CHARACTERIZING FINITE $p$-GROUPS
BY THEIR SCHUR MULTIPLIERS, $t(G) = 5$

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Let $G$ be a finite $p$-group of order $p^n$. It is known that $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$ and $t(G) \geq 0$. The structure of $G$ for $t(G) \leq 4$ was determined by several authors. In this paper we will describe all the possible structures of $G$ for $t(G) = 5$.

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1. INTRODUCTION AND PRELIMINARIES

Let $G$ be a finite $p$-group and let $\mathcal{M}(G)$ denote the Schur multiplier of $G$. It is known that $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$, where $t(G) \geq 0$ by the result of Green in [8].

The structure of $G$ for $t(G) = 0,1$ was determined in [1]. In the case $t(G) = 2$ and 3, Zhou in [18] and Ellis in [5] determined the structure of $G$, respectively. Recently, the author described in [13] all the finite $p$-groups with $t(G) = 4$. In the present paper, we will describe the structure of all finite non-abelian $p$-groups $G$ with $t(G) = 5$. Our method is quite different to that of [1, 5, 18] and depends on the results of [11, 12]. We will use the notations and the terminology in [5, 13]. In this paper, $D_8$ and $Q_8$ denote the dihedral and quaternion group of order 8, $E_1$ and $E_2$ denote the extra special $p$-groups of order $p^3$ of exponent $p$ and $p^2$, respectively. $E_4$ denotes the unique central product of a cyclic group of order $p^2$ and a non-abelian group of order $p^3$. Also $\mathbb{Z}_{p^n}^{(m)}$ denotes the direct product of $m$ copies of the cyclic group of order $p^n$.

We say that $G$ has the property $t(G) = 5$ or briefly with $t(G) = 5$ if the order of its Schur multiplier is equal to $p^{\frac{1}{2}n(n-1)-5}$.

We will state without proof some theorems which play an important role in the proof of our main result.

Theorem 1.1 (See [11], Main Theorem). Let $G$ be a non-abelian finite $p$-group of order $p^n$. If $|G'| = p^k$, then we have

$$|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)+1}.$$
In particular, 
\[ |\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1}, \]
and the equality holds in this last bound if and only if \( G = E_1 \times Z \), where \( Z \) is an elementary abelian \( p \)-group.

The following theorem is a consequence of ([12], Main Theorem).

**Theorem 1.2.** Let \( G \) be a non-abelian \( p \)-group of order \( p^n \). Then \( |\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)} \) if and only if \( G \) is isomorphic to one of the following groups.

(i) \( G \cong D_8 \times Z \), where \( Z \) is an elementary abelian \( p \)-group.

(ii) \( G \cong \mathbb{Z}_p(4) \times \mathbb{Z}_p \) (\( p \neq 2 \)).

**Theorem 1.3** (See [10], Theorem 2.2.10). For every finite groups \( H \) and \( K \), we have

\[ \mathcal{M}(H \times K) \cong \mathcal{M}(H) \times \mathcal{M}(K) \times \frac{H}{H'} \otimes \frac{K}{K'}. \]

**Theorem 1.4** (See [10], Theorem 3.3.6). Let \( G \) be an extra special \( p \)-group of order \( p^{2m+1} \). Then

(i) If \( m \geq 2 \), then \( |\mathcal{M}(G)| = p^{2m^2-m-1} \).

(ii) If \( m = 1 \), then the orders of the Schur multipliers of \( D_8, Q_8, E_1 \) and \( E_2 \) are equal to 2, 1, \( p^2 \) and 1, respectively.

2. **MAIN THEOREM**

In this section, we will characterize all the finite non-abelian \( p \)-groups \( G \) with the property \( t(G) = 5 \). In fact, we have

**Theorem 2.1** (Main Theorem). Let \( G \) be a non-abelian \( p \)-group of order \( p^n \). Then

\[ |\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-5} \]

if and only if \( G \) is isomorphic to one of the following groups.

1. \( D_8 \times \mathbb{Z}_2^{(3)} \),
2. \( E_1 \times \mathbb{Z}_p^{(4)} \),
3. \( E_2 \times \mathbb{Z}_p^{(2)} \),
4. \( E_4 \times \mathbb{Z}_p \),
5. extra special \( p \)-group of order \( p^5 \),
6. \( \langle a, b \mid a^{p^2} = 1, b^{p^2} = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^p \rangle \) (\( p \neq 2, 3 \)),
7. \( \langle a, b \mid a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = a^p, [a, b, b, b] = 1 \rangle \) (\( p \neq 2, 3 \)),
8. \( \langle a, b \mid a^{p^2} = b^p = 1, [a, b, a] = 1, [a, b, b] = a^{np}, [a, b, b, b] = 1 \rangle \), where \( n \) is a fixed quadratic non-residue of \( p \) and \( p \neq 2, 3 \),
9. \( \langle a, b \mid a^9 = 1, b^3 = a^3, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1 \rangle \),
Characterizing finite $p$-groups by their Schur multipliers

(10) $\langle a, b \mid a^9 = 1, b^3 = a^3, [a, b, a] = 1, [a, b, b] = a^3, [a, b, b, b] = 1 \rangle$,
(11) $\langle a, b \mid a^p = 1, b^p = [a, b, b], [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1 \rangle$ $(p \neq 2)$,
(12) $\langle a, b \mid a^p = b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1 \rangle$ $(p \neq 2)$. For $p = 3$, (11) and (12) are isomorphic.

(13) $D_{16}$,
(14) $\langle a, b \mid a^4 = b^4 = 1, a^{-1}ba = b^{-1} \rangle$,
(15) $Q_8 \times \mathbb{Z}_2^{(2)}$,
(16) $(D_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$,
(17) $(Q_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$,
(18) $\mathbb{Z}_2 \times \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$.

We separate the proof of it into several steps as follows.

**Lemma 2.2.** Let $G$ be a $p$-group of order $p^n$ and $|G'| = p^k (k \geq 2)$ with $t(G) = 5$. Then $n \leq 4$ unless $k = 2$, in this case $n \leq 6$.

**Proof.** By virtue of Theorem 1.1, we have

$$\frac{1}{2}(n^2 - n - 10) \leq \frac{1}{2}(n + k - 2)(n - k - 1) + 1 \leq \frac{1}{2}n(n - 3) + 1,$$

and the conclusion follows. $\square$

**Theorem 2.3.** Let $G$ be a non-abelian finite $p$-group of order $p^n$ with $t(G) = 5$. Then $|G| \leq p^7$. In the case that $n = 6$ and $n = 7$, $G$ is isomorphic to $D_8 \times \mathbb{Z}_2^{(3)}$ and $E_1 \times \mathbb{Z}_p^{(4)}$, respectively.

**Proof.** One can easily check that $n \leq 7$ by using Theorem 1.1.

In the case $n = 7$, Lemma 2.2 shows that $|G'| = p$. Since $|\mathcal{M}(G)| = p^{16}$ and equality holds in Theorem 1.1, we should have $G \cong E_1 \times \mathbb{Z}_p^{(4)}$. When $n = 6$, $|\mathcal{M}(G)| = p^{10}$ and by a consequence of ([12], Main Theorem), we have $G \cong D_8 \times \mathbb{Z}_2^{(3)}$. $\square$

As mentioned in Lemma 2.2 and Theorem 2.3, we may assume that $n \leq 5$. First assume that $p \neq 2$.

**Theorem 2.4.** Let $|G| = p^5 (p \neq 2)$ and $|G'| \geq p^2$. Then there is no such group $G$ with $t(G) = 5$.

**Proof.** Using Lemma 2.2, we may assume that $|G'| = p^2$.

For each central subgroup $K$ of order $p$, ([10], Theorem 4.1) implies that

$$p^5 = |\mathcal{M}(G)| \leq p^2 |\mathcal{M}(G/K)|.$$

If for every central subgroup $K$, $|\mathcal{M}(G/K)| = p^4$ the proof of ([12], Main Theorem) shows that $G \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$ and hence, $|\mathcal{M}(G)| = p^6$, which is a
contradiction. Thus, there exists a central subgroup $K$ such that $|\mathcal{M}(G/K)| \leq p^3$. Since $p \neq 2$ and $|G/K| = p^4$, Theorem 1.2 shows that $|\mathcal{M}(G/K)| \leq p^2$, and so $|\mathcal{M}(G)| \leq p^4$, which contradicts the assumption. □

**Theorem 2.5.** Let $|G| = p^5$ ($p \neq 2$) and $|Z(G)| = p^3$ with $t(G) = 5$. Then $G$ is isomorphic to

$$E_2 \times \mathbb{Z}_p^{(2)} \text{ or } E_4 \times \mathbb{Z}_p.$$ 

**Proof.** It is known by ([10], Theorem 4.1) that,

$$|\mathcal{M}(G)||G'| \leq |\mathcal{M}(G/G')||\mathcal{M}(G')||G' \otimes G/Z(G)|.$$ 

We know that $|G'| = p$ by Theorem 2. Now, if $G/G'$ is not elementary abelian, then $|\mathcal{M}(G/G')| \leq p^3$, and so $|\mathcal{M}(G)| \leq p^4$, which is impossible. Therefore, $G/G'$ is elementary abelian. On the other hand, ([9], Theorem 2.2) implies that $Z(G)$ is of exponent at most $p^2$. Thus, two cases may be considered.

**Case I.** First suppose that $Z(G)$ is of exponent $p$. By ([11], Lemma 2.1), we should have $G \cong H \times \mathbb{Z}_p^{(2)}$, where $H$ is extra special of order $p^3$. Since $|\mathcal{M}(G)| = p^5$, Theorems 1.3 and 1.4 imply that $H \cong E_2$.

**Case II.** In the case that $Z(G)$ is of exponent $p^2$, as in previous part one can see that $G \cong H \times \mathbb{Z}_{p^2}$, where $H$ is extra special of order $p^3$ or $G \cong E_4 \times \mathbb{Z}_p$. By invoking Theorems 1.3 and 1.4, the order of the Schur multiplier of $H \times \mathbb{Z}_{p^2}$ is at most $p^4$, and hence, does not have the property $t(G) = 5$. On the other hand, by ([13], Lemma 2.8) and Theorem 1.3, we should have $|\mathcal{M}(E_4 \times \mathbb{Z}_p)| = p^5$, as required. □

**Theorem 2.6.** Let $|G| = p^5$ ($p \neq 2$) and $|Z(G)| = p^2$. Then there is no such group $G$ with $t(G) = 5$.

**Proof.** Theorem implies $|G'| = p$. Now we may assume that $G/G'$ is not elementary abelian by using ([11], Lemma 2.1). Using ([6], Proposition 1), we have $p |\mathcal{M}(G)| \leq |\mathcal{M}(G/G')||G' \otimes G/Z(G)|$, and so $p^6 \leq |\mathcal{M}(G/G')||G' \otimes G/Z(G)|$. Thus, we should have $G/Z(G) \cong \mathbb{Z}_p^{(3)}$ and $G/G' \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$. Hence, $Z(G)$ and the Frattini subgroup coincide, and so ([6], Proposition 1) (see also [4], Proposition 5 (i) and (ii)) shows that

$$p^2|\mathcal{M}(G)| \leq |\mathcal{M}(G/G')||G' \otimes G/Z(G)| \leq p^6.$$ 

Thus, $|\mathcal{M}(G)| \leq p^4$, which is a contradiction. □

**Lemma 2.7.** Every extra special $p$-group of order $p^5$ has the property $t(G) = 5$.

**Proof.** It is straightforward by Theorem 1.4. □
Theorem 2.8. Let $|G| = p^4$ ($p \neq 2$) and $|G'| = p$ with $t(G) = 5$. Then $G$ is isomorphic to

$$\langle a, b \mid a^{p^2} = 1, b^{p^2} = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^p \rangle.$$ 

Proof. First suppose that $G/G'$ is elementary. By ([11], Lemma 2.1), we have $G \cong H \times \mathbb{Z}_p$ or $G \cong E_4$. The order of Schur multipliers of both of them is at least $p^2$ by using ([13], Lemma 2.8) and Theorem 1.4. Thus, $G/G'$ cannot be elementary abelian. Since $G^p$ and $G'$ are contained in $Z(G)$, we consider two cases.

Case I. Assuming first that $G' \cap G^p = 1$, then $G/G^p \cong E_1$, and so $|\mathcal{M}(G)| \geq |\mathcal{M}(E_1)| = p^2$ directly by using ([10], Corollary 2.5.3 (i)), which contradicts $t(G) = 5$.

Case II. In this case, we have two possibilities for $Z(G)$. The first possibility is $Z(G) = G^p \cong \mathbb{Z}_{p^2}$, thus, $G$ is of exponent $p^3$ and similar to the proof of ([13], Lemma 2.8), we have $|\mathcal{M}(G)| = 1$. The second possibility is $Z(G) = G^p \cong \mathbb{Z}_p \times G'$. By ([2], pp. 87–88), there is a unique group of order $p^4$ with this properties, which is isomorphic to

$$\langle a, b \mid a^{p^2} = 1, b^{p^2} = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^p \rangle. \qed$$

Lemma 2.9. Let $|G| = p^4$ ($p \neq 2$) and $|G'| = p^2$ with $t(G) = 5$. Then $G$ is isomorphic to

$$\langle a, b \mid a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = a^p, [a, b, b, b] = 1 \rangle,$$

$$\langle a, b \mid a^{p^2} = b^p = 1, [a, b, a] = 1, [a, b, b] = a^{np}, [a, b, b, b] = 1 \rangle,$$

where $n$ is a fixed quadratic non-residue of $p$ and $p \neq 3$,

$$\langle a, b \mid a^9 = 1, b^3 = a^3, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1 \rangle,$$

$$\langle a, b \mid a^9 = 1, b^3 = a^3, [a, b, a] = 1, [a, b, b] = a^3, [a, b, b, b] = 1 \rangle,$$

$$\langle a, b \mid a^p = b^p = 1, [a, b, a] = [a, b, b, a] = [a, b, b] = 1 \rangle.$$

For $p = 3$, the last two groups are isomorphic.

Proof. The result is obtained from ([5], pp. 4177) and ([2], pp. 88), see also [16], pp. 196–198. \qed

Lemma 2.10. Let $G$ be a $p$-group of order 16 with $t(G) = 5$. Then $G$ is isomorphic to

$$D_{16} \text{ or } \langle a, b \mid a^4 = b^4 = 1, a^{-1}ba = b^{-1} \rangle.$$

Proof. See table I on [14]. \qed
Lemma 2.11. Let $G$ be a $p$-group of order $32$ with $t(G) = 5$. Then $G$ is isomorphic to

$$Q_8 \times \mathbb{Z}_2^{(2)}, (D_8 \times \mathbb{Z}_2) \times \mathbb{Z}_2, (Q_8 \times \mathbb{Z}_2) \times \mathbb{Z}_2 \text{ or } \mathbb{Z}_2 \times \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle.$$ 

Proof. These groups are obtained by using the HAP package [7] of GAP [17]. □

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