CHARACTERIZING FINITE p-GROUPS BY THEIR SCHUR MULTIPLIERS, $t(G) = 5$

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Let G be a finite p-group of order p^n. It is known that $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$ and $t(G) \geq 0$. The structure of G for $t(G) \leq 4$ was determined by several authors. In this paper we will describe all the possible structures of G for $t(G) = 5$.

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1. INTRODUCTION AND PRELIMINARIES

Let G be a finite p-group and let $\mathcal{M}(G)$ denote the Schur multiplier of G. It is known that $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$, where $t(G) \geq 0$ by the result of Green in [8].

The structure of G for $t(G) = 0, 1$ was determined in [1]. In the case $t(G) = 2$ and 3, Zhou in [18] and Ellis in [5] determined the structure of G, respectively. Recently, the author described in [13] all the finite p-groups with $t(G) = 4$. In the present paper, we will describe the structure of all finite non-abelian p-groups G with $t(G) = 5$. Our method is quite different to that of [1, 5, 18] and depends on the results of [11, 12]. We will use the notations and the terminology in [5, 13]. In this paper, D_8 and Q_8 denote the dihedral and quaternion group of order 8, E_1 and E_2 denote the extra special p-groups of order p^3 of exponent p and p^2, respectively. E_4 denotes the unique central product of a cyclic group of order p^2 and a non-abelian group of order p^3. Also $\mathbb{Z}_{p^n}^{(m)}$ denotes the direct product of m copies of the cyclic group of order p^n. We say that G has the property $t(G) = 5$ or briefly with $t(G) = 5$ if the order of its Schur multiplier is equal to $p^{\frac{1}{2}n(n-1)-5}$.

We will state without proof some theorems which play an important role in the proof of our main result.

THEOREM 1.1 (See [11], Main Theorem). *Let G be a non-abelian finite p-group of order p^n. If $|G'| = p^k$, then we have*

$$|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)+1}.$$

In particular,

$$|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1},$$

and the equality holds in this last bound if and only if $G = E_1 \times Z$, where Z is an elementary abelian p-group.

The following theorem is a consequence of ([12], Main Theorem).

Theorem 1.2. *Let G be a non abelian p-group of order p^n. Then $|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)}$ if and only if G is isomorphic to one of the following groups.*

(i) *$G \cong D_8 \times Z$, where Z is an elementary abelian p-group.*

(ii) *$G \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p \quad (p \neq 2)$.*

Theorem 1.3 (See [10], Theorem 2.2.10). *For every finite groups H and K, we have*

$$\mathcal{M}(H \times K) \cong \mathcal{M}(H) \times \mathcal{M}(K) \times \frac{H}{H'} \otimes \frac{K}{K'}.$$

Theorem 1.4 (See [10], Theorem 3.3.6). *Let G be an extra special p-group of order p^{2m+1}. Then*

(i) *If $m \geq 2$, then $|\mathcal{M}(G)| = p^{2m^2-m-1}$.*

(ii) *If $m = 1$, then the orders of the Schur multipliers of D_8, Q_8, E_1 and E_2 are equal to $2, 1, p^2$ and 1, respectively.*

2. MAIN THEOREM

In this section, we will characterize all the finite non-abelian p-groups G with the property $t(G) = 5$. In fact, we have

Theorem 2.1 (Main Theorem). *Let G be a non-abelian p-group of order p^n. Then*

$$|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-5}$$

if and only if G is isomorphic to one of the following groups.

(1) $D_8 \times \mathbb{Z}_2^{(3)}$,

(2) $E_1 \times \mathbb{Z}_p^{(4)}$,

(3) $E_2 \times \mathbb{Z}_p^{(2)}$,

(4) $E_4 \times \mathbb{Z}_p$,

(5) *extra special p-group of order p^5,*

(6) $\langle a, b \mid a^{p^2} = 1, b^{p^2} = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^p \rangle$ $(p \neq 2, 3)$,

(7) $\langle a, b \mid a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = a^p, [a, b, b, b] = 1 \rangle$ $(p \neq 2, 3)$,

(8) $\langle a, b \mid a^{p^2} = b^p = 1, [a, b, a] = 1, [a, b, b] = a^{np}, [a, b, b, b] = 1 \rangle$, *where n is a fixed quadratic non-residue of p and $(p \neq 2, 3)$,*

(9) $\langle a, b \mid a^9 = 1, b^3 = a^3, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1 \rangle$,

(10) $\langle a, b \mid a^9 = 1, b^3 = a^3, [a, b, a] = 1, [a, b, b] = a^3, [a, b, b, b] = 1\rangle$,
(11) $\langle a, b \mid a^p = 1, b^p = [a, b, b], [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1\rangle$ $(p \neq 2)$,
(12) $\langle a, b \mid a^p = b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1\rangle$ $(p \neq 2)$, *For* $p = 3$, (11) *and* (12) *are isomorphic.*
(13) D_{16},
(14) $\langle a, b \mid a^4 = b^4 = 1, a^{-1}ba = b^{-1}\rangle$,
(15) $Q_8 \times \mathbb{Z}_2^{(2)}$,
(16) $(D_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$,
(17) $(Q_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$,
(18) $\mathbb{Z}_2 \times \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab\rangle$.

We separate the proof of it into several steps as follows.

Lemma 2.2. *Let G be a p-group of order p^n and $|G'| = p^k (k \geq 2)$ with $t(G) = 5$. Then $n \leq 4$ unless $k = 2$, in this case $n \leq 6$.*

Proof. By virtue of Theorem 1.1, we have

$$\frac{1}{2}(n^2 - n - 10) \leq \frac{1}{2}(n + k - 2)(n - k - 1) + 1 \leq \frac{1}{2}n(n - 3) + 1,$$

and the conclusion follows. □

Theorem 2.3. *Let G be a non-abelian finite p-group of order p^n with $t(G) = 5$. Then $|G| \leq p^7$. In the case that $n = 6$ and $n = 7$, G is isomorphic to*

$$D_8 \times \mathbb{Z}_2^{(3)} \text{ and } E_1 \times \mathbb{Z}_p^{(4)},$$

respectively.

Proof. One can easily check that $n \leq 7$ by using Theorem 1.1.

In the case $n = 7$, Lemma 2.2 shows that $|G'| = p$. Since $|\mathcal{M}(G)| = p^{16}$ and equality holds in Theorem 1.1, we should have $G \cong E_1 \times \mathbb{Z}_p^{(4)}$. When $n = 6$, $|\mathcal{M}(G)| = p^{10}$ and by a consequence of ([12], Main Theorem), we have $G \cong D_8 \times \mathbb{Z}_2^{(3)}$. □

As mentioned in Lemma 2.2 and Theorem 2.3, we may assume that $n \leq 5$. First assume that $p \neq 2$.

Theorem 2.4. *Let $|G| = p^5$ $(p \neq 2)$ and $|G'| \geq p^2$. Then there is no such group G with $t(G) = 5$.*

Proof. Using Lemma 2.2, we may assume that $|G'| = p^2$.

For each central subgroup K of order p, ([10], Theorem 4.1) implies that

$$p^5 = |\mathcal{M}(G)| \leq p^2\ |\mathcal{M}(G/K)|.$$

If for every central subgroup K, $|\mathcal{M}(G/K)| = p^4$ the proof of ([12], Main Theorem) shows that $G \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$ and hence, $|\mathcal{M}(G)| = p^6$, which is a contradiction. Thus, there exists a central subgroup K such that $|\mathcal{M}(G/K)| \leq p^3$. Since $p \neq 2$ and $|G/K| = p^4$, Theorem 1.2 shows that $|\mathcal{M}(G/K)| \leq p^2$, and so $|\mathcal{M}(G)| \leq p^4$, which contradicts the assumption. $\square$

THEOREM 2.5. *Let* $|G| = p^5$ $(p \neq 2)$ *and* $|Z(G)| = p^3$ *with* $t(G) = 5$. *Then* G *is isomorphic to*

$$E_2 \times \mathbb{Z}_p^{(2)} \text{ or } E_4 \times \mathbb{Z}_p.$$

Proof. It is known by ([10], Theorem 4.1) that,

$$|\mathcal{M}(G)||G'| \leq |\mathcal{M}(G/G')||\mathcal{M}(G')||G' \otimes G/Z(G)|.$$

We know that $|G'| = p$ by Theorem 2. Now, if G/G' is not elementary abelian, then $|\mathcal{M}(G/G')| \leq p^3$, and so $|\mathcal{M}(G)| \leq p^4$, which is impossible. Therefore, G/G' is elementary abelian. On the other hand, ([9], Theorem 2.2) implies that $Z(G)$ is of exponent at most p^2. Thus, two cases may be considered.

Case I. First suppose that $Z(G)$ is of exponent p. By ([11], Lemma 2.1), we should have $G \cong H \times \mathbb{Z}_p^{(2)}$, where H is extra special of order p^3. Since $|\mathcal{M}(G)| = p^5$, Theorems 1.3 and 1.4 imply that $H \cong E_2$.

Case II. In the case that $Z(G)$ is of exponent p^2, as in pervious part one can see that $G \cong H \times \mathbb{Z}_{p^2}$, where H is extra special of order p^3 or $G \cong E_4 \times \mathbb{Z}_p$. By invoking Theorems 1.3 and 1.4, the order of the Schur multiplier of $H \times \mathbb{Z}_{p^2}$ is at most p^4, and hence, does not have the property $t(G) = 5$. On the other hand, by ([13], Lemma 2.8) and Theorem 1.3, we should have $|\mathcal{M}(E_4 \times \mathbb{Z}_p)| = p^5$, as required. $\square$

THEOREM 2.6. *Let* $|G| = p^5$ $(p \neq 2)$ *and* $|Z(G)| = p^2$. *Then there is no such group* G *with* $t(G) = 5$.

Proof. Theorem implies $|G'| = p$. Now we may assume that G/G' is not elementary abelian by using ([11], Lemma 2.1). Using ([6], Proposition 1), we have $p\, |\mathcal{M}(G)| \leq |\mathcal{M}(G/G')||G' \otimes G/Z(G)|$, and so $p^6 \leq |\mathcal{M}(G/G')||G' \otimes G/Z(G)|$. Thus, we should have $G/Z(G) \cong \mathbb{Z}_p^{(3)}$ and $G/G' \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$. Hence, $Z(G)$ and the Frattini subgroup coincide, and so ([6], Proposition 1) (see also [4], Proposition 5 (i) and (ii)) shows that

$$p^2|\mathcal{M}(G)| \leq |\mathcal{M}(G/G')||G' \otimes G/Z(G)| \leq p^6.$$

Thus, $|\mathcal{M}(G)| \leq p^4$, which is a contradiction. $\square$

LEMMA 2.7. *Every extra special* p*-group of order* p^5 *has the property* $t(G) = 5$.

Proof. It is straightforward by Theorem 1.4. $\square$

THEOREM 2.8. *Let $|G| = p^4$ $(p \neq 2)$ and $|G'| = p$ with $t(G) = 5$. Then G is isomorphic to*

$$\langle a, b \mid a^{p^2} = 1, b^{p^2} = 1, [a,b,a] = [a,b,b] = 1, [a,b] = a^p\rangle.$$

Proof. First suppose that G/G' is elementary. By ([11], Lemma 2.1), we have $G \cong H \times \mathbb{Z}_p$ or $G \cong E_4$. The order of Schur multipliers of both of them is at least p^2 by using ([13], Lemma 2.8) and Theorem 1.4. Thus, G/G' can not be elementary abelian. Since G^p and G' are contained in $Z(G)$, we consider two cases.

Case I. Assuming first that $G' \cap G^p = 1$, then $G/G^p \cong E_1$, and so $|\mathcal{M}(G)| \geq |\mathcal{M}(E_1)| = p^2$ directly by using ([10], Corollary 2.5.3 (i)), which contradicts $t(G) = 5$.

Case II. In this case, we have two possibilities for $Z(G)$. The first possibility is $Z(G) = G^p \cong \mathbb{Z}_{p^2}$, thus, G is of exponent p^3 and similar to the proof of ([13], Lemma 2.8), we have $|\mathcal{M}(G)| = 1$. The second possibility is $Z(G) = G^p \cong \mathbb{Z}_p \times G'$. By ([2], pp. 87–88), there is a unique group of order p^4 with this properties, which is isomorphic to

$$\langle a, b \mid a^{p^2} = 1, b^{p^2} = 1, [a,b,a] = [a,b,b] = 1, [a,b] = a^p\rangle. \quad \square$$

LEMMA 2.9. *Let $|G| = p^4$ $(p \neq 2)$ and $|G'| = p^2$ with $t(G) = 5$. Then G is isomorphic to*

$$\langle a, b \mid a^{p^2} = b^p = 1, [a,b,a] = [a,b,b] = a^p, [a,b,b,b] = 1\rangle,$$

$$\langle a, b \mid a^{p^2} = b^p = 1, [a,b,a] = 1, [a,b,b] = a^{np}, [a,b,b,b] = 1\rangle,$$

where n is a fixed quadratic non-residue of p and $p \neq 3$,

$$\langle a, b \mid a^9 = 1, b^3 = a^3, [a,b,a] = 1, [a,b,b] = a^6, [a,b,b,b] = 1\rangle,$$

$$\langle a, b \mid a^9 = 1, b^3 = a^3, [a,b,a] = 1, [a,b,b] = a^3, [a,b,b,b] = 1\rangle,$$

$$\langle a, b \mid a^p = b^p = 1, [a,b,a] = [a,b,b,a] = [a,b,b,b] = 1\rangle.$$

$$\langle a, b \mid a^p = 1, b^p = [a,b,b], [a,b,a] = [a,b,b,a] = [a,b,b,b] = 1\rangle.$$

For $p = 3$, the last two groups are isomorphic.

Proof. The result is obtained from ([5], pp. 4177) and ([2], pp. 88), see also [16], pp. 196–198. $\square$

LEMMA 2.10. *Let G be a p-group of order 16 with $t(G) = 5$. Then G is isomorphic to*

$$D_{16} \text{ or } \langle a, b \mid a^4 = b^4 = 1, a^{-1}ba = b^{-1}\rangle.$$

Proof. See table I on [14]. $\square$

Lemma 2.11. *Let G be a p-group of order 32 with $t(G) = 5$. Then G is isomorphic to*

$$Q_8 \times \mathbb{Z}_2^{(2)}, (D_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2, (Q_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \text{ or}$$
$$\mathbb{Z}_2 \times \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle.$$

Proof. These groups are obtained by using the HAP package [7] of GAP [17]. □

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