ON THE NUMBER OF SOLUTIONS OF THE DIOPHANTINE EQUATION $x^2 + 2^a \cdot p^b = y^4$

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Let p be a fixed odd prime. In this paper, we study the integer solutions (x, y, a, b) of the equation $x^2 + 2^a \cdot p^b = y^4, \gcd(x, y) = 1, x > 0, y > 0, a \geq 0, b \geq 0$, and we derive upper bounds for the number of such solutions.

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1. INTRODUCTION

Let $\mathbb{Z}, \mathbb{N}$ be the sets of all integers and positive integers respectively. Let p be a fixed odd prime. There are many recent papers related to the equation

$$x^2 + 2^a p^b = y^n, x, y, n, a, b \in \mathbb{Z}, \gcd(x, y) = 1, x > 0, y > 0, n > 2, a \geq 0, b \geq 0, \tag{1.1}$$

which has been solved for some small values of p (see [2, 4, 8–10, 13]).

Since $n > 2$, for a general p, (1.1) can be classified into the following two equations:

$$x^2 + 2^a p^b = y^4, x, y, a, b \in \mathbb{Z}, \gcd(x, y) = 1, x > 0, y > 0, a \geq 0, b \geq 0 \tag{1.2}$$

and

$$x^2 + 2^a p^b = y^q, x, y, a, b \in \mathbb{Z}, \gcd(x, y) = 1, x > 0, y > 0, a \geq 0, b \geq 0, \tag{1.3}$$

where q is an odd prime.

Obviously, (1.2) has only the solution $(x, y, a, b) = (7, 3, 5, 0)$ with $b = 0$ (see [3]). Thus it can be seen that we only need to solve the equations

$$x^2 + p^b = y^4, \quad x, y, b \in \mathbb{N}, \quad \gcd(x, y) = 1 \tag{1.4}$$

and

$$x^2 + 2^a p^b = y^4, \quad x, y, a, b \in \mathbb{N}, \quad \gcd(x, y) = 1. \tag{1.5}$$

In [15], H. Zhu, G. Soydan and W. Qin gave all solutions of the equations (1.4) and (1.5) under the assumption that the famous equation

$$x^2 - 2 = y^n, \quad x > 1, y \geq 1, n \geq 3, \gcd(x, y) = 1 \tag{1.6}$$

has no solutions. Recently, in [14], H. Zhu, M. Le, G. Soydan and A.Togbé restudied the equations (1.4) and (1.5) and firstly, they gave all solutions of the equation (1.4) without using the assumption that equation (1.6) has no solutions, in the following theorem:

THEOREM A ([14]). *All solutions of (1.4) are given as follows:*
(i) $p = 23, (x, y, b) = (6083, 78, 3)$.
(ii) $p = u_{2k+1}, (x, y, b) = (v_{2k+1}^2 - 1, v_{2k+1}, 2)$, *where* k *is a positive integer,* $u_{2k+1} = \frac{1}{2}\Big((1+\sqrt{2})^{2k+1} + (1-\sqrt{2})^{2k+1}\Big)$, $v_{2k+1} = \frac{1}{2\sqrt{2}}\Big((1+\sqrt{2})^{2k+1} - (1-\sqrt{2})^{2k+1}\Big)$.
(iii) $p = 2l^2 - 1, (x, y, b) = (l^2 - 1, l, 1)$, *where* l *is a positive integer with* $l > 1$.

Secondly, they gave all solutions of the equation (1.5) for a general prime p with $p \not\equiv 7 \pmod 8$ without using the assumption that equation (1.6) has no solutions, in the following theorem:

THEOREM B ([14]). *If* $p \not\equiv 7 \pmod 8$, *then equation* (1.5) *has only the following solutions:*
(i) $p = 3,\ (x, y, a, b) = (7, 5, 6, 2)$.
(ii) $p = 3,\ (x, y, a, b) = (47, 7, 6, 1)$.
(iii) $p = 3,\ (x, y, a, b) = (287, 17, 7, 2)$.
(iv) $p = 17,\ (x, y, a, b) = (4785, 71, 9, 3)$.
(v) $p = 2^{2^{r-1}} + 1,\ (x, y, a, b) = (2^{2^r+2} + 2^{2^{r-1}+2} - 1, 2^{2^{r-1}+1} + 1, 2^{r-1} + 4, 1)$, *where* $r \in \mathbb{N}$, *and if* $2^{2^{r-1}} + 1$ *is a prime.*
(vi) $p = 2^r + 1,\ (x, y, a, b) = (|2^{2r-2} - 2^r - 1|, 2^{r-1} + 1, 2r, 1)$, *where* $r \in \mathbb{N}$, *and if* $2^r + 1$ *is a prime.*
(vii) $p = f^2 - 2^{2r-1},\ (x, y, a, b) = (|f^2 - 2^{2r}|, f, 2r + 1, 1)$, *where* $r \in \mathbb{N}$, $2 \nmid f$, *and if* $f^2 - 2^{2r-1}$ *is a prime.*

In this paper, we give partial classification of integer solutions (x, y, a, b) of the equations (1.4) and (1.5), and we obtain some upper bounds for the number of such solutions.

THEOREM 1. *(1.4) has at most one solution* (x, y, b) *except* $(x, y, b) = (3, 2, 1)$ *and* $(24, 5, 2)$ *for* $p = 7$.

THEOREM 2. *If* $p \equiv 3 \pmod 8$, *then (1.5) has no solution except for* $(x, y, a, b) = (23, 5, 5, 1), (47, 7, 6, 1), (7, 5, 6, 2)$ *and* $(287, 17, 7, 2)$ *when* $p = 3$.

THEOREM 3. *If $p \equiv 5 \pmod 8$, then (1.5) has no solution except for $(x, y, a, b) = (1, 3, 4, 1)$ and $(79, 9, 6, 1)$ when $p = 5$.*

THEOREM 4. *If $p \equiv 1 \pmod 4$, then (1.5) has at most one solution (x, y, a, b) except for the following cases:*

(i) $(x, y, a, b) = (9, 5, 5, 1)$, $(15, 7, 7, 1)$, $(47, 9, 8, 1)$, $(1087, 33, 8, 1)$, $(4785, 71, 9, 3)$ and $(495, 23, 11, 1)$ when $p = 17$.

(ii) $(x, y, a, b) = (2^{2^{r-1}} - 2^{2^{r-2}+1} + 1, 2^{2^{r-2}} + 1, 2^{r-2} + 3, 1)$, $(2^{2^r-2} - 2^{2^{r-1}} - 1, 2^{2^{r-1}-1} + 1, 2^r, 1)$ and $(2^{2^r+2} + 2^{2^{r-1}+2} - 1, 2^{2^{r-1}+1} + 1, 2^{r-1} + 4, 1)$, where r is a positive integer with $r > 3$ and $p = 2^{2^{r-1}} + 1$.

(iii) $(x, y, a, b) = (|(2l_1 + 1)^2 - 2^{2s_1}|, 2l_1 + 1, 2s_1 + 1, 1)$ and $(|(2l_2 + 1)^2 - 2^{2s_2}|, 2l_2 + 1, 2s_2 + 1, 1)$, where r, l_1, l_2, s_1, s_2 are positive integers with $s_2 > s_1 > 1$ and $p \neq 2^{2^{r-1}} + 1$, $p = (2l_1 + 1)^2 - 2^{2s_1-1} = (2l_2 + 1)^2 - 2^{2s_2-1}$.

We organize this paper as follows. In Section 2, we recall some useful lemmas and prove Theorem 1, by using Theorem A. In Section 3, we prove Theorems 2–3 by using Theorem B. In Section 4, by means of some lemmas at the beginning of this section and Theorem B, we prove Theorem 4.

2. PROOF OF THEOREM 1

For any nonnegative integer t, let

$$u_t = \frac{1}{2}(\alpha^t + \beta^t), \quad v_t = \frac{1}{2\sqrt{2}}(\alpha^t - \beta^t), \tag{2.1}$$

where

$$\alpha = 1 + \sqrt{2}, \quad \beta = 1 - \sqrt{2}. \tag{2.2}$$

By the properties of Pell equations and related sequences (see, Chapter 8 of [12]), we have the following lemma:

LEMMA 1. *(i) Every solution (u, v) of the equation*

$$u^2 - 2v^2 = 1, \quad u, v \in \mathbb{N} \tag{2.3}$$

can be expressed as $(u, v) = (u_{2k}, v_{2k})$, where k is a positive integer.

(ii) Every solution (U, V) of the equation

$$U^2 - 2V^2 = -1, \quad U, V \in \mathbb{N} \tag{2.4}$$

can be expressed as $(U, V) = (u_{2k+1}, v_{2k+1})$, where k is a nonnegative integer.

(iii) For any nonnegative integer k, $\gcd(u_k, u_{k+1}) = \gcd(v_k, v_{k+1}) = 1$.

LEMMA 2 ([6]). *The equation*

$$X^4 - 2Y^2 = 1, \quad X, Y \in \mathbb{N} \tag{2.5}$$

has no solution (X, Y).

Lemma 3 ([7]). *The equation*

$$X^2 - 2Y^4 = -1, \quad X, Y \in \mathbb{N} \tag{2.6}$$

only has the solutions $(X, Y) = (1, 1)$ *and* $(239, 13)$.

Lemma 4. *The equation*

$$u_{2k+1} = 2l^2 - 1, \quad k, l \in \mathbb{N} \tag{2.7}$$

has only the solution $(k, l) = (1, 2)$.

Proof. Let (k, l) be a solution of (2.7). If $2|k$, then $k = 2s$, where s is a positive integer. By (2.1) and (2.2), we have

$$\begin{aligned} u_{2k+1} + 1 = u_{4s+1} + 1 &= \frac{\alpha^{4s+1} + \beta^{4s+1}}{\alpha + \beta} + (\alpha\beta)^{2s} \\ &= \frac{(\alpha^{2s} + \beta^{2s})(\alpha^{2s+1} + \beta^{2s+1})}{\alpha + \beta} = 2u_{2s}u_{2s+1}. \end{aligned} \tag{2.8}$$

By substituting (2.8) into (2.7), we get

$$u_{2s}u_{2s+1} = l^2. \tag{2.9}$$

By (iii) of Lemma 1, we have $\gcd(u_{2s}, u_{2s+1}) = 1$. Hence, we get from (2.9) that

$$u_{2s} = f^2, \quad u_{2s+1} = g^2, \quad l = fg, \quad f, g \in \mathbb{N}. \tag{2.10}$$

Further, by (i) of Lemma 1, we see from the first equality of (2.10) that (2.5) has the solution $(X, Y) = (f, v_{2s})$. But, by Lemma 2, it is impossible.

If $2 \nmid k$, then $k = 2s + 1$, where s is a nonnegative integer. By (2.1) and (2.2), we have

$$\begin{aligned} u_{2k+1} + 1 = u_{4s+3} + 1 &= \frac{\alpha^{4s+3} + \beta^{4s+3}}{\alpha + \beta} - (\alpha\beta)^{2s+1} \\ &= \frac{(\alpha^{2s+1} - \beta^{2s+1})(\alpha^{2s+2} - \beta^{2s+2})}{\alpha + \beta} = 4v_{2s+1}v_{2s+2}. \end{aligned} \tag{2.11}$$

By substituting (2.11) into (2.7), we get

$$2v_{2s+1}v_{2s+2} = l^2. \tag{2.12}$$

Further, since $\gcd(v_{2s+1}, v_{2s+2}) = 1$ and $2 \nmid v_{2s+1}$ by (2.4), we obtain from (2.12) that

$$v_{2s+1} = f^2, \quad v_{2s+2} = 2g^2, \quad l = 2fg, \quad f, g \in \mathbb{N}. \tag{2.13}$$

Furthermore, by (ii) of Lemma 1, we see from the first equality of (2.13) that (2.6) has the solution $(X, Y) = (u_{2s+1}, f)$. Therefore, by Lemma 3, we get either $(s, f) = (0, 1)$ or $(3, 13)$. Since $v_2 = 2$ and $v_8 = 408$, by the second and the third equality of (2.13), we determine that (2.7) has only the solution $(k, l) = (1, 2)$ which completes the proof. $\square$

2.1. PROOF OF THEOREM 1

We now assume that p is an odd prime such that (1.4) has two solutions. Since $23 \neq u_{2k+1}$ and $23 \neq 2l^2 - 1$, by Theorem A, we have

$$p = u_{2k+1} = 2l^2 - 1, \quad k, l \in \mathbb{N}. \tag{2.14}$$

Applying Lemma 4 to (2.14), we get $k = 1, l = 2$ and $p = 7$. It implies that (1.4) has at most one solution except for $p = 7, (x, y, b) = (3, 2, 1)$ and $(24, 5, 2)$. This completes the proof of the Theorem.

3. **PROOF OF THEOREMS 2 AND 3**

3.1. PROOF OF THEOREM 2

We consider the case that $p \equiv 3 \pmod 8$.

When $p = 3$, since $3 = 2^1 + 1$ and $3 = 2^{2^0} + 1$, by the solutions of types (i), (ii), (iii), (v) and (vi) given in Theorem B, (1.5) has exactly four solutions $(23, 5, 5, 1), (47, 7, 6, 1), (7, 5, 6, 2)$ and $(287, 17, 7, 2)$, respectively.

When $p > 3$, by Theorem B, (1.5) has no solution which completes the proof.

3.2. PROOF OF THEOREM 3

When $p = 5$, since $5 = 2^2 + 1$, by the solution of type (v) and (vi) given in Theorem B, then (1.5) has exactly two solutions $(x, y, a, b) = (79, 9, 6, 1)$ and $(4, 3, 4, 1)$ respectively.

When $p > 5$, by Theorem B, (1.5) has no solution. This completes the proof of the Theorem.

4. **PROOF OF THEOREM 4**

Let D be a positive odd integer.

LEMMA 5 ([12], Chapter 8). *If D is an odd prime with $D \equiv 1 \pmod 4$, then the equation*

$$U^2 - DV^2 = 1, \quad U, V \in \mathbb{N} \tag{4.1}$$

has solutions (U, V).

Lemma 6. *If D is an odd prime and the equation*

$$X^2 - D = 2^n, \quad X, n \in \mathbb{N}, \quad n > 2 \tag{4.2}$$

has solutions (X,n) with $2|n$, then $D = 2^{2^{r-1}} + 1$, where r is a positive integer with $r > 2$.

Proof. Let (X,n) be a solution of (4.2) with $2|n$. Since D is an odd prime, by (4.2), we have

$$X + 2^{\frac{n}{2}} = D, \quad X - 2^{\frac{n}{2}} = 1, \tag{4.3}$$

whence we get

$$D = 2^{\frac{n}{2}+1} + 1. \tag{4.4}$$

Therefore, we see from (4.4) that $D = 2^{2^{r-1}} + 1$, where r is a positive integer with $r > 2$ which completes the proof. □

Lemma 7 ([5], Theorem 1 and 2). *(i) If $D = 2^{2r} - 3 \cdot 2^{r+1} + 1$, where r is a positive integer with $r > 2$, then (4.2) has exactly four solutions $(X,n) = (2^r - 3, 3), (2^r - 1, r+2), (2^r + 1, r+3)$ and $(3 \cdot 2^r - 1, 2r+3)$.*

(ii) If $D = \left(\frac{1}{3}(2^{2r+1} - 17)\right)^2 - 32$, where r is a positive integer with $r > 2$, then (4.2) has exactly three solutions $(X,n) = \left(\frac{1}{3}(2^{2r+1} - 17), 5\right), \left(\frac{1}{3}(2^{2r+1} + 1), 2r+3\right)$ and $\left(\frac{1}{3}(17 \cdot 2^{2r+1} - 1), 4r+7\right)$.

(iii) If $D = 2^{2r_1} + 2^{2r_2} - 2^{r_1+r_2+1} - 2^{r_1+1} - 2^{r_2+1} + 1$, where r_1, r_2 are positive integers with $r_2 > r_1 > 2$, then (4.2) has exactly three solutions $(X,n) = (2^{r_2} - 2^{r_1} - 1, r_1+2), (2^{r_2} - 2^{r_1} + 1, r_2+2)$ and $(2^{r_2} + 2^{r_1} - 1, r_1+r_2+2)$.

(iv) If D is not of one of the above types and (5.1) has solutions (U,V), then (4.2) has at most two solutions (X,n).

Lemma 8. *The equation*

$$2^{2r} - 3 \cdot 2^{r+1} + 1 = 2^{2^{s-1}} + 1, \quad r, s \in \mathbb{N}, \quad r > 2, \quad s > 2 \tag{4.5}$$

has only the solution $(r,s) = (3,3)$.

Proof. Let (r,s) be a solution of (4.5). Then we have $2^{r-1} - 3 = 2^{2^s - r - 1}$, whence we get $r = 3$ and $s = 3$. This completes the proof of the Lemma. □

Lemma 9. *If $D = \left(\frac{1}{3}(2^{2r+1} - 17)\right)^2 - 32$, where r is a positive integer with $r > 2$, then D is not a prime.*

Proof. Under the assumption, we have

$$D = \frac{1}{9}\left(2^{4r+2} - 17 \cdot 2^{2r+2} + 1\right) = \frac{1}{9}\left((2^{2r+1} - 1)^2 - 2^{2r+6}\right)$$

$$(4.6) \qquad = \frac{1}{9}\left(2^{2r+1} + 2^{r+3} - 1\right)\left(2^{2r+1} - 2^{r+3} - 1\right).$$

Since $r \geq 3$, we have $2^{2r+1} + 2^{r+3} - 1 > 2^{2r+1} - 2^{r+3} - 1 \geq 63$. Therefore, by (4.6), D is not a prime which completes the proof. □

Lemma 10. *The equation*

$$(4.7) \quad 2^{2r_1} + 2^{2r_2} - 2^{r_1+r_2+1} - 2^{r_1+1} - 2^{r_2+1} + 1 = 2^{2^{s-1}} + 1, \quad r_1, r_2, s \in \mathbb{N}, \quad r_2 > r_1 + 1 > 2, \quad s > 3$$

has no solution (r_1, r_2, s).

Proof. Let (r_1, r_2, s) be a solution of (4.7). Then we have

$$(4.8) \qquad 2^{r_1-1} + 2^{2r_2-r_1-1} - 2^{r_2} - 2^{r_2-r_1} - 1 = 2^{2^{s-1}-r_1-1}.$$

But since $2r_2 - r_1 - 1 > r_2 > \max\{r_1 - 1, r_2 - r_1\} \geq \min\{r_1 - 1, r_2 - r_1\} \geq 1$, (4.8) is impossible. This completes the proof of the Lemma. □

Lemma 11. *If D is an odd prime, then (4.2) has at most two solutions except for*

(i) $D = 17, (X, n) = (5, 3), (7, 5), (9, 6)$ and $(23, 9)$. Further, if (4.2) has exactly two solutions (X_1, n_1) and (X_2, n_2) with $n_1 < n_2$, then we have

(ii) $D = 2^{2^{r-1}} + 1, (X_1, n_1) = \left(2^{2^{r-2}} + 1, 2^{r-2} + 1\right)$ and $(X_2, n_2) = \left(2^{2^{r-1}-1} + 1, 2^r - 2\right)$, where r is a positive integer with $r > 2$.

(iii) $D \neq 2^{2^{r-1}} + 1, (X_1, n_1) = (2l_1 + 1, 2r_1 + 1)$ and $(X_2, n_2) = (2l_2 + 1, 2r_2 + 1)$, where l_1, l_2, r_1, r_2 are positive integers with $r_1 < r_2$.

Proof. We now assume that D is an odd prime such that (4.2) has at least three solutions. By Lemma 7, D must be of the types (i), (ii) and (iii).

When D is of type (i), since one of $r + 2$ and $r + 3$ is even, (4.2) has a solution (X, n) with $2|n$. Hence, by Lemma 6, we have $D = 2^{2^{r-1}} + 1$, where r is a positive integer with $r > 2$. Further, by Lemma 8, we get $D = 17$ and the solutions (i).

When D is of type (ii), by Lemma 9, it is impossible.

When D is of type (iii), since one of $r_1 + 2, r_2 + 2$ and $r_1 + r_2 + 2$ is even, by Lemma 6, we have $D = 2^{2^{r-1}} + 1$. But, by Lemma 10, it is impossible. Therefore, by Lemma 5 and (iv) of Lemma 7, (4.2) has at most two solutions except for $D = 17$.

Finally, we assume that D is an odd prime which makes (4.2) has exactly two solutions. Then we have $D \neq 17$.

When $D = 2^{2^{r-1}} + 1$, since $D \neq 17$, we have $r > 3$ and the solutions of type (ii).

When $D \neq 2^{2^{r-1}}+1$, by Lemma 5, (4.2) has no solution (X, n) with $2|n$. Therefore, we obtain the solutions of type (iii). This completes the proof of the Lemma. $\square$

4.1. PROOF OF THEOREM 4.

We first consider the case that $p = 2^{2^{r-1}}+1$, where r is a positive integer with $r > 2$.

When $r = 3$, we have $p = 17$. By the case (i) of Lemma 11 and the solutions of types (iv),(v),(vi) and (vii) given in Theorem B, (1.5) has exactly six solutions $(x, y, a, b) = (9, 5, 5, 1)$, $(15, 7, 7, 1)$, $(47, 9, 8, 1)$, $(1087, 33, 8, 1)$, $(4785, 71, 9, 3)$ and $(495, 23, 11, 1)$, respectively.

When $r > 3$, by the case (ii) of Lemma 11 and the solutions of types (v),(vi) and (vii) given in Theorem B, we obtain the solutions of type (ii).

We next consider the case that $p \neq 2^{2^{r-1}}+1$. Since $p \equiv 1 \pmod 8$, by Theorem B, (1.5) has only solutions belong to the type (vii) with $r > 1$. Therefore, by Lemma 11,(1.5) has at most one solution (x, y, a, b) except for the case (iii) which completes the proof.

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