THE NON-CENTRALIZER GRAPH OF A FINITE GROUP

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In this paper, we define the non-centralizer graph associated to a finite group G, as the graph whose vertices are the elements of G, and whose edges are obtained by joining two distinct vertices if their centralizers are not equal. We denote this graph by Υ_G. The non-centralizer graph is used to study the properties of the non-commuting graph of an AC-group. We prove that the non-centralizer graphs associated to two isoclinic groups for which the order of their centers are equal are isomorphic. Moreover, we observe that the converse holds for two isomorphic 4-partite graphs. We finally prove that if $\Upsilon_G \cong \Upsilon_S$, then $G \cong S$, where S is a simple group which is not $B_n(q)$ or $C_n(q)$.

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1. INTRODUCTION

Graphs play an important role in the mathematics, providing visual means that help us to better understand other mathematical objects that they are connected with, like algebraic structures, for instance. Associating a graph to a group and using information on one of the two objects to solve a problem for the other is an interesting research topic. On the other hand, many recent problems in group theory are related to the notion of commutativity, like for instance the problem to determine the probability that two elements of a group commute, or to find how many centralizers can a group have. For results related to the problem of counting the centralizers of a group, we refer the reader to the work of S.M. Belcastro and G.J. Sherman [8], A.R. Ashrafi [5], A.R. Ashrafi and B. Taeri [6], A. Abdollahi, S.M. Jafarian Amiri, and A. Mohammadi Hassanabadi [2], M. Zarrin [17] and S.J. Baishya [7]. The non-commuting graph of a group G was first considered by Paul Erdös in 1975. We denote this graph by Γ_G, and recall that the vertices of Γ_G are the elements of G, and that two distinct vertices are joined by an edge whenever they do not commute. Of course, there are some other ways to construct a graph associated to a given group or semigroup. In this paper, we will define the non-centralizer graph Υ_G of the group G to be the graph whose vertices are the elements of G and whose edges are obtained by joining two distinct vertices if their centralizers are not equal. The non-centralizer graph and the non-commuting graph of a group G are closely related. By studying the non-centralizer graph, one may describe the structure of the non-commuting graph associated to large classes of groups.

We organize this paper in three parts. In the next section we discuss general properties of the graph such as diameter, girth, domination number, chromatic number and independent set. We prove that if Υ_G has a vertex of degree m and $|Z(G)| = m$, then G is nilpotent. Moreover, we observe that the non-centralizer and the non-commuting graph associated to an AC-group are isomorphic. Since the non-centralizer graph is a complete k-partite graph, the non-commuting graph of an AC-group is completely determined. In the third section we prove that two non-centralizer graphs associated to two isoclinic groups such that the order of their centers are equal are isomorphic. Furthermore, the converse holds if the graphs are 4-partite. Finally, we prove that if $\Upsilon_G \cong \Upsilon_S$, then $G \cong S$, where S is a simple group not isomorphic to $B_n(q)$ or $C_n(q)$.

Throughout the paper, graphs are simple and all the notations and terminologies about the graphs are standard (for instance see [9, 10, 13]).

2. MAIN RESULTS

For a group G, $C_G(x) = \{y \in G : xy = yx\}$ is the centralizer of the element $x \in G$. Let us start with the following definition.

Definition 2.1. Let G be a group. We construct a graph whose vertices are the elements of G and whose edges are obtained by joining any two vertices x and y whenever $C_G(x) \neq C_G(y)$. We call this graph the non-centralizer graph of G, and we denote it by Υ_G.

If G is an abelian group, then Υ_G is an empty graph. Therefore throughout the paper all the groups are finite non-abelian unless otherwise mentioned.

It is clear that for a non-central element x we have $\deg(x) \geq |G|-|C_G(x)|$ and deg(z) $= |G| - |Z(G)|$ for a central element z.

If we consider the induced subgraph of Υ_G associated to the non-abelian group G with vertex set $G \setminus Z(G)$, then we have non-central vertices with degree $\deg(x) \geq |G| - |C_G(x)|$. Let us denote this subgraph by $\Upsilon_{G\setminus Z(G)}$.

Clearly $\mathrm{Aut}(G) \subseteq \mathrm{Aut}(\Upsilon_G)$. The converse inclusion does not generally hold. Consider the non-centralizer graph of the symmetric group S_3 and β as the graph automorphism such that $\beta((1\ 2)) = (2\ 3)$, $\beta((2\ 3)) = (1\ 3)$, $\beta((1\ 3)) = (1\ 2)$, $\beta((1\ 3\ 2)) = (1\ 2\ 3)$ and $\beta((1\ 2\ 3)) = (1\ 3\ 2)$. Since $\beta((1\ 2)(2\ 3)) \neq \beta((1\ 2))\beta((2\ 3))$, β is not a group automorphism.

THEOREM 2.2. *Let Υ_G be the non-centralizer graph of the non-abelian group G. Then we conclude* $\mathrm{diam}(\Upsilon_G) = \mathrm{diam}(\Upsilon_{G\setminus Z(G)}) = 2$ *and* $\mathrm{girth}(\Upsilon_G) = \mathrm{girth}(\Upsilon_{G\setminus Z(G)}) = 3$.

Proof. Let x and y be two non-central elements of the group which are not adjacent in Υ_G. It is clear that both are adjacent to the identity element. Therefore $d(x, y) = 2$. It is obvious that central elements are not adjacent but they join to a non-central element. Thus $\mathrm{diam}(\Upsilon_G) = 2$. If g, h are two adjacent vertices, then they are not central elements, so they join to a central element. Hence $\mathrm{girth}(\Upsilon_G) = 3$.

Let x and y be two non-adjacent vertices in $\Upsilon_{G\setminus Z(G)}$. Since x and y are non-central there is an element t which does not commute with x and y. Thus x and y joins t. Moreover, if g and h are adjacent, then $\{g, h, gh\}$ is a triangle. □

By Theorem 2.2 we deduce that Υ_G and $\Upsilon_{G\setminus Z(G)}$ are connected. Suppose $\mathrm{Cent}(G) = \{C_G(g) | g \in G\}$. A group G is called n-centralizer if $|\mathrm{Cent}(G)| = n$. It is clear that Υ_G is a complete $|\mathrm{Cent}(G)|$-partite graph.

Example 2.3. We provide here examples of non-centralizer graphs for non-commutative groups of small order.

(i) Let $D_8 = \langle a, b : a^4 = b^2 = 1, a^b = a^{-1} \rangle$ be the dihedral group of order 8. Then Υ_{D_8} is $K_{2,2,2,2}$ such that $\{1, a^2\}, \{a, a^3\}, \{b, a^2b\}$ and $\{ab, a^3b\}$ are its parts.

(ii) Let S_3 be the symmetric group on three objects. Then Υ_{S_3} is $K_{1,1,1,1,2}$ such that $\{(1)\}, \{(1\ 2)\}, \{(2\ 3)\}, \{(1\ 3)\}$ and $\{(1\ 2\ 3), (1\ 3\ 2)\}$ are its parts.

As we mentioned Erdös associated the non-commuting graph Γ_G to the group G, with vertex set the elements of G, and two vertices joined by an edge whenever they do not commute. Let us denote the subgraph of the non-commuting graph with the vertex set $G \setminus Z(G)$ by $\Gamma_{G\setminus Z(G)}$. For an edge $\{x, y\}$ in the non-centralizer graph we have $C_G(x) \neq C_G(y)$ but not necessarily $[x, y] \neq 1$. Thus the non-centralizer graph and the non-commuting graph are in general not the same.

In the above example, we see that $\Gamma_{D_8\setminus Z(D_8)} \cong \Upsilon_{D_8\setminus Z(D_8)}$ and $\Gamma_{S_3\setminus Z(S_3)} \cong \Upsilon_{S_3\setminus Z(S_3)}$. To see what particular property of these groups forces the above isomorphisms, we recall that a group G all of whose centralizers are abelian is called an *AC*-group. It is now easy to see that in view of Theorem 2.11, the above isomorphism holds for the class of *AC*-groups.

A dominating set for a graph Γ is a subset D of $V(\Gamma)$ such that every vertex outside D is adjacent to at least one member of D. The domination number $\gamma(\Gamma)$ is the size of the smallest dominating set of Γ.

Proposition 2.4. *Let G be a non-abelian group. Then $\gamma(\Upsilon_G) \leq 2$.*

Proof. Suppose x is a non-central element of G. Thus $\{e, x\}$ is the dominating set for Υ_G, where e is identity element of G. $\square$

Proposition 2.5. *If $\{x\}$ is a dominating set for Υ_G, then $Z(G) = 1$ and $x^2 = 1$.*

Proof. Suppose $1 \neq z \in Z(G)$. Thus xz is a vertex which is not adjacent to x, which is a contradiction. Assume $x \neq x^{-1}$. It is clear that if $[x,t] = 1$, then $[x^{-1}, t] = [t, x]^{x^{-1}} = 1$. This shows that $C_G(x) = C_G(x^{-1})$, so x does not join x^{-1}, again a contradiction. $\square$

It is clear that for an abelian group G, $\chi(\Upsilon_G) = 1$ and for a non-abelian group G we have $\chi(\Upsilon_G) \leq |G| - |Z(G)|$. We note that this upper bound is sharp, since $\chi(\Upsilon_{S_3}) = 5$. It is obvious that the chromatic number of the non-centralizer graph is equal to the number of its part. For instance $\chi(\Upsilon_{D_8}) = 4$. We note here that for the groups $\mathrm{PSL}(2, q)$, where q is a prime-power, and for $Sz(q)(q = 2^{2m+1}, m > 0)$, $|\mathrm{Cent}(G)|$ was determined in [17], so for these groups one can easily study their associated non-centralizer graphs.

For an element x of a group we define the set $\mathcal{C}_G(x) = \{y \in G : C_G(y) = C_G(x)\}$. This notion can be generalized to an arbitrary set T of G by letting $\mathcal{C}_G(T) = \{y \in G : C_G(y) = C_G(t), \text{ for all } t \in T\}$. Obviously $\mathcal{C}_G(T) \subseteq C_G(T)$.

Proposition 2.6. *Let G be a non-abelian group and $T \subseteq V(\Upsilon_{G \setminus Z(G)})$. Then T is a dominating set if and only if $\mathcal{C}_G(T) \subseteq T \cup Z(G)$.*

Proof. Suppose T is a dominating set and $g \in \mathcal{C}_G(T)$. Thus $C_G(g) = C_G(t)$ for all $t \in T$ and so g is not adjacent to all elements of T. As T is a dominating set $g \in Z(G)$. The converse follows immediately. $\square$

Proposition 2.7. *Let T be the maximal independent set for the graph $\Upsilon_{G \setminus Z(G)}$. Then $T = \mathcal{C}_G(T) - Z(G)$.*

Proof. Suppose $x \in T$. Since T is the maximal independent set, x does not join to all the other vertices in T. Thus $T \subseteq \mathcal{C}_G(T) - Z(G)$. Now assume $g \in \mathcal{C}_G(T) - Z(G)$. Therefore $C_G(g) = C_G(t)$, for all $t \in T$ which means g is not adjacent to all vertices of T. Thereby $T \cup \{g\}$ is an independent set and since T is maximal independent set for the graph $\Upsilon_{G \setminus Z(G)}$ which shows that $g \in T$ and completes the proof. $\square$

The above proposition implies that $T \subseteq C_G(T) - Z(G)$, for the maximal independent set of $\Upsilon_{G \setminus Z(G)}$.

PROPOSITION 2.8. *If φ is an isomorphism between the graphs Υ_G, Υ_H and $x \in G\backslash Z(G)$ is a vertex of degree i, then $\varphi(x) \in H\backslash Z(H)$ and $\deg(\varphi(x)) = i$, where $i = 1, 2, 3$.*

Proof. Suppose $\varphi(x) \in Z(H)$. Therefore $\varphi(x)$ is adjacent to all non-central elements of H, $\deg(\varphi(x)) = |H| - |Z(H)| = i$. Hence H is an abelian group or S_3 which both are contradiction. □

For a finite group G, let $\{n_1, \cdots, n_r\}$ be the set of integers each of which is the index of the centralizer of some element of G. We may assume that $n_1 > n_2 > \cdots > n_r$. Then the vector $(n_1, \cdots, n_r)$ is called the conjugate type vector of G. Itô proved that any group of type $(n_1, 1)$ or $(n_1, n_2, 1)$ is nilpotent or soluble, respectively (see [15, 16] for more details).

PROPOSITION 2.9. *Let Υ_G be the non-centralizer graph associated to the non-abelian group G. Then we have*

(i) There is no central vertex of degree 1, 2 or 3.

(ii) If Υ_G has a non-central vertex of degree 1, then G is nilpotent with $Z(G) = 1$.

(iii) If Υ_G has a non-central vertex of degree 2 and $|Z(G)| = 1$, then G is a solvable group.

(iv) If Υ_G has a non-central vertex of degree 2 and $|Z(G)| = 2$, then G is a nilpotent group.

(v) If $|Z(G)| > n$, then Υ_G does not have any non-central vertex of degree n, where n is a positive integer.

Proof. (i) It is easy.

(ii) Let $x \in G\backslash Z(G)$ be a vertex of degree one. Thus x is exactly adjacent to the identity element of the group. Consequently the conjugate type vector for G is $(n, 1)$ which proves our assertion.

(iii) Suppose x is a non-central vertex of degree 2. Since $|Z(G)| = 1$, x joins the identity element of the group and another non-central element y. This means that the centralizer of x is equal to all other centralizers except $C_G(y)$ and $Z(G)$. It is clear that G has at most three conjugacy class sizes, so the assertion is clear by Itô's result.

(iv) The proof is similar to that of (ii).

(v) Since $|Z(G)| > n$ the degree of the non-central vertices exeeds n. □

In a similar way, one may prove the following result.

PROPOSITION 2.10. *Let Υ_G be the non-centralizer graph associated to the group G and x a non-central vertex of degree m. Then*

(i) If $|Z(G)| = m$, then G is a nilpotent group.

(ii) If $|Z(G)| = m - 1$, then G is a soluble group.

(iii) If $|Z(G)| = i$ and x is an element of minimum degree, then G has at most $m - i + 2$ conjugacy class sizes, where $i < m$.

Clearly if $\Upsilon_{G \setminus Z(G)}$ has a vertex of minimum degree m, then G has at most $m + 2$ conjugacy class sizes. Thus for the non-centralizer induced subgraph associated to the group G with a vertex of degree one we deduce G is soluble.

The non-commuting graph and the non-centralizer graph of a given group are not isomorphic in general. In the following theorem we present a property for the group such that its non-centralizer graph coincides to its non-commuting graph.

THEOREM 2.11. *Let G be an AC-group. Then $\Upsilon_{G \setminus Z(G)} \cong \Gamma_{G \setminus Z(G)}$.*

Proof. Suppose $\{x, y\}$ is an edge of $\Upsilon_{G \setminus Z(G)}$. Therefore $C_G(x) \neq C_G(y)$. We claim $[x, y] \neq 1$, because if x and y commutes, then $[y, t] = 1$ for all $t \in C_G(x)$ as G is an AC-group. Thus the elements of $C_G(x)$ lie inside $C_G(y)$, which implies $C_G(x) = C_G(y)$, a contradiction. Hence $\{x, y\}$ is an edge of Γ_G. We also immediately deduce that if $\{x, y\}$ is an edge of $\Gamma_{G \setminus Z(G)}$, then it is an edge of $\Upsilon_{G \setminus Z(G)}$. □

By the above result and the non-centralizer induced subgraph definition, we deduce that non-commuting graph $\Gamma_{G \setminus Z(G)}$ associated to a non-abelian AC-group is a complete $|\mathrm{Cent}(G)|$-partite graph. Furthermore, all the results which were proved in [1] for the non-commuting induced subgraph of an AC-group is valid for the non-centralizer induced subgraph.

PROPOSITION 2.12. *Assume that Υ_G is the non-centralizer graph associated to the group G. Then we have*

(i) If the central factor of G is of order p^2, then Υ_G is a complete $(p + 2)$-partite graph.

(ii) Let p be the smallest prime dividing $|G|$. If $|G : Z(G)| = p^3$, then Υ_G is complete $(p^2 + p + 2)$-partite or $(p^2 + 2)$-partite graph.

Proof. (i) Belcastro and Sherman proved that if $|G : Z(G)| = p^2$, then $|\mathrm{Cent}(G)| = p + 2$ (see [8]). Therefore (i) immediately follows.

(ii) It is clear by [7, Proposition 2.2]. □

THEOREM 2.13. *Let G be a group which satisfies one of the hypothesis (i) or (ii) of Proposition 2.12. Then Γ_G is either a $(p + 2)$-partite graph or a $(p^2 + p + 2)$-partite, or a $(p^2 + 2)$-partite graph, respectively.*

Proof. By [7, Lemma 2.1] we see that a group whose central factor is of order pqr, where p, q, r are primes not necessarily distinct, is an AC-group. The proof follows now by Theorem 2.11 and Proposition 2.12. □

Let G be a non-abelian group such that $|Z(G)| \geq 3$. Then Υ_G is not planar. Since by $x, xz_1, xz_2, e, z_1, z_2$ we can make a $K_{3,3}$ as the induced subgraph of Υ_G, where e, z_1 and z_2 are three distinct elements of the center of the group and $x \in V(\Upsilon_G)$ is a non-central element of G.

In the following example we observe that non-abelian groups with small center exist such that their non-centralizer graph is not planar.

Example 2.14. Let $D_{2n} = \langle a, b : a^n = b^2 = 1, a^b = a^{-1}\rangle$ be the dihedral group of order $2n$ and $n \geq 4$. Then

(i) $\Upsilon_{D_{2n}}$ is a complete $(n/2+2)$-partite or $(n+2)$-partite graph, where n is an even or an odd number, respectively (see [5] for more details).

(ii) $\Upsilon_{D_{2n}}$ is not planar for odd numbers n. The vertices $1, a, b, a^i b, a^j b$ form the complete graph K_5, where i, j are distinct, $1 \leq i, j \leq n-1$.

(iii) If $n \geq 4$ is an even number, then $\Upsilon_{D_{2n}}$ is not planar.

(iv) Υ_{S_n} is not a planar graph for $n \geq 4$, since $\{(1\ 2), (2\ 3), (1\ 3), (1\ 4), (2\ 4)\}$ is complete induced subgraph of Υ_{A_n}, it is not a planar graph, where $n \geq 4$.

We are interested to find a general result about the planarity of the graph. If $\{x, y\}$ is an edge in Γ_G, then $[x, y] \neq 1$. Thus $x \notin C_G(y)$ and $\{x, y\}$ is an edge in Υ_G. Hence $\Gamma_{G\setminus Z(G)} \subseteq \Gamma_G \subseteq \Upsilon_G$. As we have seen in [1], the non-commuting graph $\Gamma_{G\setminus Z(G)}$ is planar whenever G is isomorphic to S_3 or D_8 or Q_8.

Theorem 2.15. *Υ_G is planar, if and only if G is an abelian group.*

Proof. It is clear that the non-centralizer graph of an abelian group is planar. Suppose Υ_G is the non-centralizer graph associated to a non-abelian group G, and assume that it is a planar graph. Then by the above discussion $\Gamma_{G\setminus Z(G)}$ is planar so $G \cong S_3$ or D_8 or Q_8. Since Υ_{S_3}, Υ_{D_8} and Υ_{Q_8} have $K_{3,3}$ as their induced subgraphs, they are not planar. □

The following corollary is direct result of the above discussion.

Corollary 2.16. *$\Upsilon_{G\setminus Z(G)}$ is planar if and only if $G \cong S_3$, D_8 or Q_8.*

Theorem 2.17. *Let G be a group. Then*

(i) Υ_G is not a 4-regular graph.

(ii) Υ_G is a 6-regular graph if and only if $G \cong D_8$ or Q_8.

(iii) Υ_G is not a 8-regular graph.

(iv) Υ_G is not a p-regular graph, where $p = 5, 7, 11, 13, 17$.

Proof. (i) Suppose Υ_G is 4-regular, and recall that Υ_G is a k-partite graph. Let each part have a_i vertices and let x_i be the representative of the i-th part, $1 \leq i \leq k$. Thus $\deg(x_i) = a_1 + a_2 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_k = 4 =$

$\deg(x_{i+1}) = a_1 + \cdots + a_i + a_{i+2} + \cdots + a_k$, so we must have $a_i = a_{i+1}$. Since the degree of the vertices are all equal we conclude that all the parts have an equal number of vertices, say a. Thus $\deg(x_i) = (k-1)a$, and we distinguish three cases. If $k = 3$ and $a = 2$, then $G \cong S_3$, which is not regular. If $k = 5$ and $a = 1$, then G must be an abelian group, which is a contradiction. If $k = 2$ and $a = 4$, then $G \cong D_8$ or $G \cong Q_8$, groups whose associated non-centralizer graphs are 6-regular graphs.

(ii) Let Υ_G be a 6-regular graph. Similarly we deduce $(k-1)a = 6$, where k is the number of parts, each part having a vertices. Therefore we have $(k = 3,\ a = 3)$, $(k = 4,\ a = 2)$, $(k = 2,\ a = 6)$ or $(k = 7,\ a = 1)$. The first and the last cases imply that G is an abelian group, which can not hold. If $(k = 2,\ a = 6)$, then G is isomorphic to A_4, to D_{12} or to $T = \langle a, b : a^6 = 1, b^2 = a^3, a^b = a^{-1}\rangle$ whose non-centralizer graphs are not regular. Finally, if $(k = 4,\ a = 2)$, then $|G| = 8$ and the result follows.

(iii) Suppose $(k-1)a = 8$. Similarly $|G| = 9,\ 10,\ 12$ or 16. Clearly, the first three cases are not possible. If $|G| = 16$, $(k = 2,\ a = 8)$, then Υ_G is a regular graph whenever $|Z(G)| = 8$. But there is no group of order 16 with center of order 8.

(iv) If Υ_G is a p-regular graph, then G is a non-abelian group and the degree of the identity element of G is p. This means $\deg(e) = |G| - |Z(G)| = p$. Therefore $|Z(G)| = 1$ or p. It is not possible that $|Z(G)| = p$, because $G \cong D_{2p}$ whose center is trivial. Therefore assume $|Z(G)| = 1$ so $|G| = p + 1$. It is clear that degree of each vertices are complete. Similarly we deduce $(k-1)a = p$, where k is the number of parts, each part having a vertices. Thus $k - 1 = 1$ or $a = 1$. If $k = 2$, then we have a contradiction since degree of vertices are complete. Thus we have $p + 1$ parts with one vertex in each of them. If $p = 5$, $p = 7$, $p = 11$ or $p = 13$, then $G \cong S_3$, $G \cong D_8,\ Q_8$, $G \cong T,\ D_{12},\ A_4$ or $G \cong D_{14}$ which is a contradiction. Moreover for $p = 17$, there are two groups of order 18 such that their centers are trivial but their associate non-centralizer graphs are not regular. □

We guess there is no group of order $p + 1$ such that its non-centralizer graph is p-regular, where p is a prime number. Thus we have the following conjecture.

Conjecture. *Υ_G is not a p-regular graph, where p is a prime number.*

Proposition 2.18. *Let G be a group. Then*

(i) There is no complete bipartite and 3-partite non-centralizer graph.

(ii) The non-centralizer graph Υ_G is complete 4-partite if and only if $G/Z(G) \cong C_2 \times C_2$.

(iii) *The non-centralizer graph* Υ_G *is complete* 5*-partite if and only if* $G/Z(G) \cong C_3 \times C_3$ *or* S_3.

(iv) *If the non-centralizer graph* Υ_G *is complete* 6*-partite, then* $G/Z(G) \cong D_8, A_4, C_2 \times C_2 \times C_2$ *or* $C_2 \times C_2 \times C_2 \times C_2$.

(v) *The non-centralizer graph* Υ_G *is complete* 7*-partite if and only if* $G/Z(G) \cong C_5 \times C_5, D_{10}$ *or* $\langle x, y | x^5 = y^4 = 1, y^{-1}xy = x^3 \rangle$.

(vi) *If the non-centralizer graph* Υ_G *is complete* 8*-partite, then* $G/Z(G) \cong C_2 \times C_2 \times C_2, A_4$ *or* D_{12}.

(vii) *If* $|G|$ *is odd, then* Υ_G *is complete* 9*-partite if and only if* $G/Z(G) \cong C_7 \rtimes C_3$ *or* $C_7 \times C_7$.

(viii) *If* $G/Z(G) \cong A_5$, *then* Υ_G *is complete* 22*-partite or* 32*-partite.*

Proof. The proof follows by using the results in [2, 5–8]. □

Abdollahi *et al.* proved that for a finite group G such that $G/Z(G) \cong D_{2n}$ with $n \geq 2$, $|\text{Cent}(G)| = n + 2$ (see [2]). Therefore we deduce the following result.

PROPOSITION 2.19. *If* G *is a finite group such that* $G/Z(G) \cong D_{2n}$, *then* Υ_G *is complete* $(n + 2)$*-partite.*

3. GROUPS WITH THE SAME GRAPHS

In this section, we try to find conditions under which two graphs are isomorphic.

PROPOSITION 3.1. *If* $G_1 \cong G_2$, *then* $\Upsilon_{G_1} \cong \Upsilon_{G_2}$.

Proof. Suppose φ is the isomorphism between two groups. Let $\{x, y\}$ be an edge in Υ_{G_1}. Therefore $C_{G_1}(x) \neq C_{G_1}(y)$. Without loss of generality we may assume that there is an element $g \in C_{G_1}(x)$ which does not belong to $C_{G_1}(y)$. Thus $[\varphi(g), \varphi(y)] \neq 1$, $[\varphi(g), \varphi(x)] = 1$ and so $\{\varphi(x), \varphi(y)\}$ is an edge of Υ_{G_2}. □

The converse of the above result is not valid. For instance $\Upsilon_{\mathbb{Z}_4} \cong \Upsilon_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ but $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\Upsilon_G \cong \Upsilon_{D_8}$ implies $G \cong D_8$ or Q_8.

PROPOSITION 3.2. *Let* G *and* H *be two groups with* $|Z(G)| = |Z(H)|$. *If* $\Upsilon_{G \setminus Z(G)} \cong \Upsilon_{H \setminus Z(H)}$, *then* $\Upsilon_G \cong \Upsilon_H$.

Proof. Suppose that ψ is an isomorphism between graphs $\Upsilon_{G \setminus Z(G)}$ and $\Upsilon_{H \setminus Z(H)}$ and θ is a bijection between the centers of the groups. Define $\varphi : G \to H$ such that a non-central element x maps to $\psi(x)$, and a central element y maps to $\theta(y)$. It is clear that φ preserves edges. □

P. Hall defined in 1939 an equivalence relation called isoclinism, in order to classify p-groups (see [14] for more details). Let us recall the definition of this notion.

Definition 3.3. Let G and H. Then the pair (α, β) is called a isoclinism from G to H whenever

(i) α is an isomorphism from $G/Z(G)$ to $H/Z(H)$.

(ii) β is an isomorphism from G' to H', with the law $[g_1, g_2] \mapsto [h_1, h_2]$ in which $h_i \in \alpha(g_i Z(G))$, $i = 1, 2$.

If there is such a pair (α, β) with the above properties, then we say that G and H are isoclinic, fact denoted by $G \sim H$.

THEOREM 3.4. *If G and H are two isoclinic groups such that the order of their centers are equal, then $\Upsilon_G \cong \Upsilon_H$.*

Proof. Suppose $\alpha : G/Z(G) \to H/Z(H)$ and $\beta : G' \to H'$ are isomorphisms that satisfy the isoclinism definition. Let $\alpha(g_i Z(G)) = h_i Z(H)$. It is clear that $|G| = |H|$. Assume $\theta : Z(G) \to Z(H)$ is a bijection. We define $\varphi : V(\Upsilon_G) \to V(\Upsilon_H)$ which maps $g_i z$ to $h_i \theta(z)$. The fact that the map φ is a bijection which preserve edges follows by using the properties of the isomorphisms α and β. $\square$

The following proposition presents circumstance such that the converse of the above theorem is valid.

PROPOSITION 3.5. *Let G and H be groups such that $\Upsilon_G \cong \Upsilon_H$, and assume that Υ_G and Υ_H are 4-partite graphs. Then G and H are isoclinic.*

Proof. The proof follows by Proposition 2.18. $\square$

LEMMA 3.6. *If $\Upsilon_G \cong \Upsilon_H$, then $N(G) = N(H)$, where $N(X)$ is the set $\{n \in \mathbb{N} | \ X$ has a conjugacy class C, such that $|C| = n\}$ and X is a group.*

Proof. Since $\Upsilon_G \cong \Upsilon_H$ we conclude that $\mathrm{Cent}(G) = \mathrm{Cent}(H)$. Hence the result follows. $\square$

We also note that if $\Upsilon_G \cong \Upsilon_H$ and G is an abelian group, then H is an abelian group.

PROPOSITION 3.7. *Let G be a group.*

(i) If $\Upsilon_G \cong \Upsilon_{S_3}$, then $G \cong S_3$.

(ii) If $\Upsilon_G \cong \Upsilon_{A_n}$, G is a simple group and $n \geq 5$, then $G \cong A_n$.

(iii) If $\Upsilon_G \cong \Upsilon_H$, $G \neq B_n(q)$ or $C_n(q)$ is a simple group, then $G \cong H$.

Proof. (i) It is clear.

(ii) Suppose $\Upsilon_G \cong \Upsilon_{A_n}$, G is a simple group and $n \geq 5$, $n \neq 8$. Since $|G| = |A_n|$, the assertion follows by [11, Corollary 2.4]. Now assume $\Upsilon_G \cong \Upsilon_{A_8}$. Thus $|G| = 8!/2$ and $G \cong A_8$ or $G \cong L_3(4)$. By Lemma 3.6 we have $N(G) = N(A_8) = \{1, 210, 105, 112, 1680, 1120, 2520, 1260, 1344, 1344, 1344, 3360, 2880, 2880\}$ and $N(L_3(4)) = \{1, 4032, 4032, 2240, 1260, 315, 1260, 1260, 2880, 2880\}$ by group theory package GAP [12]. Hence $G \cong A_8$.

(iii) The proof follows by using the results of E. Artin in [3, 4] and the classification of finite simple groups. □

We end by noting that by the second part of the above proposition we deduce $\Upsilon_G \cong \Upsilon_{L_3(4)}$ implies $G \cong L_3(4)$.

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