A UNIFIED REPRESENTATION OF P-MAX STABLE DISTRIBUTIONS

ZUOXIANG PENG, QIN JIANG and SARALEES NADARAJAH

Communicated by Marius Iosifescu

Let (X_n) be a sequence of independent and identically distributed random variables with common distribution function F. In this note, the universal form of the possible limiting extreme value distributions under power normalization is derived. We also discuss some relationships between D_l and D_p, the domains of attraction under linear normalization and power normalization.

AMS 2010 Subject Classification: Primary 62G32, Secondary 65C05.

Key words: linear and power normalization, l-max stable laws, Max domains of attraction; p-max stable laws.

1. INTRODUCTION

Let (X_n) be a sequence of independent and identically distributed (i.i.d.) random variables with common distribution function (df) F. Let $M_n = \max(X_1, X_2, \ldots, X_n)$ denote the partial maximum. Limiting distributions of extreme values under linear normalization have attracted considerable attention, c.f. Resnick [18], de Haan and Ferreira [8] and Falk *et al.* [10]. The notation $F \in D_l(G)$ means that there exist some suitable normalizing constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\mathrm{P}\left\{M_n \le a_n x + b_n\right\} \to G(x)$$

as $n \to \infty$ for all continuity points x of G, where G is a non-degenerate df. We call $G(x)$ a max stable distribution under linear normalization or simply a l-max stable distribution. It is well known that $G(x)$ must belong to one l-type of the following three classes of extreme value distributions:

Type I Gumbel:

$$\Lambda(x) = \exp\left\{-\exp(-x)\right\}, \qquad x \in \mathbb{R};$$

Type II Fréchet:

$$\Phi_\alpha(x) = \begin{cases} 0, & x < 0, \\ \exp\left\{-x^{-\alpha}\right\}, & x \ge 0 \end{cases} \qquad \text{for some} \quad \alpha > 0; \tag{1.1}$$

Type III Weibull:

$$\Psi_\alpha(x) = \begin{cases} \exp\left\{-(-x)^\alpha\right\}, & x < 0, \\ 1, & x \ge 0 \end{cases} \qquad \text{for some} \quad \alpha > 0.$$

Here, two dfs H and K are of the same l-type if there exist $A > 0$ and $B \in \mathbb{R}$ such that $H(x) = K(Ax+B)$ for all $x \in \mathbb{R}$. Moreover, the three classes of extreme value distributions in (1.1) can be rewritten to take the same l-type as

$$G(x) = G_\kappa(x) = \exp\left\{-(1+\kappa x)^{-1/\kappa}\right\}, \qquad 1+\kappa x > 0 \tag{1.2}$$

for $\kappa \in \mathbb{R}$. For details on necessary and sufficient conditions for $F \in D_l(G)$ and the choices of normalizing constants a_n and b_n, see Resnick [18] and de Haan and Ferreira [8].

Pancheva [16], Mohan and Ravi [13] and Pantcheva [17] extended the above results to a kind of power normalization for M_n. A df F is said to belong to the max domain of attraction of $H(x)$ under power normalization if there exist some suitable normalizing constants $a_n > 0$ and $b_n > 0$ such that

$$\mathrm{P}\left\{\left|\frac{M_n}{a_n}\right|^{\frac{1}{b_n}} \operatorname{sign}(M_n) \le x\right\} = F^n\left(a_n|x|^{b_n}\operatorname{sign}(x)\right) \to H(x) \tag{1.3}$$

weakly as $n \to \infty$, where $\operatorname{sign}(x) = -1$ if $x < 0$, $\operatorname{sign}(x) = 0$ if $x = 0$, and $\operatorname{sign}(x) = 1$ if $x > 0$. In this case, we denote $F \in D_p(H)$ and call $H(x)$ a max stable df under power normalization or simply a p-max stable df. Pancheva [16] proved that $H(x)$ belongs to one p-type of the following six classes of extreme value distributions:

Type I) : $$H_{1,\beta}(x) = \begin{cases} 0, & x \le 1, \\ \exp\left\{-(\log x)^{-\beta}\right\}, & x > 1 \end{cases}$$

for some $\beta > 0$;

Type II) : $$H_{2,\beta}(x) = \begin{cases} 0, & x \le 0, \\ \exp\left\{-(-\log x)^{\beta}\right\}, & 0 < x \le 1, \\ 1, & x > 1 \end{cases}$$

for some $\beta > 0$;

Type III) : $$H_{3,\beta}(x) = \begin{cases} 0, & x \le -1, \\ \exp\left\{-(-\log(-x))^{-\beta}\right\}, & -1 < x \le 0, \\ 1, & x > 0 \end{cases}$$

for some $\beta > 0$;

Type IV) : $$H_{4,\beta}(x) = \begin{cases} \exp\left\{-(\log(-x))^{\beta}\right\}, & x \le -1, \\ 1, & x > -1 \end{cases}$$

for some $\beta > 0$;

Type V) : $$H_5(x) = \begin{cases} 0, & x \le 0, \\ \exp\left(-x^{-1}\right), & x > 0; \end{cases}$$

Type VI) : $$H_6(x) = \begin{cases} \exp(x), & x \le 0, \\ 1, & x > 0. \end{cases}$$

Here, two dfs H and K are of the same p-type if there exist $A > 0$ and $B > 0$ such that $H(x) = K(A|x|^B \operatorname{sign}(x))$ for all $x \in \mathbb{R}$. The power max stable or p-max stable df H means that there exist positive constants a_n and b_n such that

$$H^n\left(a_n|x|^{b_n} \operatorname{sign}(x)\right) = H(x)$$

for all $x \in \mathbb{R}$. The necessary and sufficient conditions for a df to belong to $D_p(H)$ for each of the six p-max stable laws were obtained by Mohan and Ravi [13] and Subramanya [19]. Barakat and Nigm [1] studied the limiting distribution of extreme order statistics under power normalization and random index. For other papers on power normalization, we mention Christoph and Falk [5], Nigm [15], Barakat *et al.* [2], and Barakat and Omar [3]. In these papers, the results of Gnedenko [12] and de Haan [7] concerning linear normalization were extended to p-stable laws.

However, the universal form of extreme value distributions under power normalization like (1.2) for linear normalization has not been studied. Nasri-Roudsari [14] attempted to find a universal form, but their equation (5) is not in universal form. This note derives a universal form by employing the theory of general regularly varying functions (de Haan [6], Geluk and de Haan [11], Bingham *et al.* [4], de Haan and Resnick [9]), see Theorem 2.2. Moreover, in the process of proving Theorem 2.2, equivalent results which are more meaningful and simple are derived in Corollary 2.1. We build a significant bridge between power normalization and general regularly varying functions. Consequently, applying the universal formulation, we can easily find the relationship between l-max stable dfs and p-max stable dfs, see Theorems 3.1 and 3.2. The normalizing constants for power normalization can be easily found through known results for linear normalization, see Corollary 3.1. We will see that some of the results in Section 3 coincide with those in Mohan and Ravi [13].

The contents of this note are organized as follows. In Section 2, we obtain the universal form of extreme value distributions under power normalization. In Section 3, we compare $D_l(\cdot)$ and $D_p(\cdot)$. Related proofs are provided in Section 4. Finally, we provide some examples in Section 5.

2. LIMIT DISTRIBUTION OF MAXIMA AND EQUIVALENT CONDITIONS

In this section, firstly, we obtain an alternative formulation of (1.3). Taking logarithms on both sides of (1.3), we have

$$\lim_{n\to\infty} n \log F\left(a_n|x|^{b_n} \operatorname{sign}(x)\right) = \log H(x) \tag{2.1}$$

for each continuity point x for which $0 < H(x) < 1$. Obviously, (2.1) is equivalent to

$$\lim_{n\to\infty} n\left(1 - F\left(a_n|x|^{b_n}\operatorname{sign}(x)\right)\right) = -\log H(x). \tag{2.2}$$

Define $U(x) = (1/(1-F))^{\leftarrow}(x) = \inf\{y : \frac{1}{1-F(y)} \geq x\}$ for $x > 1$. By properties of the inverse function, (2.2) is equivalent to

$$\begin{cases} \lim_{n\to\infty} -\left(\dfrac{-U(nx)}{a_n}\right)^{1/b_n} = H^{\leftarrow}\left(e^{-1/x}\right), & \text{if} \quad x(F) \leq 0, \\ \lim_{n\to\infty} \left(\dfrac{U(nx)}{a_n}\right)^{1/b_n} = H^{\leftarrow}\left(e^{-1/x}\right), & \text{if} \quad x(F) > 0 \end{cases} \tag{2.3}$$

for each point x, where $x(F) = \sup\{x : F(x) < 1\}$. We are now going to make (2.3) more explicit in the following sense:

Theorem 2.1. *Suppose (X_n) is a sequence of i.i.d. random variables with common df F and let $a_n, b_n > 0$ be real sequences of constants and H a non-degenerate df. The following statements are equivalent:*

(1). *For each continuity point x of H,*

$$\lim_{n\to\infty} F^n\left(a_n|x|^{b_n}\operatorname{sign}(x)\right) = H(x).$$

(2). *For each continuity point x of H satisfying $0 < H(x) < 1$,*

$$\lim_{t\to\infty} t\left(1 - F\left(a(t)|x|^{b(t)}\operatorname{sign}(x)\right)\right) = -\log H(x),$$

where $a(t) := a_{[t]}$ and $b(t) := b_{[t]}$ with $[t]$ denoting the integer part of t.

(3). *Let $D(x) := H^{\leftarrow}(e^{-1/x})$ and $a(t) := a_{[t]}, b(t) := b_{[t]}$.*

$$\text{If} \quad x(F) \leq 0, \qquad \lim_{t\to\infty} -\left(-\frac{U(tx)}{a(t)}\right)^{1/b(t)} = D(x);$$

$$\text{If} \quad x(F) > 0, \qquad \lim_{t=\infty} \left(\frac{U(tx)}{a(t)}\right)^{1/b(t)} = D(x)$$

for each continuity point x of $D(x)$.

(4). *Let $D(x) := H^{\leftarrow}(e^{-1/x})$ and $a(t) := a_{[t]}$, $b(t) := b_{[t]}$.*

$$\text{If} \quad x(F) \leq 0, \qquad \lim_{t\to\infty} \frac{\log(-U(tx)) - \log a(t)}{b(t)} = \log(-D(x));$$

$$\text{If} \quad x(F) > 0, \qquad \lim_{t\to\infty} \frac{\log U(tx) - \log a(t)}{b(t)} = \log D(x)$$

for each continuity point x of $D(x)$.

Remark 2.1. Theorem 2.1(4) shows that $\log |U(t)|$ is a general regularly varying function at infinity on $(0, +\infty)$.

Next, using general regularly varying functions (Bingham *et al.* [4]), the six power extreme value distribution classes in Section 1 can be written in a universal form.

THEOREM 2.2. *Let (X_n) be a sequence of i.i.d. random variables with common df F. If there exist constants a_n, $b_n > 0$ such that*

$$\lim_{n\to\infty} F^n\left(a_n|x|^{b_n}\operatorname{sign}(x)\right) = H(x)$$

as $n \to \infty$ for some non-degenerate df $H(x)$ then $H(x)$ must be of the same p-type as one of the following two classes:

(1). *for $x(F) \leq 0$,*

$$H_\gamma(x) = \exp\left\{-\left(1-\gamma\log(-x)\right)^{-1/\gamma}\right\},\ x<0 \textit{ and } 1-\gamma\log(-x)>0,$$

(2). *for $x(F) > 0$,*

$$H_\gamma(x) = \exp\left\{-\left(1+\gamma\log x\right)^{-1/\gamma}\right\},\ x>0 \textit{ and } 1+\gamma\log x>0,$$

where γ is real.

For $\gamma = 0$, the right hand sides of (1) *and* (2) *should be interpreted as* $\exp\{x\}$, *$x \leq 0$ and* $\exp\{-x^{-1}\}$, *$x > 0$ for $x(F) \leq 0$ and $x(F) > 0$, respectively.*

Remark 2.2. From the proof of Theorem 2.2, one can obtain the following results:

(i). For $x(F) \leq 0$, let $W(t) = \log(-U(t))$, then $-W(t) \in GRV(\gamma, b(t))$; that is, $-W(t)$ is a general regularly varying function with parameters γ and $b(t)$. Additionally, $x(F) < 0$ if $\gamma < 0$, and $x(F) = 0$ if $\gamma > 0$.

(ii). For $x(F) > 0$, let $W^*(t) = \log U(t)$, then $W^*(t) \in GRV(\gamma, b(t))$. Additionally, $0 < x(F) < \infty$ if $\gamma < 0$, and $x(F) = \infty$ if $\gamma > 0$.

The following remark links H_γ in Theorem 2.2 with the H_1 and H_2 given by equation (5) in Nasri-Roudsari [14].

Remark 2.3. For $x > 0$ and $1+\gamma(\log x - a)/b > 0$, $H_1(x) = H_\gamma\left(e^{-a/b}x^{1/b}\right)$; For $x < 0$ and $1-\gamma(\log(-x)-a)/b > 0$, $H_2(x) = H_\gamma\left(-e^{-a/b}(-x)^{1/b}\right)$. The p-max dfs, H_1 and H_2, are defined by

$$H_1(x) = \exp\left\{-\left[1+\gamma(\log x - a)/b\right]^{-1/\gamma}\right\}$$
$$\text{for } x>0 \text{ and } 1+\gamma\left(\log x - a\right)/b > 0;$$
$$H_2(x) = \exp\left\{-\left[1-\gamma\left(\log(-x) - a\right)/b\right]^{-1/\gamma}\right\}$$

$$\text{for } x < 0 \text{ and } 1 + \gamma\left(\log(-x) - a\right)/b > 0,$$

respectively.

Considering the sign of γ and $x(F)$, we obtain the following six extreme value distribution classes.

Corollary 2.1. *Under the conditions of Theorem 2.2, $F \in D_p(\cdot)$ if and only if*

$$\lim_{t\to\infty} \frac{\log|U(tx)| - \log|U(t)|}{\widehat{b}(t)} = \operatorname{sign}\left(x(F)\right)\frac{x^\gamma - 1}{\gamma}, \qquad \gamma \in \mathbb{R}$$

for $x > 0$, where $\widehat{b}(t)/b(t) \to c > 0$ as $t \to \infty$. Moreover,
(1). *For $x(F) < 0$ and $\gamma < 0$, $F \in D_p(H_{4,\beta})$ with $\beta = -1/\gamma$;*
(2). *For $x(F) \le 0$ and $\gamma = 0$, $F \in D_p(H_6)$;*
(3). *For $x(F) = 0$ and $\gamma > 0$, $F \in D_p(H_{3,\beta})$ with $\beta = 1/\gamma$;*
(4). *For $x(F) \in (0, \infty)$ and $\gamma < 0$, $F \in D_p(H_{2,\beta})$ with $\beta = -1/\gamma$;*
(5). *For $0 < x(F) \le \infty$ and $\gamma = 0$, $F \in D_p(H_5)$;*
(6). *For $x(F) = \infty$ and $\gamma > 0$, $F \in D_p(H_{1,\beta})$ with $\beta = 1/\gamma$.*

Applying the properties of general regularly varying functions (Bingham *et al.* [4]), we obtain more general results.

Corollary 2.2. *Under the conditions of Theorem 2.2, $F \in D_p(\cdot)$ if and only if $\log|U(t)| \in GRV(\widehat{b}(t), \varphi)$, i.e.,*

$$\lim_{t\to\infty} \frac{\log|U(tx)| - \log|U(t)|}{\widehat{b}(t)} = \varphi(x)$$

holds for all $x > 0$ with φ not constant, where $\widehat{b}(t)/b(t) \to c > 0$ as $t \to \infty$. Moreover, $\varphi(x)$ is decreasing if and only if $x(F) \le 0$, and $\varphi(x)$ is increasing if and only if $x(F) > 0$.

3. RELATIONSHIPS BETWEEN $D_l(G)$ AND $D_p(H)$

In this section, results coinciding with those obtained by Mohan and Ravi [13] are given. All proofs are given in Section 4 by the equivalence of Corollary 2.1.

Theorem 3.1. *Suppose (X_n) is a sequence of i.i.d. random variables with common df F. Let $F_1(x) = F(-\exp(-x))$ for $x < 0$, and let $F_2(x) = F(\exp(x))$ for $x > 0$.*

(1). *For* $x(F) \le 0$ *and* $\gamma < 0$, $F \in D_p(\cdot)$ *if and only if* $F_1 \in D_l(\cdot)$ *with* $\alpha = \beta = -1/\gamma$*; For* $x(F) \le 0$ *and* $\gamma > 0$, $F \in D_p(\cdot)$ *if and only if* $F_1 \in D_l(\cdot)$ *with* $\alpha = \beta = 1/\gamma$*; For* $x(F) \le 0$ *and* $\gamma = 0$, $F \in D_p(H_6)$ *if and only if* $F_1 \in D_l(\Lambda)$.

(2). *For* $x(F) > 0$ *and* $\gamma < 0$, $F \in D_p(\cdot)$ *if and only if* $F_2 \in D_l(\cdot)$ *with* $\alpha = \beta = -1/\gamma$*; For* $x(F) > 0$ *and* $\gamma > 0$, $F \in D_p(\cdot)$ *if and only if* $F_2 \in D_l(\cdot)$ *with* $\alpha = \beta = 1/\gamma$*; For* $x(F) > 0$ *and* $\gamma = 0$, $F \in D_p(H_5)$ *if and only if* $F_2 \in D_l(\Lambda)$.

In view of the derived results on $D_l(\cdot)$, the normalizing constants under power normalization can be obtained.

Corollary 3.1. *Under the conditions of Corollary 2.1,*

(1). *For* $x(F) < 0$ *and* $\gamma < 0$*, we have* $F \in D_p(H_{4,\beta})$, $\beta = -1/\gamma$ *with normalizing constants* $a_n = -x(F)$, $b_n = \log(F^{\leftarrow}(1 - \frac{1}{n})/x(F))$*;*

(2). *For* $x(F) \le 0$ *and* $\gamma = 0$*, we have* $F \in D_p(H_6)$ *with normalizing constants* $a_n = -F^{\leftarrow}(1 - \frac{1}{n})$ *and* $b_n = f(-a_n)$*, where* $f(t) = -\int_t^{x(F)}(1 - F(s))\frac{1}{s}ds/(1 - F(t))$*;*

(3). *For* $x(F) = 0$ *and* $\gamma > 0$*, we have* $F \in D_p(H_{3,\beta})$, $\beta = 1/\gamma$ *with normalizing constants* $a_n = 1$, $b_n = -\log(-F^{\leftarrow}((1 - \frac{1}{n})))$*;*

(4). *For* $x(F) \in (0, \infty)$ *and* $\gamma < 0$*, we have* $F \in D_p(H_{2,\beta})$, $\beta = -1/\gamma$ *with normalizing constants* $a_n = x(F)$, $b_n = \log(x(F)/F^{\leftarrow}(1 - \frac{1}{n}))$*;*

(5). *For* $0 < x(F) \le \infty$ *and* $\gamma = 0$*, we have* $F \in D_p(H_5)$ *with normalizing constants* $a_n = F^{\leftarrow}(1 - \frac{1}{n})$ *and* $b_n = f(a_n)$*, where* $f(t) = \int_t^{x(F)}(1 - F(s))\frac{1}{s}ds/(1 - F(t))$*;*

(6). *For* $x(F) = \infty$ *and* $\gamma > 0$*, we have* $F \in D_p(H_{1,\beta})$, $\beta = 1/\gamma$ *with normalizing constants* $a_n = 1$, $b_n = \log F^{\leftarrow}(1 - \frac{1}{n})$.

Considering the extreme value distributions under linear normalization, Φ_α, Ψ_α and Λ_α, we have the following result.

Theorem 3.2. *Suppose* (X_n) *is a sequence of i.i.d. random variables with common df* F.

(1). *If* $F \in D_l(\Lambda)$ *then we have* $F \in D_p(H_6)$ *for* $x(F) \le 0$ *and* $F \in D_p(H_5)$ *for* $0 < x(F) \le \infty$*;*

(2). *If* $F \in D_l(\Phi_\alpha)$ *then* $F \in D_p(H_5)$*;*

(3). *If* $F \in D_l(\Psi_\alpha)$ *then we have* $F \in D_p(H_{4,\beta})$ *with* $\beta = 1/\alpha$ *for* $x(F) < 0$, $F \in D_p(H_6)$ *for* $x(F) = 0$ *and* $F \in D_p(H_{2,\beta})$ *with* $\beta = 1/\alpha$ *for* $x(F) > 0$.

4. PROOFS

In this section, we prove results provided in Section 2 and Section 3. The convergence-type theorem from Subramanya [19] is needed.

LEMMA 4.1. *Suppose $U(x)$ and $V(x)$ are non-degenerate dfs on $\mathbb{R}$. For $n \geq 1$, let F_n be a df on $\mathbb{R}$ such that for all continuity points x of U and V*

$$\lim_{n\to\infty} F_n\left(a_n|x|^{b_n}\operatorname{sign}(x)\right) = U(x),$$

$$\lim_{n\to\infty} F_n\left(\alpha_n|x|^{\beta_n}\operatorname{sign}(x)\right) = V(x), \tag{4.1}$$

where a_n, b_n, α_n and β_n are positive for all n. Then, there exist $A > 0$ and $B > 0$ such that

$$\lim_{n\to\infty}\frac{\beta_n}{b_n} = B, \qquad \lim_{n\to\infty}\left(\frac{\alpha_n}{a_n}\right)^{1/b_n} = A \tag{4.2}$$

and

$$V(x) = U\left(A|x|^B\operatorname{sign}(x)\right). \tag{4.3}$$

Also, if (4.2) *holds then either of the two relations in* (4.1) *implies the other and* (4.3) *is true.*

Proof of Theorem 2.1. The equivalence of (2) and (3) follows by virtue of Lemma 1.1.1 in de Haan and Ferreira [8]. It is clear that (3) and (4) are equivalent by taking logarithms. Here, we only need to show that (1) is equivalent to (3). We have already checked that (1) is equivalent to (2.3). So, it is sufficient to prove that (2.3) implies (3). Let x be a continuity point of D. For $t \geq 1$, we have

$$-\left(-\frac{U([t]x)}{a_{[t]}}\right)^{1/b_{[t]}} \leq -\left(\frac{-U(tx)}{a_{[t]}}\right)^{1/b_{[t]}} \leq -\left(-\frac{U\left([t]x\left(1+1/[t]\right)\right)}{a_{[t]}}\right)^{1/b_{[t]}}$$

if $x(F) \leq 0$. The right-hand side is eventually less than $D(x')$ for any continuity point $x' > x$ with $D(x') > D(x)$. Since D is continuous at x, we obtain

$$\lim_{t\to\infty} -\left(\frac{-U(tx)}{a(t)}\right)^{\frac{1}{b(t)}} = D(x).$$

If $x(F) > 0$, the proof is similar. The proof is complete. □

Proof of Theorem 2.2. We consider two cases.

(i) Consider the case of $x(F) \leq 0$.

For $x > 0$ and any a_n, $b_n > 0$, obviously

$$F^n\left(a_n|x|^{b_n}\operatorname{sign}(x)\right) = \mathrm{P}\left\{M_n \leq a_n x^{b_n}\right\} = 1.$$

For $x = 0$, large n and all $x' < 0$,

$$F^n\left(a_n\left|x'\right|^{b_n}\operatorname{sign}(x)\right) = F^n\left(-a_n\left|x'\right|^{b_n}\right) \le F^n(0) \le H(z)$$

for arbitrary $z > 0$. We must have $F^n(0) \to 1$ as $n \to \infty$ since H is non-degenerate. So, $H(x) \equiv 1$ for $x \ge 0$ and we only need to consider $x < 0$.

Let us consider the class of limit functions D in (4) of Theorem 2.1. First, we suppose 1 is a continuity point of D. Then note that for continuing points $x > 0$,

$$\frac{\log(-U(tx)) - \log(-U(t))}{b(t)} \to \log(-D(x)) - \log(-D(1)) =: E(x) \tag{4.4}$$

as $t \to \infty$. Setting $\log(-U(t)) = W(t)$, we obtain

$$\frac{W(tx) - W(t)}{b(t)} \to E(x)$$

as $t \to \infty$. By arguments similar to the proof of Theorem 1.1.3 in de Haan and Ferreira [8], we obtain

$$E(x) = H'(0)\frac{x^\gamma - 1}{\gamma}, \qquad \gamma \in \mathbb{R},$$

where $H(x) := E(e^x)$ ($E(x)$ should be interpreted as $H'(0)\log x$ if $\gamma = 0$). Note by (4.4) that E is decreasing and cannot be constant since H is non-degenerate, so $H'(0) < 0$. So, by (4.4), we conclude that

$$D(x) = D(1)\exp\left\{H'(0)\cdot\frac{x^\gamma - 1}{\gamma}\right\}.$$

So,

$$D^{\leftarrow}(x) = \left\{1 - \gamma\frac{\log(x/D(1))}{-H'(0)}\right\}^{1/\gamma} > 0, \tag{4.5}$$

where $D(1) < 0$ and $H'(0) < 0$. Note that $D(x) = H^{\leftarrow}(e^{-1/x})$, implying

$$D^{\leftarrow}(x) = \frac{1}{-\log H(x)}. \tag{4.6}$$

Combining with (4.5) and (4.6), we have

$$H(x) = \exp\left\{-\left(1 - \gamma\log(-x)\right)^{-1/\gamma}\right\}$$

with $x < 0$ and $1 - \gamma\log(-x) > 0$.

If 1 is not a continuity point of D, we follow the proof with the function $U(tx_0)$, where x_0 is a continuity point of D. In this case, we have

$$\frac{W(tx) - W(tx_0)}{b(t)} \to \log(-D(x)) - \log(-D(x_0)) =: Q(x).$$

So,

$$\frac{W(tx)-W\left(tx_0\right)}{b\left(tx_0\right)}=\frac{W(tx)-W\left(tx_0\right)}{b(t)}\frac{b(t)}{b\left(tx_0\right)}$$

and

$$\frac{W(tx)-W\left(tx_0\right)}{b\left(tx_0\right)}\rightarrow\frac{Q(x)}{A\left(x_0\right)}=:P(x)$$

since $A(x_0)$ cannot be zero: F cannot be constant because G is non-degenerate.

It follows that

$$\frac{W(txy)-W\left(tx_0\right)}{b\left(tx_0\right)}=\frac{W(txy)-W\left(tyx_0\right)}{b\left(tyx_0\right)}\cdot\frac{b\left(tyx_0\right)}{b\left(tx_0\right)}+\frac{W\left(tx_0y\right)-W\left(tx_0\right)}{b\left(tx_0\right)}.$$

So,

$$P(xy)=P(x)\cdot A(y)+P(y).$$

The else proof is similar to the case that 1 is a continuity point of D. The details are omitted here.

(ii) Consider the case of $x(F)>0$. By a similar method, we obtain

$$\frac{\log U(tx)-\log U(t)}{b(t)}\rightarrow\log D(x)-\log D(1)=:E^*(x).$$

Putting $\log U(t)=W^*(t)$, we see that

$$\frac{W^*(tx)-W^*(t)}{b(t)}\rightarrow E^*(x)$$

for $x>0$. Similarly, we obtain

$$E^*(x)=H^{*\prime}(0)\frac{x^\gamma-1}{\gamma},\qquad\gamma\in\mathbb{R},$$

where $H^*(x):=E^*(e^x)$ ($E^*(x)$ should be interpreted as $H^{*\prime}(0)\log x$ if $\gamma=0$). Note that E^* is increasing and cannot be constant since H^* is non-degenerate. So, $H^{*\prime}(0)>0$. So,

$$D^{\leftarrow}(x)=\left\{1+\gamma\frac{\log\left(x/D(1)\right)}{H^{*\prime}(0)}\right\}^{1/\gamma}>0,$$

where $D(1)>0$ and $H^{*\prime}(0)>0$. Equation (4.6) and transformations similar to those in (i) imply that

$$H(x)=\exp\left\{-\left(1+\gamma\log(x)\right)^{-1/\gamma}\right\}$$

with $x>0$ and $1+\gamma\log x>0$. If 1 is not a continuity point of D, we obtain the desired result by a similar proof to the case above that 1 is a continuity point of D. The details are omitted here. $\square$

Proof of Corollary 2.1. The proof follows directly from the equivalence of (1) and (4) in Theorem 2.1, Theorem 2.2 and Remark 2.2. $\square$

Proof of Theorem 3.1. We only prove the necessity of the case $x(F) \leq 0$ since others are similar.

If $x(F) \leq 0$, Theorem 2.2 implies that $F^n(a_n|x|^{b_n}\operatorname{sign}(x)) = 1$ for any $x \geq 0$, and

$$F^n\left(-a_n(-x)^{b_n}\right) = F^n\left(a_n|x|^{b_n}\operatorname{sign}(x)\right) \to \exp\left\{-\left(1-\gamma\log(-x)\right)^{-1/\gamma}\right\}$$

weakly for $x < 0$. Note that $F(-a_n(-x)^{b_n}) = F(-\exp(\log a_n + b_n\log(-x))$ and let $y = -\log(-x)$. We obtain

$$F_1^n\left(b_n y - \log a_n\right) \to \exp\left\{-(1+\gamma y)^{-1/\gamma}\right\}$$

weakly. So,

(i). If $\gamma < 0$, it is easy to conclude that $F_1 \in D_l(\Psi_\alpha)$ with $\alpha = -1/\gamma$ and the normalizing constants

$$\begin{cases} -\log a_n = x\left(F_1\right); \\ \qquad b_n = x\left(F_1\right) - \left(1/\left(1-F_1\right)\right)^{\leftarrow}(n). \end{cases}$$

(ii). If $\gamma = 0$ then $F_1 \in D_l(\Lambda)$ and the normalizing constants satisfy

$$\begin{cases} -\log a_n = \left(1/\left(1-F_1\right)\right)^{\leftarrow}(n); \\ \qquad b_n = f\left(-\log a_n\right), \end{cases}$$

where $f_1(t) = \int_t^{x(F_1)}(1-F_1(s))\mathrm{d}s/(1-F_1(t))$.

(iii). If $\gamma > 0$ then $F_1 \in D_l(\Phi_\alpha)$ with $\alpha = 1/\gamma$ and the normalizing constants

$$\begin{cases} -\log a_n = 1; \\ \qquad b_n = \left(1/\left(1-F_1\right)\right)^{\leftarrow}(n). \end{cases}$$

The proof is complete. $\square$

Proof of Corollary 3.1. From the proof of Theorem 3.1, we can see that $x(F_1) = -\log(-x(F))$, where $x(F_1) = \sup\{x : F_1(x) < 1\}$ and $(1/(1-F_1))^{\leftarrow}(n) = -\log(-1/(1-F))^{\leftarrow}(n))$. Moreover,

$$\begin{aligned} f_1(t) &= \int_t^{x(F_1)}\left(1-F_1(s)\right)\mathrm{d}s/\left(1-F_1(t)\right) \\ &= -\int_{-\exp(-t)}^{x(F)}\left(1-F(y)\right)/y\mathrm{d}y/\left(1-F\left(-\exp(-t)\right)\right). \end{aligned}$$

So, (1), (2) and (3) hold. Also, (4), (5) and (6) can be obtained by applying the result that $F_2 \in D_l(\cdot)$. So, (i). If $\gamma < 0$, we have

$$\begin{cases} \log a_n = x(F_2); \\ \qquad b_n = x(F_2) - (1/(1-F_2))^{\leftarrow}(n). \end{cases}$$

(ii). If $\gamma = 0$, we have

$$\begin{cases} \log a_n = (1/(1-F_2))^{\leftarrow}(n); \\ \qquad b_n = f(\log a_n), \end{cases}$$

where $f_2(t) = \int_t^{x(F_2)} (1-F_2(s))\mathrm{d}s/(1-F_2(t))$.

(iii). If $\gamma > 0$, we have

$$\begin{cases} \log a_n = 1; \\ \qquad b_n = (1/(1-F_2))^{\leftarrow}(n). \end{cases}$$

Note that $x(F_2) = \log x(F)$, where $x(F_2) = \sup\{x : F_2(x) < 1\}$ and $(1/(1-F_2))^{\leftarrow}(n) = \log(1/(1-F))^{\leftarrow}(n)$. Additionally,

$$f_2(t) = \int_t^{x(F_2)} (1-F_2(s))\,\mathrm{d}s/(1-F_2(t)) = \int_{\exp(t)}^{x(F)} (1-F(y))/y\mathrm{d}y/(1-F(\exp(t))).$$

The proof is complete. □

Proof of Theorem 3.2. (*i*) When $F \in D_l(\Lambda)$, we consider two cases. It is known that there exists a positive function $\alpha(t)$ such that $\frac{U(tx)-U(t)}{\alpha(t)} \to \log x$ as $t \to \infty$. So,

$$\frac{U(tx)}{U(t)} = 1 + \frac{\alpha(t)}{U(t)} \log x\,(1+o(1))$$

as $t \to \infty$. Note that $U(tx)/U(t) \to 1$ also holds for $x > 0$ as $t \to \infty$. So,

$$\log \frac{U(tx)}{U(t)} \sim \frac{U(tx)}{U(t)} - 1 = \frac{\alpha(t)}{U(t)} \log x\,(1+o(1))$$

as $t \to \infty$.

If $x(F) \leq 0$, letting $\widehat{b}_3(t) = -\frac{\alpha(t)}{U(t)} > 0$ for sufficiently large t, we obtain

$$\frac{\log \frac{U(tx)}{U(t)}}{\widehat{b}_3(t)} \to -\log x$$

as $t \to \infty$. So, Theorem 2.2 with $x(F) \leq 0$ and $\gamma = 0$ implies $F \in D_p(H_6)$.

If $0 < x(F) \leq \infty$, letting $\widehat{b}_3(t) = \frac{\alpha(t)}{U(t)} > 0$ for sufficiently large t, we obtain

$$\frac{\log \frac{U(tx)}{U(t)}}{\widehat{b}_3(t)} \to \log x$$

as $t \to \infty$. So, Theorem 2.2 with $0 < x(F) \le \infty$ and $\gamma = 0$ implies $F \in D_p(H_5)$.
(ii) When $F \in D_l(\Phi_\alpha)$, we have $U \in RV_{1/\alpha}$, that is to say for $x > 0$

$$\frac{U(tx)}{U(t)} \to x^{1/\alpha}$$

as $t \to \infty$. So, for $x > 0$

$$\frac{\log \frac{U(tx)}{U(t)}}{1/\alpha} \to \log x$$

as $t \to \infty$. Applying Theorem 2.2 with $x(F) = \infty$ and $\gamma = 0$, we have $F \in D_p(H_5)$.
(iii) For the case $F \in D_l(\Psi_\alpha)$, we have $x(F) < \infty$. We discuss three cases.

If $x(F) = 0$, using $x(F) - U(t) \in RV_{-1/\alpha}$, we obtain

$$\frac{\log \frac{U(tx)}{U(t)}}{1/\alpha} \to -\log x$$

as $t \to \infty$. Applying Theorem 2.2 with $x(F) = 0$ and $\gamma = 0$, we have $F \in D_p(H_6)$.

If $x(F) < 0$, we obtain for $x > 0$

$$\frac{x(F) - U(tx)}{x(F) - U(t)} \to x^{-1/\alpha}$$

as $t \to \infty$. So, for $x > 0$

$$\frac{U(tx)}{U(t)} = \frac{x(F)}{U(t)} - x^{-1/\alpha}\left(\frac{x(F)}{U(t)} - 1\right)(1 + o(1))$$

as $t \to \infty$. Note that $U(tx)/U(t) \to 1$ holds as $t \to \infty$ and $\frac{x(F)}{U(t)}$ is increasing.

So,

$$\log \frac{U(tx)}{U(t)} \sim \frac{U(tx)}{U(t)} - 1 = \frac{x(F)}{U(t)} - 1 - x^{-1/\alpha}\left(\frac{x(F)}{U(t)} - 1\right)(1 + o(1))$$

as $t \to \infty$. Let $\widehat{b}_1(t) = -\frac{1}{\alpha}(\frac{x(F)}{U(t)} - 1) > 0$ for sufficiently large t. We obtain

$$\frac{\log \frac{U(tx)}{U(t)}}{\widehat{b}_1(t)} \to -\frac{x^{-1/\alpha} - 1}{-1/\alpha}$$

as $t \to \infty$. Applying Theorem 2.2 with $x(F) < 0$ and $\gamma = -1/\alpha < 0$, we have $F \in D_p(H_{4,\beta})$ with $\beta = \alpha$.

If $x(F) > 0$, note that $U(tx)/U(t) \to 1$ holds as $t \to \infty$ and $\frac{x(F)}{U(t)}$ is decreasing. So, putting $\widehat{b}_2(t) = \frac{1}{\alpha}(\frac{x(F)}{U(t)} - 1) > 0$, we obtain

$$\frac{\log \frac{U(tx)}{U(t)}}{\widehat{b}_2(t)} \to \frac{x^{-1/\alpha} - 1}{-1/\alpha}$$

as $t \to \infty$. Applying Theorem 2.2 with $x(F) \in (0, \infty)$ and $\gamma = -1/\alpha < 0$, we conclude that $F \in D_p(H_{2,\beta})$ with $\beta = \alpha$.

The proof is complete. □

5. EXAMPLES

In this section, some examples are given. Examples 1, 2, 3 and 7 are also considered in Christoph and Falk [5]. Corollary 2.2 and Corollary 3.1 are very useful for us to show whether a df F belongs to power attraction domains or not and to identify the power attraction domain.

Example 1. If F is the uniform df on $(-1, 0)$ then $F \in D_p(H_6)$ with the normalizing constants $a_n = \frac{1}{n}$ and $b_n = 1$.

Proof. Obviously, $x(F) = 0$. It is easy to see that $U(x) = (1/(1 - F))^{\leftarrow}(x) = -1/x$ for $x > 1$. So,

$$\log \frac{U(tx)}{U(t)} \sim -\log x, \qquad x > 0$$

as $t \to \infty$. Let $\widehat{b}(t) = 1$. By Corollary 2.1 and Corollary 3.1 with $x(F) = \gamma = 0$, we obtain the desired result. □

Example 2. If F is the uniform df on $(-1 + \varepsilon, \varepsilon)$, where $0 < \varepsilon \leq 1$, then $F \in D_p(H_{2,\beta})$ with $\beta = 1$ and the normalizing constants $a_n = \varepsilon$ and $b_n = \frac{1}{n\varepsilon}$.

Proof. Obviously, $x(F) = \varepsilon > 0$. It is easy to see that $U(x) = (1/(1 - F))^{\leftarrow}(x) = \varepsilon - 1/x$ for $x > 1$. So,

$$\log \frac{U(tx)}{U(t)} \sim \frac{1 - x^{-1}}{t\varepsilon}, \qquad x > 0$$

as $t \to \infty$. Let $\widehat{b}(t) = 1/(t\varepsilon)$. By Corollary 2.1 and Corollary 3.1 with $x(F) > 0$, $\gamma = -1 < 0$, we obtain the desired result. □

Note that Example 1 is the particular case of Example 2 for $\varepsilon \to 0$.

Example 3. If F is the uniform df on $(-2, -1)$ then $F \in D_p(H_{4,\beta})$ with $\beta = 1$ and the normalizing constants $a_n = 1$ and $b_n = \frac{1}{n}$.

Proof. Obviously, $x(F) = -1 < 0$. It is easy to see that $U(x) = -1/x - 1$ for $x > 1$. So,

$$\log \frac{U(tx)}{U(t)} \sim \frac{x^{-1} - 1}{t}, \qquad x > 0$$

as $t \to \infty$. Let $\widehat{b}(t) = 1/t$. By Corollary 2.1 and Corollary 3.1 with $x(F) < 0$, $\gamma = -1 < 0$, we obtain the desired result. $\square$

Example 4. If $F = \Phi_\alpha$ then $F \in D_p(H_5)$ with the normalizing constants $a_n = n^{1/\alpha}$ and $b_n = 1/\alpha$ for sufficiently large n.

Proof. Obviously, $x(F) = \infty$. It is easy to see that $U(x) = (-\log(1 - x^{-1}))^{-1/\alpha}$ for $x > 1$. So,

$$\log \frac{U(tx)}{U(t)} = -1/\alpha \log \left[1 + \frac{\log \frac{1-(tx)^{-1}}{1-t^{-1}}}{\log\left(1 - t^{-1}\right)} \right] \sim 1/\alpha \log x, \qquad x > 0$$

as $t \to \infty$. Let $\widehat{b}(t) = 1/\alpha$. By Corollary 2.1 and Corollary 3.1 with $x(F) = \infty$, $\gamma = 0$, we obtain the desired result. $\square$

Example 5. If

$$F(x) = \begin{cases} 0, & \text{if} \quad x \le e, \\ 1 - (\log x)^{-\alpha}, & \text{if} \quad e \le x \end{cases}$$

for $\alpha > 0$ then $F \in D_p(H_{1,\beta})$ with $\beta = \alpha$ and the normalizing constants $a_n = 1$ and $b_n = n^{1/\alpha}$.

Proof. Obviously, $x(F) = \infty$. It is easy to see that $U(x) = -\exp(x^{1/\alpha})$ for $x > 1$. So,

$$\frac{\log \frac{U(tx)}{U(t)}}{\alpha^{-1} t^{-1/\alpha}} \sim \frac{x^{1/\alpha} - 1}{1/\alpha}, \qquad x > 0$$

as $t \to \infty$. Let $\widehat{b}(t) = \alpha^{-1} t^{-1/\alpha}$. By Corollary 2.1 and Corollary 3.1 with $x(F) = \infty$, $\gamma = 1/\alpha > 0$, we obtain the desired result. $\square$

Example 6. If

$$F(x) = \begin{cases} 0, & \text{if } x < -e^{-1}, \\ 1 - [-\log(-x)]^{-\alpha}, & \text{if } -e^{-1} \le x < 0, \\ 1, & \text{if } x \ge 0 \end{cases}$$

for $\alpha > 0$ then $F \in D_p(H_{3,\beta})$ with $\beta = \alpha$ and the normalizing constants $a_n = 1$ and $b_n = n^{1/\alpha}$.

Proof. Obviously, $x(F) = 0$. It is easy to see that $U(x) = -\exp(-x^{-1/\alpha})$ for $x > 1$. So,

$$\frac{\log \frac{U(tx)}{U(t)}}{\alpha^{-1} t^{1/\alpha}} \sim -\frac{x^{1/\alpha} - 1}{1/\alpha}, \qquad x > 0$$

as $t \to \infty$. Let $\widehat{b}(t) = \alpha^{-1} t^{1/\alpha}$. By Corollary 2.1 and Corollary 3.1 with $x(F) = 0$, $\gamma = 1/\alpha > 0$, we obtain the desired result. $\square$

Finally, we provide an example which does not belong to any class of $D_p(\cdot)$.

Example 7. If

$$F(x) = \begin{cases} 1 - (\log \log x)^{-1}, & \text{if} \quad x \geq \exp(e), \\ 0, & \text{otherwise} \end{cases}$$

then F does not belong to any of $D_p(\cdot)$ since $U(x) = (1/(1-F))^{\leftarrow}(x) = \exp(e^x)$ for $x > 1$, and $\log U(t) = \exp(x)$ is not a general regularly varying function.

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Received 9 September 2013

Southwest University,
School of Mathematics and Statistics,
Chongqing, 400715, China

Southwest University,
School of Mathematics and Statistics,
Chongqing, 400715, China

University of Manchester
School of Mathematics,
Manchester, United Kingdom
Saralees.Nadarajah@manchester.ac.uk