, $k \in \underline{r}$, are continuously differentiable at $x^*$, and that any one of the constraint qualifications [8, 12] holds at $x^*$. Then there exist $\lambda^* \in R, (p, \overline{y}^*, u^*, \alpha^*, \beta^*) \in \mathbb{K}$, $v^* \in R^q_+$, $w^* \in R^r$, and $\gamma^{*j} \in R^{n_j}$, $j \in \underline{q}$, such that*

$$\sum_{i=1}^{p} u_i^* \{\nabla f(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda^* [\nabla g(x^*, y^{*i}) - B(y^{*i})^T \beta^{*i}]\}$$
$$+ \sum_{j=1}^{q} v_j^* [\nabla G_j(x^*) + C_j^T \gamma^{*j}] + \sum_{k=1}^{r} w_k^* \nabla H_k(x^*) = 0,$$
$$u_i^* \{f(x^*, y^{*i}) + \langle \alpha^{*i}, A(y^{*i})x^* \rangle - \lambda^* [g(x^*, y^{*i}) - \langle \beta^{*i}, B(y^{*i})x^* \rangle]\} = 0, i \in \underline{p},$$
$$v_j^* [G_j(x^*) + \|C_j x^*\|_{c(j)}] = 0, j \in \underline{q},$$
$$\left\|\alpha^{*i}\right\|_a^* \leq 1, \left\|\beta^{*i}\right\|_b^* \leq 1, i \in \underline{p},$$
$$\left\|\gamma^{*j}\right\|_{c(j)}^* \leq 1, j \in \underline{q},$$

$$< \alpha^{*i}, A(y^{*i})x^* > = \left\|A(y^{*i})x^*\right\|_a,$$
$$< \beta^{*i}, B(y^{*i})x^* > = \left\|B(y^{*i})x^*\right\|_b, i \in \underline{p},$$
$$< \gamma^{*j}, C_j x^* > = \|C_j x^*\|_{c(j)}, j \in \underline{q},$$

*where*

$$\mathbb{K} = \{(p, \overline{y}, u, \alpha, \beta) : 1 \leq p \leq n+1;\ \overline{y} = (y^1, y^2, ..., y^p),\ y^i \in Y;\ u \in R^p_+$$

*with*

$$\sum_{i=1}^{p} u_i = 1;\ \alpha = (\alpha^1, \alpha^2, ..., \alpha^p), \alpha^i \in R^l;\ \beta = (\beta^1, \beta^2, ..., \beta^p), \beta^i \in R^m\}$$

.

LEMMA 2.1 ([9]). *For each $a, b \in R^m$, $\langle a, b \rangle \leq \|a\|^* \|b\|$.*

LEMMA 2.2 ([18]). *For each $x \in X$,*

$$\varphi_0(x) \equiv \max_{y\in Y} \frac{f(x,y) + \|A(y)x\|_a}{g(x,y) - \|B(y)x\|_b} = \max_{p\in \underline{n+1}} \max_{\substack{u\in U\\ y^i\in Y}} \frac{\sum_{i=1}^{p} u_i[f(x,y^i) + \|A(y^i)x\|_a]}{\sum_{i=1}^{p} u_i[g(x,y^i) - \|B(y^i)x\|_b]},$$

*where* $U = \{u \in R_+^p : \sum_{i=1}^{p} u_i = 1\}$.

## 3. SUFFICIENT OPTIMALITY CONDITIONS

In this section, we derive sufficient optimality conditions for (P) under the assumption of $(\Phi,\rho)$-invex functions introduced in the previous section. Denote

$$\Theta(.) = \sum_{i=1}^{p} u_i^*\{f(.,y^{*i}) + \langle \alpha^{*i}, A(y^{*i}).\rangle - \lambda^*[g(.,y^{*i}) - \langle \beta^{*i}, B(y^{*i}).\rangle]\}.$$

THEOREM 3.1 (Sufficiency). *Let* $x^*$ *be a feasible solution to* (P) *and let* $\lambda^* = \varphi_0(x^*) \geq 0$, *and assume that the functions* $f(.,y), g(.,y), y \in Y,\ G_j, j \in \underline{q}$ *and* $H_k, k \in \underline{r}$, *are differentiable at* $x^*$ *and there exist* $(p, \bar{y}^*, u^*, \alpha^*, \beta^*) \in \mathbb{K}$, $v^* \in R_+^q$, $w^* \in R^r$, *and* $\gamma^{*j} \in R^{\,n_j}, j \in \underline{q}$, *satisfying*

(3.1)
$$\sum_{i=1}^{p} u_i^*\{\nabla f(x^*,y^{*i}) + A(y^{*i})^T\alpha^{*i} - \lambda^*[\nabla g(x^*,y^{*i}) - B(y^{*i})^T\beta^{*i}]\}$$
$$+\sum_{j=1}^{q} v_j^*[\nabla G_j(x^*) + C_j^T\gamma^{*j}] + \sum_{k=1}^{r} w_k^*\nabla H_k(x^*) = 0,$$

(3.2)
$$u_i^*\{f(x^*,y^{*i}) + \langle \alpha^{*i}, A(y^{*i})x^*\rangle - \lambda^*[g(x^*,y^{*i}) - \langle \beta^{*i}, B(y^{*i})x^*\rangle]\} = 0, i \in \underline{p},$$

(3.3)
$$v_j^*[G_j(x^*) + \|C_jx^*\|_{c(j)}] = 0, j \in \underline{q},$$

(3.4)
$$\|\alpha^{*i}\|_a^* \leq 1, \|\beta^{*i}\|_b^* \leq 1, i \in \underline{p},$$

(3.5)
$$\|\gamma^{*j}\|_{c(j)}^* \leq 1, j \in \underline{q}.$$

*Furthermore, assume that the following conditions hold:*

(i) *for each* $i \in \underline{p}$, $f(.,y^{*i}) + \langle \alpha^{*i}, A(y^{*i}).\rangle - \lambda^*[g(.,y^{*i}) - \langle \beta^{*i}, B(y^{*i}).\rangle$ *is* $(\Phi, \bar{\rho}_i)$- *invex at* $x^*$,

(ii) *for each* $j \in J_+ \equiv J_+(v^*)$, $G_j(.) + \langle \gamma^{*j}, C_j.\rangle$ *is* $(\Phi, \hat{\rho}_j)$- *invex at* $x^*$,

(iii) *for each* $k \in K_* \equiv K_*(w^*)$, $H_k(.)$ *is* $(\Phi, \breve{\rho}_k)$- *invex at* $x^*$,

(iv) $\sum\limits_{i=1}^{p} u_i^* \; \overline{\rho}_i + \sum\limits_{j=1}^{q} v_j^* \; \hat{\rho}_j + \sum\limits_{k=1}^{r} w_k^* \; \breve{\rho}_k \geq 0.$

*Then $x^*$ is an optimal solution to (P).*

*Proof.* Suppose to the contrary that $x^*$ is not an optimal solution to (P). Then there exists a feasible solution $x$ to (P) such that

$$\varphi_0(x) < \varphi_0(x^*) = \lambda^*,$$

equivalently,

$$\max_{y \in Y} \frac{f\,(x, y^*) + \; \|A(y^*)x\|_a}{g\,(x, y^*) - \; \|B(y^*)x\|_b} < \lambda^*.$$

Therefore for each $i \in \underline{p}$, we have

$$\frac{f\left(x, y^{*i}\right) + \; \left\|A(y^{*i})x\right\|_a}{g\,(x, y^{*i}) - \; \|B(y^{*i})x\|_b} < \lambda^*,$$

or,

$$f\left(x, y^{*i}\right) + \; \left\|A(y^{*i})x\right\|_a - \lambda^*[g\left(x, y^{*i}\right) - \; \left\|B(y^{*i})x\right\|_b] < 0, \quad \forall i \in \underline{p}. \tag{3.6}$$

Now,

$$\Theta(x) = \sum_{i=1}^{p} u_i^* \{f(x, y^{*i}) + \left\langle \alpha^{*i}, A(y^{*i})x \right\rangle - \lambda^*[g(x, y^{*i}) - \left\langle \beta^{*i}, B(y^{*i})x \right\rangle]\}$$

$$\leq \sum_{i=1}^{p} u_i^* \{f(x, y^{*i}) + \left\|\alpha^{*i}\right\|_a^* \left\|A(y^{*i})x\right\|_a - \lambda^*[g(x, y^{*i}) - \left\|\beta^{*i}\right\|_b^* \left\|B(y^{*i})x\right\|_b]\}$$

(by Lemma 2.1)

$$\leq \sum_{i=1}^{p} u_i^* \{f(x, y^{*i}) + \left\|A(y^{*i})x\right\|_a - \lambda^*[g(x, y^{*i}) - \left\|B(y^{*i})x\right\|_b]\} \quad \text{(by (3.4))}$$

$$< 0 = \sum_{i=1}^{p} u_i^* \{f(x^*, y^{*i}) + \left\langle \alpha^{*i}, A(y^{*i})x^* \right\rangle - \lambda^*[g(x^*, y^{*i}) - \left\langle \beta^{*i}, B(y^{*i})x^* \right\rangle]\}$$

(by (3.2) & (3.6))

$$= \Theta(x^*).$$

That is,

$$\Theta(x) - \Theta(x^*) < 0. \tag{3.7}$$

From the $(\Phi, \bar{\rho}_i)$-invexity of

$$\{f(., y^{*i}) + \left\langle \alpha^{*i}, A(y^{*i}). \right\rangle - \lambda^*[g(., y^{*i}) - \left\langle \beta^{*i}, B(y^{*i}). \right\rangle]\}$$

at $x^*$, we have

$$\{f(x,y^{*i})+\langle\alpha^{*i},A(y^{*i})x\rangle-\lambda^*[g(x,y^{*i})-\langle\beta^{*i},B(y^{*i})x\rangle]\}$$

$$-\{f(x^*,y^{*i})+\langle\alpha^{*i},A(y^{*i})x^*\rangle-\lambda^*[g(x^*,y^{*i})-\langle\beta^{*i},B(y^{*i})x^*\rangle]\}$$

$$\geq \Phi(x,x^*,(\nabla f(x^*,y^{*i})+A(y^{*i})^T\alpha^{*i}-\lambda^*[\nabla g(x^*,y^{*i})-B(y^{*i})^T\beta^{*i}],\bar{\rho}_i)),\quad \forall\, i\in \underline{p}\,.$$

Multiplying by $u_i^*$ and then summing over $i$, we have

$$\sum_{i=1}^{p}u_i^*\{f(x,y^{*i})+\langle\alpha^{*i},A(y^{*i})x\rangle-\lambda^*[g(x,y^{*i})-\langle\beta^{*i},B(y^{*i})x\rangle]\}$$

$$-\sum_{i=1}^{p}u_i^*\{f(x^*,y^{*i})+\langle\alpha^{*i},A(y^{*i})x^*\rangle-\lambda^*[g(x^*,y^{*i})-\langle\beta^{*i},B(y^{*i})x^*\rangle]\}$$

$$\geq \sum_{i=1}^{p}u_i^*\{\Phi(x,x^*,(\nabla f(x^*,y^{*i})+A(y^{*i})^T\alpha^{*i}-\lambda^*[\nabla g(x^*,y^{*i})-B(y^{*i})^T\beta^{*i}],\bar{\rho}_i))\},\quad \forall\, i\in \underline{p}\,.$$

The above inequality together with (3.7), gives

$$\sum_{i=1}^{p}u_i^*\{\Phi(x,x^*,(\nabla f(x^*,y^{*i})+A(y^{*i})^T\alpha^{*i}-\lambda^*[\nabla g(x^*,y^{*i})-B(y^{*i})^T\beta^{*i}],\bar{\rho}_i))\}<0. \tag{3.8}$$

On the other hand, by using Lemma 2.1, feasibility of $x$ to (P), (3.3) and (3.5), for each $j\in J_+$, we obtain

$$\begin{aligned}G_j(x)+\langle\gamma^{*j},C_jx\rangle &\leq G_j(x)+\|\gamma^{*j}\|^*_{c(j)}\|C_jx\|_{c(j)}\\ &\leq G_j(x)+\|C_jx\|_{c(j)}\\ &\leq 0\\ &= G_j(x^*)+\langle\gamma^{*j},C_jx^*\rangle\end{aligned}$$

Therefore,

$$\{G_j(x)+\langle\gamma^{*j},C_jx\rangle\}-\{G_j(x^*)+\langle\gamma^{*j},C_jx^*\|\}\leq 0,$$

which by $(\Phi,\hat{\rho}_j)$-invexity of $G_j(.)+\langle\gamma^{*j},C_j.\rangle$ at $x^*$ for each $j\in J_+$, we have

$$\Phi(x,x^*,(\nabla G_j(x^*)+C_j^T\gamma^{*j},\hat{\rho}_j))\leq 0.$$

Since $v_j^*\geq 0$ for each $j\in\underline{q}$ and $v_j^*=0$ for each $j\in\underline{q}\setminus J_+$, multiplying $v_j^*$ and summing over $j$, the above inequality gives

$$\sum_{j=1}^{q}v_j^*\{\Phi(x,x^*,(\nabla G_j(x^*)+C_j^T\gamma^{*j},\hat{\rho}_j))\}\leq 0. \tag{3.9}$$

Similarly, by using the $(\Phi, \hat{\rho}_j)$-invexity of $H_k(.)$ at $x^*$, for each $k \in K_* \equiv K_*(w^*)$, we have

$$\sum_{k=1}^{r} w_k^* \{\Phi(x, x^*, (\nabla H_k(x^*), \breve{\rho}_k))\} \leq 0. \tag{3.10}$$

On adding (3.8), (3.9) and (3.10), we have

(3.11)

$$\sum_{i=1}^{p} u_i^* \{\Phi(x, x^*, (\nabla f(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda^* [\nabla g(x^*, y^{*i}) - B(y^{*i})^T \beta^{*i}], \bar{\rho}_i))$$

$$+ \sum_{j=1}^{q} v_j^* \{\Phi(x, x^*, (\nabla G_j(x^*) + C_j^T \gamma^{*j}, \hat{\rho}_j))\} + \sum_{k=1}^{r} w_k^* \{\Phi(x, x^*, (\nabla H_k(x^*), \breve{\rho}_k))\} < 0.$$

Now we introduce the following notations:

$$\mu_i^* = \frac{u_i^*}{\sum_{i=1}^{p} u_i^* + \sum_{j=1}^{q} v_j^* + \sum_{k=1}^{r} w_k^*}, i \in \underline{p}, \tag{3.12}$$

$$\eta_j^* = \frac{v_j^*}{\sum_{i=1}^{p} u_i^* + \sum_{j=1}^{q} v_j^* + \sum_{k=1}^{r} w_k^*}, j \in J_+, \tag{3.13}$$

$$\xi_k^* = \frac{w_k^*}{\sum_{i=1}^{p} u_i^* + \sum_{j=1}^{q} v_j^* + \sum_{k=1}^{r} w_k^*}, k \in K_*. \tag{3.14}$$

Note that $0 \leq \mu_i^* \leq 1, i \in \underline{p}, 0 \leq \eta_j^* \leq 1, j \in J_+, 0 \leq \xi_k^* \leq 1, k \in K_*$, and moreover

$$\sum_{i=1}^{p} \mu_i^* + \sum_{j=1}^{q} \eta_j^* + \sum_{k=1}^{r} \xi_k^* = 1. \tag{3.15}$$

On combining (3.11)–(3.14), we have

$$\sum_{i=1}^{p} \mu_i^* \{\Phi(x, x^*, (\nabla f(x^*, y^{*i}) + A(y^{*i})^T \alpha^{*i} - \lambda^* [\nabla g(x^*, y^{*i}) - B(y^{*i})^T \beta^{*i}], \bar{\rho}_i))\}$$

$$+ \sum_{j=1}^{q} \eta_j^* \{\Phi(x, x^*, (\nabla G_j(x^*) + C_j^T \gamma^{*j}, \hat{\rho}_j))\} + \sum_{k=1}^{r} \xi_k^* \{\Phi(x, x^*, (\nabla H_k(x^*), \breve{\rho}_k))\} < 0.$$

Thus, by Φ-convexity on $R^{n+1}$ and (3.15), we conclude that

(3.16)
$$\Phi(x,x^*,(\sum_{i=1}^{p}\mu_i^*\{\nabla f(x^*,y^{*i})+A(y^{*i})^T\alpha^{*i}-\lambda^*[\nabla g(x^*,y^{*i})-B(y^{*i})^T\beta^{*i}]\}$$
$$+\sum_{j=1}^{q}\eta_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}]+\sum_{k=1}^{r}\xi_k^*\nabla H_k(x^*),\sum_{i=1}^{p}\mu_i^*\bar{\rho}_i+\sum_{j=1}^{q}\eta_j^*\hat{\rho}_j+\sum_{k=1}^{r}\xi_k^*\breve{\rho}_k))<0.$$

On the other hand, using the hypothesis specified in $(iv)$ and (3.1) together with (3.12)–(3.14), we get

$$\sum_{i=1}^{p}\mu_i^*\ \bar{\rho}_i+\sum_{j=1}^{q}\eta_j^*\ \hat{\rho}_j+\sum_{k=1}^{r}\xi_k^*\ \breve{\rho}_k\geq 0,$$

and

$$\sum_{i=1}^{p}\mu_i^*\{\nabla f(x^*,y^{*i})+A(y^{*i})^T\alpha^{*i}-\lambda^*[\nabla g(x^*,y^{*i})-B(y^{*i})^T\beta^{*i}]\}$$
$$+\sum_{j=1}^{q}\eta_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}]+\sum_{k=1}^{r}\xi_k^*\nabla H_k(x^*)=0.$$

From the above two inequalities and the fact $\Phi(x,x^*,(0,r))\geq 0, r\geq 0$, it follows that

$$\Phi(x,x^*,(\sum_{i=1}^{p}\mu_i^*\{\nabla f(x^*,y^{*i})+A(y^{*i})^T\alpha^{*i}-\lambda^*[\nabla g(x^*,y^{*i})-B(y^{*i})^T\beta^{*i}]\}$$
$$+\sum_{j=1}^{q}\eta_j^*[\nabla G_j(x^*)+C_j^T\gamma^{*j}]+\sum_{k=1}^{r}\xi_k^*\nabla H_k(x^*),\sum_{i=1}^{p}\mu_i^*\bar{\rho}_i+\sum_{j=1}^{q}\eta_j^*\hat{\rho}_j+\sum_{k=1}^{r}\xi_k^*\breve{\rho}_k))\geq 0,$$

which contradicts (3.16). This completes the proof. □

## 4. DUALITY

In this section, we consider the following dual [19] for (P):

$$\textbf{(D)}\qquad \max_{(p,\bar{y},u,\alpha,\beta)\in\mathbb{K}}\quad \sup_{(s,v,w,\gamma,\lambda)\in\mathbb{L}}\quad \lambda$$

subject to

(4.1)
$$\sum_{i=1}^{p}u_i\{\nabla f(s,y^i)+A(y^i)^T\alpha^i-\lambda[\nabla g(s,y^i)-B(y^i)^T\beta^i]\}$$

$$+\sum_{j=1}^{q} v_j[\nabla G_j(s) + C_j^T\gamma^j] + \sum_{k=1}^{r} w_k \nabla H_k(s) = 0,$$

$$u_i\{f(s,y^i) + \langle \alpha^i, A(y^i)s\rangle - \lambda[g(s,y^i) - \langle \beta^i, B(y^i)s\rangle]\} \geq 0, i \in \underline{p}, \tag{4.2}$$

$$v_j[G_j(s) + \langle \gamma^j, C_j s\rangle] \geq 0, j \in \underline{q}, \tag{4.3}$$

$$w_k H_k(s) \geq 0, k \in \underline{r}, \tag{4.4}$$

$$\left\|\alpha^i\right\|_a^* \leq 1, \qquad \left\|\beta^i\right\|_b^* \leq 1,\ i \in \underline{p}, \tag{4.5}$$

$$\left\|\gamma^j\right\|_{c(j)}^* \leq 1, \quad j \in \underline{q}; \tag{4.6}$$

where

$$\mathbb{L} = \{(s,v,w,\gamma,\lambda) : s \in R^n, v \in R^q_+, w \in R^r, \gamma = (\gamma^1,\gamma^2,...,\gamma^q), \gamma^j \in R^{n_j}, j \in \underline{q}, \lambda \in R_+\}.$$

In this Section, we denote

$$\Theta_1(.) = \sum_{i=1}^{p} u_i\{f(.,y^i) + \langle \alpha^i, A(y^i).\rangle - \lambda[g(.,y^i) - \langle \beta^i, B(y^i).\rangle]\} + \sum_{j=1}^{q} v_j[G_j(.) + \langle \gamma^j, C_j.\rangle] + \sum_{k=1}^{r} w_k H_k(.).$$

Now, we derive the following weak, strong and strict converse duality theorems.

THEOREM 4.1 (Weak duality). *Let $x$ and $(p,u,\bar{y},\alpha,\beta;s,v,w,\gamma,\lambda)$ be the feasible solutions to* (P) *and* (D). *Furthermore, assume that the following conditions hold:*

(i) *for each $i \in \underline{p}$, $f(.,y^i) + \langle \alpha^i, A(y^i).\rangle - \lambda[g(.,y^i) - \langle \beta^i, B(y^i).\rangle]$ is $(\Phi,\overline{\rho}_i)$-invex at $s$,*
(ii) *for each $j \in \underline{q}$, $G_j(.) + \langle \gamma^j, C_j.\rangle$ is $(\Phi,\hat{\rho}_j)$- invex at $s$,*
(iii) *for each $k \in \underline{r}$, $H_k(.)$ is $(\Phi,\breve{\rho}_k)$- invex at $s$,*
(iv) $\sum_{i=1}^{p} u_i\ \overline{\rho}_i + \sum_{j=1}^{q} v_j\ \hat{\rho}_j + \sum_{k=1}^{r} w_k\ \breve{\rho}_k \geq 0.$

*Then* $\varphi_0 \geq \lambda$.

*Proof.* Suppose contrary to the result that

$$\varphi_0(x) < \lambda,$$

equivalently,

$$\max_{y\in Y}\frac{f(x,y)+\ \|A(y)x\|_a}{g(x,y)-\ \|B(y)x\|_b}<\lambda.$$

Therefore, for each $i\in \underline{p}$, we have

$$\frac{f\left(x,y^i\right)+\ \left\|A(y^i)x\right\|_a}{g\left(x,y^i\right)-\ \|B(y^i)x\|_b}<\lambda,$$

or

$$f\left(x,y^i\right)+\ \left\|A(y^i)x\right\|_a-\lambda[g\left(x,y^i\right)-\ \left\|B(y^i)x\right\|_b]<0,\text{for each } i\in\underline{p}.$$

It follows from $u\in R^p_+$, with $\sum_{i=1}^{p}u_i=1$, that

$$u_i\{f\left(x,y^i\right)+\ \left\|A(y^i)x\right\|_a-\lambda[g\left(x,y^i\right)-\ \left\|B(y^i)x\right\|_b]\}\leq 0, \tag{4.7}$$

with at least one strict inequality because $u=(u_1,u_2,...,u_p)\neq 0$.

Now,

$$\Theta_1(x)=\sum_{i=1}^{p}u_i\{f(x,y^i)+\left\langle\alpha^i,A(y^i)x\right\rangle-\lambda[g(x,y^i)-\left\langle\beta^i,B(y^i)x\right\rangle]\}$$

$$+\sum_{j=1}^{q}v_j[G_j(x)+\left\langle\gamma^j,C_jx\right\rangle]+\sum_{k=1}^{r}w_kH_k(x)$$

$$\leq\sum_{i=1}^{p}u_i\{f(x,y^i)+\left\|\alpha^i\right\|_a^*\left\|A(y^i)x\right\|_a-\lambda[g(x,y^i)-\left\|\beta^i\right\|_b^*\left\|B(y^i)x\right\|_b]\}$$

$$+\sum_{j=1}^{q}v_i\{G_j(x)+\left\|\gamma^j\right\|_{c(j)}^*\left\|C_jx\right\|_{c(j)}\}\quad\text{(by Lemma 2.1 and feasibility of } x)$$

$$\leq\sum_{i=1}^{p}u_i\{f(x,y^i)+\left\|A(y^i)x\right\|_a-\lambda[g(x,y^i)-\left\|B(y^i)x\right\|_b]\}$$

$$+\sum_{j=1}^{q}v_i\{G_j(x)+\left\|C_jx\right\|_{c(j)}\}\ (by\ (4.5)\ and\ (4.6))$$

$$\leq\sum_{i=1}^{p}u_i\{f(x,y^i)+\left\|A(y^i)x\right\|_a-\lambda[g(x,y^i)-\left\|B(y^i)x\right\|_b]\}\qquad\text{(by feasibility of } x)$$

$$<0\leq\sum_{i=1}^{p}u_i\{f(s,y^i)+\left\langle\alpha^i,A(y^i)s\right\rangle-\lambda[g(s,y^i)-\left\langle\beta^i,B(y^i)s\right\rangle]\}$$

$$+\sum_{j=1}^{q}v_j[G_j(s)+\left\langle\gamma^j,C_js\right\rangle]+\sum_{k=1}^{r}w_kH_k(s)\ \text{(by (4.2),(4.3),(4.4) and(4.7))}=\Theta_1(s).$$

That is,

$$\Theta_1(x) - \Theta_1(s) < 0. \tag{4.8}$$

From the $(\Phi, \bar{\rho}_i)$-invexity of $\{f(.,y^i)+\langle\alpha^i, A(y^i).\rangle-\lambda^*[g(.,y^i)-\langle\beta^i, B(y^i.)\rangle]\}$ at $s$, we obtain

$$\begin{aligned}&\{f(x,y^i) + \langle\alpha^i, A(y^i)x\rangle - \lambda[g(x,y^i) - \langle\beta^i, B(y^i)x\rangle]\}\\ &-\{f(s,y^i) + \langle\alpha^i, A(y^i)s\rangle - \lambda[g(s,y^i) - \langle\beta^i, B(y^i)s\rangle]\}\\ &\geq \Phi(x,s,(\nabla f(s,y^i) + A(y^i)^T\alpha^i - \lambda[\nabla g(s,y^i) - B(y^i)^T\beta^i], \bar{\rho}_i)),\ \forall\ i \in \underline{p}.\end{aligned}$$

Multiplying by $u_i$ and then summing over $i$, we get

$$\begin{aligned}&\sum_{i=1}^{p} u_i\{f(x,y^i) + \langle\alpha^i, A(y^i)x\rangle - \lambda[g(x,y^i) - \langle\beta^i, B(y^i)x\rangle]\}\\ &-\sum_{i=1}^{p} u_i\{f(s,y^i) + \langle\alpha^i, A(y^i)s\rangle - \lambda[g(s,y^i) - \langle\beta^i, B(y^i)s\rangle]\}\\ &\geq \sum_{i=1}^{p} u_i\{\Phi(x,s,(\nabla f(s,y^i) + A(y^i)^T\alpha^i - \lambda[\nabla g(s,y^i) - B(y^i)^T\beta^i], \bar{\rho}_i))\}.\end{aligned} \tag{4.9}$$

On the other hand, by using $(\Phi, \hat{\rho}_j)$-invexity of $G_j(.) + \langle\gamma^j, C_j.\rangle$ at $s$, for each $j \in J_+$ we have

$$G_j(x) + \langle\gamma^j, C_jx\rangle - \{G_j(s) + \langle\gamma^j, C_js\rangle\} \geq \Phi(x,s,(\nabla G_j(s) + C_j^T\gamma^j, \hat{\rho}_j)).$$

Since $v_j \geq 0$ for each $j \in \underline{q}$ and $v_j = 0$ for each $j \in \underline{q} \setminus J_+$, multiplying $v_j$ and summing over $j$, the above inequality gives

$$\begin{aligned}&\sum_{j=1}^{q} v_j\{G_j(x) + \langle\gamma^j, C_jx\rangle\} - \sum_{j=1}^{q} v_j\{G_j(s) + \langle\gamma^j, C_js\rangle\}\\ &\geq \sum_{j=1}^{q} v_j\{\Phi(x,s,(\nabla G_j(s) + C_j^T\gamma^j, \hat{\rho}_j))\}.\end{aligned} \tag{4.10}$$

Similarly, by using the $(\Phi, \hat{\rho}_j)$-invexity of $H_k(.)$ at $s$, for each $k \in K_* \equiv K_*(w)$, we have

$$\sum_{k=1}^{r} w_kH_k(x) - \sum_{k=1}^{r} w_kH_k(s) \geq \sum_{k=1}^{r} w_k\{\Phi(x,s,(\nabla H_k(s), \breve{\rho}_k))\}. \tag{4.11}$$

On adding (4.9), (4.10) and (4.11), and by using (4.8), we have

$$\sum_{i=1}^{p} u_i\{\Phi(x,s,(\nabla f(s,y^i) + A(y^i)^T\alpha^i - \lambda[\nabla g(s,y^i) - B(y^i)^T\beta^i], \bar{\rho}_i))\} \tag{4.12}$$

$$+\sum_{j=1}^{q} v_j\{\Phi(x,s,(\nabla G_j(s)+C_j^T\gamma^j,\hat{\rho}_j))\}+\sum_{k=1}^{r} w_k\{\Phi(x,s,(\nabla H_k(s),\breve{\rho}_k))\}<0.$$

Now we introduce the following notations:

$$\mu_i=\frac{u_i}{\sum_{i=1}^{p}u_i+\sum_{j=1}^{q}v_j+\sum_{k=1}^{r}w_k}, i\in \underline{p}, \tag{4.13}$$

$$\eta_j=\frac{v_j}{\sum_{i=1}^{p}u_i+\sum_{j=1}^{q}v_j+\sum_{k=1}^{r}w_k}, j\in J_+, \tag{4.14}$$

$$\xi_k=\frac{w_k}{\sum_{i=1}^{p}u_i+\sum_{j=1}^{q}v_j+\sum_{k=1}^{r}w_k}, k\in K_*. \tag{4.15}$$

Note that $0\le \mu_i\le 1, i\in \underline{p}, 0\le \eta_j\le 1, j\in J_+, 0\le \xi_k\le 1, k\in K_*$, and moreover

$$\sum_{i=1}^{p}\mu_i+\sum_{j=1}^{q}\eta_j+\sum_{k=1}^{r}\xi_k=1. \tag{4.16}$$

On combining (4.12)–(4.15), we have

$$\sum_{i=1}^{p}\mu_i\{\Phi(x,s,(\nabla f(s,y^i)+A(y^i)^T\alpha^i-\lambda[\nabla g(s,y^i)-B(y^i)^T\beta^i],\bar{\rho}_i))\}$$
$$+\sum_{j=1}^{q}\eta_j\{\Phi(x,s,(\nabla G_j(s)+C_j^T\gamma^j,\hat{\rho}_j))\}+\sum_{k=1}^{r}\xi_k\{\Phi(x,s,(\nabla H_k(x^*),\breve{\rho}_k))\}<0.$$

Thus, by Φ-convexity on $R^{n+1}$ and (4.16), we conclude that

$$\Phi(x,s,(\sum_{i=1}^{p}\mu_i\{\nabla f(s,y^i)+A(y^i)^T\alpha^i-\lambda[\nabla g(s,y^i)-B(y^i)^T\beta^i]\} \tag{4.17}$$
$$+\sum_{j=1}^{q}\eta_j[\nabla G_j(s)+C_j^T\gamma^j]+\sum_{k=1}^{r}\xi_k\nabla H_k(s),\sum_{i=1}^{p}\mu_i\bar{\rho}_i+\sum_{j=1}^{q}\eta_j\hat{\rho}_j+\sum_{k=1}^{r}\xi_k\breve{\rho}_k))<0.$$

On the other hand, using the hypothesis specified in $(iv)$ and (4.1) together with (4.13)–(4.15), we get

$$\sum_{i=1}^{p}\mu_i\ \bar{\rho}_i+\sum_{j=1}^{q}\eta_j\ \hat{\rho}_j+\sum_{k=1}^{r}\xi_k\ \breve{\rho}_k\ge 0,$$

and

$$\sum_{i=1}^{p}\mu_i\{\nabla f(s,y^i)+A(y^i)^T\alpha^i-\lambda[\nabla g(s,y^i)-B(y^i)^T\beta^i]\}$$
$$+\sum_{j=1}^{q}\eta_j[\nabla G_j(s)+C_j^T\gamma^j]+\sum_{k=1}^{r}\xi_k\nabla H_k(s)=0.$$

From the above two inequalities and the fact $\Phi(x,s,(0,r))\geq 0, r\geq 0$, it follows that

$$\Phi(x,s,(\sum_{i=1}^{p}\mu_i\{\nabla f(s,y^i)+A(y^i)^T\alpha^i-\lambda[\nabla g(s,y^i)-B(y^i)^T\beta^i]\}$$
$$+\sum_{j=1}^{q}\eta_j[\nabla G_j(s)+C_j^T\gamma^j]+\sum_{k=1}^{r}\xi_k\nabla H_k(s),\sum_{i=1}^{p}\mu_i\bar{\rho}_i+\sum_{j=1}^{q}\eta_j\hat{\rho}_j+\sum_{k=1}^{r}\xi_k\breve{\rho}_k))\geq 0,$$

which contradicts (4.17). This completes the proof. □

Theorem 4.2 (Strong Duality). *Let $x^*$ be an optimal solution for* (P) *at which a suitable constraint qualification is satisfied [19]. Furthermore, assume that the weak duality Theorem* 4.1 *holds for all feasible of* (D). *Then there exist $(p^*,\bar{y}^*,u^*,\alpha^*,\ \beta^*)\in\mathbb{K},\ v^*\in R_+^q,\gamma^{*j}\in R^{n_j},j\in\underline{q}$ and $\lambda^*\in R_+$ such that $z^*=(p^*,\bar{y}^*,u^*,\alpha^*,\beta^*;x^*,\ v^*,w^*,\gamma^*,\lambda^*)$ is an optimal solution of* (D) *and $\varphi_0(x^*)=\lambda^*$.*

*Proof.* Since $x^*$ is an optimal solution of (P), by Theorem (2.1), there exist $p^*\in\underline{n+1},\bar{y}^*,u^*,\alpha^{*i},\beta^{*i},i\in\underline{p}^*,v^*,w^*,\gamma^{*j},j\in\underline{q}$ , and $\lambda^*(=\varphi_0(x^*))$, as specified above, such that $z^*$ is a feasible solution of (D). Since $\varphi_0(x^*)=\lambda^*$, the optimality of $z^*$ for (D) follows from Theorem (4.1). □

Theorem 4.3 (Strict converse duality). *Let $x^*$ and $(\tilde{p},\bar{\tilde{y}},\tilde{u},\tilde{\alpha},\tilde{\beta};\tilde{x},\tilde{v},\tilde{w},\tilde{\gamma},\tilde{\lambda})$ be an optimal solution for* (P) *and* (D), *respectively. Furthermore, assume that the following conditions hold:*

(i) $\{f(.,\bar{\tilde{y}}^{\,i})+\langle\tilde{\alpha}^i,A(\bar{\tilde{y}}^i).\rangle-\tilde{\lambda}[g(.,\bar{\tilde{y}}^i)-\left\langle\tilde{\beta}^i,B(\bar{\tilde{y}}^i).\right\rangle\}$ *is strictly $(\Phi,\overline{\rho}_i)$-invex at $\tilde{x}$, for $i\in\underline{p}$,*

(ii) $G_j(.)+\langle\tilde{\gamma}^j,C_j.\rangle$ *is strictly $(\Phi,\hat{\rho}_j)$-invex at $\tilde{x}$, for $j\in\underline{q}$,*

(iii) $H_k(.)$ *is strictly $(\Phi,\breve{\rho}_k)$-invex at $\tilde{x}$, for $k\in\underline{r}$,*

(iv) $\sum_{i=1}^{p}\tilde{u}_i\ \overline{\rho}_i+\sum_{j=1}^{q}\tilde{v}_j\ \hat{\rho}_j+\sum_{k=1}^{r}\tilde{w}_k\ \breve{\rho}_k\geq 0.$

*Then $\tilde{x}=x^*$; that is, $\tilde{x}$ is an optimal solution for $P$ and $\varphi_0(\tilde{x})=\tilde{\lambda}$.*

*Proof.* Suppose to contrary that $\tilde{x}\neq x^*$. From Theorem (4.2), we have

$$\varphi_0(x^*)=\tilde{\lambda}. \tag{4.18}$$

Similar to the proof of Theorem (4.1), we have

$$\Theta_1(x^*) - \Theta_1(\tilde{x}) > \sum_{i=1}^{p} \tilde{u}_i\{\Phi(x^*, \tilde{x}, (\nabla f(\tilde{x}, \bar{\tilde{y}}^i) + A(\bar{\tilde{y}}^{\,i})^T \tilde{\alpha}^i - \tilde{\lambda}[\nabla g(\tilde{x}, \bar{\tilde{y}}^i) - B(\bar{\tilde{y}}^i)^T \tilde{\beta}^i], \bar{\rho}_i))\} + \sum_{j=1}^{q} \tilde{v}_j\{\Phi(x^*, \tilde{x}, (\nabla G_j(\tilde{x}) + C_j^T \tilde{\gamma}^j, \hat{\rho}_j))\} + \sum_{k=1}^{r} \tilde{w}_k\{\Phi(x^*, \tilde{x}, (\nabla H_k(\tilde{x}), \breve{\rho}_k))\}. \tag{4.19}$$

Now we introduce the following notations:

$$\mu_i = \frac{\tilde{u}_i}{\sum_{i=1}^{p} \tilde{u}_i + \sum_{j=1}^{q} \tilde{v}_j + \sum_{k=1}^{r} \tilde{w}_k}, i \in \underline{p}, \tag{4.20}$$

$$\eta_j = \frac{\tilde{v}_j}{\sum_{i=1}^{p} \tilde{u}_i + \sum_{j=1}^{q} \tilde{v}_j + \sum_{k=1}^{r} \tilde{w}_k}, j \in J_+, \tag{4.21}$$

$$\xi_k = \frac{\tilde{w}_k}{\sum_{i=1}^{p} \tilde{u}_i + \sum_{j=1}^{q} \tilde{v}_j + \sum_{k=1}^{r} \tilde{w}_k}, k \in K_*. \tag{4.22}$$

Note that $0 \le \mu_i \le 1, i \in \underline{p}, 0 \le \eta_j \le 1, j \in J_+, 0 \le \xi_k \le 1, k \in K_*$, and moreover

$$\sum_{i=1}^{p} \mu_i + \sum_{j=1}^{q} \eta_j + \sum_{k=1}^{r} \xi_k = 1. \tag{4.23}$$

On combining (4.19)–(4.22), we have

$$\sum_{i=1}^{p} \mu_i\{f(x^*, \bar{\tilde{y}}^i) + \langle \tilde{\alpha}^i, A(\bar{\tilde{y}}^i)x^* \rangle - \tilde{\lambda}[g(x^*, \bar{\tilde{y}}^i) - \left\langle \tilde{\beta}^i, B(\bar{\tilde{y}}^i)x^* \right\rangle]\} + \sum_{j=1}^{q} \eta_j [G_j(x^*) + \langle \tilde{\gamma}^j, C_j x^* \rangle] + \sum_{k=1}^{r} \xi_k H_k(x^*) - [\sum_{i=1}^{p} \mu_i\{f(\tilde{x}, \bar{\tilde{y}}^i) + \langle \tilde{\alpha}^i, A(\bar{\tilde{y}}^i)\tilde{x} \rangle - \tilde{\lambda}[g(\tilde{x}, \bar{\tilde{y}}^i) - \left\langle \tilde{\beta}^i, B(\bar{\tilde{y}}^i)\tilde{x} \right\rangle]\} + \sum_{j=1}^{q} \eta_j [G_j(\tilde{x}) + \langle \tilde{\gamma}^j, C_j \tilde{x} \rangle] + \sum_{k=1}^{r} \xi_k H_k(\tilde{x})]$$
$$> \sum_{i=1}^{p} \mu_i\{\Phi(x^*, \tilde{x}, (\nabla f(\tilde{x}, \bar{\tilde{y}}^i) + A(\bar{\tilde{y}}^{\,i})^T \tilde{\alpha}^i - \tilde{\lambda}[\nabla g(\tilde{x}, \bar{\tilde{y}}^i) - B(\bar{\tilde{y}}^i)^T \tilde{\beta}^i], \bar{\rho}_i))\}$$

$$+\sum_{j=1}^{q}\eta_j\{\Phi(x^*,\tilde{x},(\nabla G_j(\tilde{x})+C_j^T\tilde{\gamma}^j,\hat{\rho}_j))\}+\sum_{k=1}^{r}\xi_k\{\Phi(x^*,\tilde{x},(\nabla H_k(\tilde{x}),\breve{\rho}_k))\}.$$

Thus, by $\Phi$-convexity on $R^{n+1}$ and (4.23), we conclude that

(4.24)

$$\sum_{i=1}^{p}\mu_i\{f(x^*,\bar{\tilde{y}}^i)+\left\langle\tilde{\alpha}^i,A(\bar{\tilde{y}}^i)x^*\right\rangle-\tilde{\lambda}[g(x^*,\bar{\tilde{y}}^i)-\left\langle\tilde{\beta}^i,B(\bar{\tilde{y}}^i)x^*\right\rangle]\}+\sum_{j=1}^{q}\eta_j[G_j(x^*)$$

$$+\left\langle\tilde{\gamma}^j,C_jx^*\right\rangle]+\sum_{k=1}^{r}\xi_kH_k(x^*)-[\sum_{i=1}^{p}\mu_i\{f(\tilde{x},\bar{\tilde{y}}^i)+\left\langle\tilde{\alpha}^i,A(\bar{\tilde{y}}^i)\tilde{x}\right\rangle-\tilde{\lambda}[g(\tilde{x},\bar{\tilde{y}}^i)$$

$$-\left\langle\tilde{\beta}^i,B(\bar{\tilde{y}}^i)\tilde{x}\right\rangle]\}+\sum_{j=1}^{q}\eta_j[G_j(\tilde{x})+\left\langle\tilde{\gamma}^j,C_j\tilde{x}\right\rangle]+\sum_{k=1}^{r}\xi_kH_k(\tilde{x})]$$

$$>\Phi(x^*,\tilde{x},(\sum_{i=1}^{p}\mu_i\{\nabla f(\tilde{x},\bar{\tilde{y}}^i)+A(\bar{\tilde{y}}^{\,i})^T\tilde{\alpha}^i-\tilde{\lambda}[\nabla g(\tilde{x},\bar{\tilde{y}}^i)-B(\bar{\tilde{y}}^i)^T\tilde{\beta}^i]\}$$

$$+\sum_{j=1}^{q}\eta_j\{\nabla G_j(\tilde{x})+C_j^T\tilde{\gamma}^j\}+\sum_{k=1}^{r}\xi_k\nabla H_k(\tilde{x}),\sum_{i=1}^{p}\mu_i\bar{\rho}_i+\sum_{j=1}^{q}\eta_j\hat{\rho}_j+\sum_{k=1}^{r}\xi_k\breve{\rho}_k)).$$

On the other hand, using the hypothesis specified in $(iv)$ and (4.1) together with (4.20)–(4.22), we get

$$\sum_{i=1}^{p}\mu_i\ \bar{\rho}_i+\sum_{j=1}^{q}\eta_j\ \hat{\rho}_j+\sum_{k=1}^{r}\xi_k\ \breve{\rho}_k\geq 0,$$

and

$$\sum_{i=1}^{p}\mu_i\{\nabla f(\tilde{x},\bar{\tilde{y}}^i)+A(\bar{\tilde{y}}^{\,i})^T\tilde{\alpha}^i-\tilde{\lambda}[\nabla g(\tilde{x},\bar{\tilde{y}}^i)-B(\bar{\tilde{y}}^i)^T\tilde{\beta}^i]\}$$
$$+\sum_{j=1}^{q}\eta_j\{\nabla G_j(\tilde{x})+C_j^T\tilde{\gamma}^j\}+\sum_{k=1}^{r}\xi_k\nabla H_k(\tilde{x})=0.$$

From the above two inequalities and the fact $\Phi(x^*,\tilde{x},(0,r))\geq 0, r\geq 0$, it follows that

$$\Phi(x^*,\tilde{x},(\sum_{i=1}^{p}\mu_i\{\nabla f(\tilde{x},\bar{\tilde{y}}^i)+A(\bar{\tilde{y}}^{\,i})^T\tilde{\alpha}^i-\tilde{\lambda}[\nabla g(\tilde{x},\bar{\tilde{y}}^i)-B(\bar{\tilde{y}}^i)^T\tilde{\beta}^i]\}$$

$$+\sum_{j=1}^{q}\eta_j\{\nabla G_j(\tilde{x})+C_j^T\tilde{\gamma}^j\}+\sum_{k=1}^{r}\xi_k\nabla H_k(\tilde{x}),\sum_{i=1}^{p}\mu_i\bar{\rho}_i+\sum_{j=1}^{q}\eta_j\hat{\rho}_j+\sum_{k=1}^{r}\xi_k\breve{\rho}_k))\geq 0.$$

Therefore, from (4.24), we conclude that

$$\sum_{i=1}^{p}\mu_i\{f(x^*,\bar{\tilde{y}}^i)+\left\langle\tilde{\alpha}^i,A(\bar{\tilde{y}}^i)x^*\right\rangle-\tilde{\lambda}[g(x^*,\bar{\tilde{y}}^i)-\left\langle\tilde{\beta}^i,B(\bar{\tilde{y}}^i)x^*\right\rangle]\}+\sum_{j=1}^{q}\eta_j[G_j(x^*)$$

$$+\left\langle\tilde{\gamma}^j, C_j x^*\right\rangle] + \sum_{k=1}^{r}\xi_k H_k(x^*) - [\sum_{i=1}^{p}\mu_i\{f(\tilde{x},\bar{\tilde{y}}^i) + \left\langle\tilde{\alpha}^i, A(\bar{\tilde{y}}^i)\tilde{x}\right\rangle - \tilde{\lambda}[g(\tilde{x},\bar{\tilde{y}}^i) - \left\langle\tilde{\beta}^i, B(\bar{\tilde{y}}^i)\tilde{x}\right\rangle]\} + \sum_{j=1}^{q}\eta_j[G_j(\tilde{x}) + \left\langle\tilde{\gamma}^j, C_j\tilde{x}\right\rangle] + \sum_{k=1}^{r}\xi_k H_k(\tilde{x})] > 0,$$

which together with the feasibility of $\tilde{x}$ yields

$$\sum_{i=1}^{p}\mu_i\{f(x^*,\bar{\tilde{y}}^i) + \left\langle\tilde{\alpha}^i, A(\bar{\tilde{y}}^i)x^*\right\rangle - \tilde{\lambda}[g(x^*,\bar{\tilde{y}}^i) - \left\langle\tilde{\beta}^i, B(\bar{\tilde{y}}^i)x^*\right\rangle]\} + \sum_{j=1}^{q}\eta_j[G_j(x^*) + \left\langle\tilde{\gamma}^j, C_j x^*\right\rangle] + \sum_{k=1}^{r}\xi_k H_k(x^*) > 0.$$

The above inequality along with (4.20)–(4.22) gives

$$\sum_{i=1}^{p}\tilde{u}_i\{f(x^*,\bar{\tilde{y}}^i) + \left\langle\tilde{\alpha}^i, A(\bar{\tilde{y}}^i)x^*\right\rangle - \tilde{\lambda}[g(x^*,\bar{\tilde{y}}^i) - \left\langle\tilde{\beta}^i, B(\bar{\tilde{y}}^i)x^*\right\rangle]\} + \sum_{j=1}^{q}\tilde{v}_j[G_j(x^*) + \left\langle\tilde{\gamma}^j, C_j x^*\right\rangle] + \sum_{k=1}^{r}\tilde{w}_k H_k(x^*) > 0.$$

Now we summarize to get

$$\begin{aligned}
0 &< \sum_{i=1}^{p}\tilde{u}_i\{f(x^*,\tilde{y}^i) + \left\langle\tilde{\alpha}^i, A(\tilde{y}^i)x^*\right\rangle - \tilde{\lambda}[g(x^*,\tilde{y}^i) - \left\langle\tilde{\beta}^i, B(\tilde{y}^i)x^*\right\rangle]\} \\
&\quad + \sum_{j=1}^{q}\tilde{v}_j[G_j(x^*) + \left\langle\tilde{\gamma}^j, C_j x^*\right\rangle] + \sum_{k=1}^{r}\tilde{w}_k H_k(x^*) \\
&\leq \sum_{i=1}^{p}\tilde{u}_i\{f(x^*,\tilde{y}^i) + \left\|\tilde{\alpha}^i\right\|_a^* \left\|A(\tilde{y}^i)x^*\right\|_a - \tilde{\lambda}[g(x^*,\tilde{y}^i) - \left\|\tilde{\beta}^i\right\|_b^* \left\|B(\tilde{y}^i)x^*\right\|_b]\} \\
&\quad + \sum_{j=1}^{q}\tilde{v}_i\{G_j(x^*) + \left\|\tilde{\gamma}^j\right\|_{c(j)}^* \left\|C_j x^*\right\|_{c(j)}\} \\
&\qquad \text{(by Lemma 2.1 and feasibility of } x^*) \\
&\leq \sum_{i=1}^{p}\tilde{u}_i\{f(x^*,\tilde{y}^i) + \left\|A(\tilde{y}^i)x^*\right\|_a - \tilde{\lambda}[g(x^*,\tilde{y}^i) - \left\|B(\tilde{y}^i)x^*\right\|_b]\} \\
&\quad + \sum_{j=1}^{q}\tilde{v}_i\{G_j(x^*) + \left\|C_j x^*\right\|_{c(j)}\} \quad \text{(by (4.5) and (4.6))}
\end{aligned}$$

$$\leq \sum_{i=1}^{p} \tilde{u}_i\{f(x^*,\tilde{y}^i)+\left\|A(\tilde{y}^i)x^*\right\|_a-\tilde{\lambda}[g(x^*,\tilde{y}^i)-\left\|B(\tilde{y}^i)x^*\right\|_b]\} \text{ (by feasibility of } x^*).$$

That is,

$$\sum_{i=1}^{p} \tilde{u}_i\{f(x^*,\bar{\tilde{y}}^i)+\left\|A(\bar{\tilde{y}}^i)x^*\right\|_a-\tilde{\lambda}[g(x^*,\bar{\tilde{y}}^i)-\left\|B(\bar{\tilde{y}}^i)x^*\right\|_b]\} > 0.$$

Using Lemma 2.2 and above inequality, we see that

$$\tilde{\lambda} < \frac{\sum_{i=1}^{p} \tilde{u}_i f\left(x^*,\tilde{y}^i\right) + \left\|A(\tilde{y}^i)x^*\right\|_a}{\sum_{i=1}^{p} \tilde{u}_i g\left(x^*,\tilde{y}^i\right) - \left\|B(\tilde{y}^i)x^*\right\|_b}$$

$$\leq \max_{p\in\underline{n+1}} \max_{\substack{d\in U\\ \tilde{y}^i\in Y}} \frac{\sum_{i=1}^{p} d_i[f\left(x^*,\tilde{y}^i\right) + \left\|A(\tilde{y}^i)x^*\right\|_a]}{\sum_{i=1}^{p} d_i[g\left(x^*,\tilde{y}^i\right) - \left\|B(\tilde{y}^i)x^*\right\|_b]} = \varphi_0(x^*),$$

which contradicts (4.18). This completes the proof. □

## 5. CONCLUSIONS

In this paper, we have discussed the sufficient optimality conditions and duality theorems for a class of minimax fractional programming problem under the assumptions of $(\Phi,\rho)$-invexity. It will be interesting to see whether or not the second and higher order duality results developed in this paper hold. This would be task of some of our forthcoming works.

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