ON ω_1-SEPARABLE AND CRAWLEY ABELIAN p-GROUPS IN ZFC

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We prove in ZFC that ω_1-separable p-groups having the weak ω-cps-elongation property are Crawley groups. We also show in ZFC that all groups possessing the strong ω-cps-elongation property are Crawley groups.

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1. INTRODUCTION AND KNOWN FACTS

Throughout the present short note, the word "*group*" will mean "*an Abelian p-group*", where p is a fixed arbitrary prime, and $n \in \mathbb{N} \cup \{0\}$. Our group-theoretic notion and notation that are not explicitly stated herein will generally follow [7], and we shall assume that the readers are familiar with the standard terminology of set theory – if not, we refer the interested reader to [8].

We will say that a group A is Σ-*cyclic* if it is isomorphic (and hence is) a direct sum of cyclic groups. It was proved via collective efforts in [2, 9, 14, 15], that a separable group A is Σ-cyclic if, and only if, for any two groups G and H with $p^\omega G \cong p^\omega H$ and $G/p^\omega G \cong A \cong H/p^\omega H$, it follows that $G \cong H$.

Moreover, a group A is said to be ω_1*-separable* if it is separable (*i.e.*, there are no non-zero elements of infinite heights) and each of its countable subsets can be embedded in its countable direct summand. Clearly Σ-cyclic groups are ω_1-separable; however the converse does not hold - in fact, there exists an example from [7] showing that there is an ω_1-separable group of cardinality $\aleph_1$ which is not Σ-cyclic.

On the other vein, to obtain a somewhat stronger converse of Σ-cyclic groups, an important class of groups was defined in [10], motivated by a work of Crawley [2], namely: A separable group A is called *a Crawley group* if for any two groups G and H with $p^\omega G \cong \mathbb{Z}(p) \cong p^\omega H$ and $G/p^\omega G \cong A \cong H/p^\omega H$, it follows that $G \cong H$. Since Σ-cyclic groups are obviously Crawley, a major question is of whether or not Crawley groups are Σ-cyclic. It was, however, resolved in the negative as there exist several comprehensive achievements in this direction and we will briefly represent below the most important of them as follows:

• ([12]) Consistently in ZFC there exists a Crawley group of cardinality $\aleph_1$ which is not ω_1-separable; in fact, it even has a countable basic subgroup.

• ([15]) With the presence of CH each Crawley group with a countable basic subgroup is Σ-cyclic.

• ([10]); ([12], [13]) With the presence of V=L every Crawley group (of cardinality $\aleph_1$) is Σ-cyclic.

In a concrete reference to [13], it is noteworthy (to indicate) that the restriction on the power $\aleph_1$ can be eliminated, that is, the theorem is valid in all generality.

Thus, in the last two cases, *i.e.* provided CH or V=L, with the cited above example of [7] at hand, there is an ω_1-separable group that is not Crawley. However, this is not the case in the next situation.

• ([10], [12]) With the presence of MA $+\neg$CH any ω_1-separable group of cardinality $\aleph_1$ is Crawley. In particular, there is a Crawley group which is not Σ-cyclic - in fact, this group is ω_1-separable as aforementioned.

The last assertion once again illustrates that the set-theoretic assumption MA $+\neg$CH sheds a dramatic effect on abelian group theory. Besides, the above discussion shows that there is no a purely group-theoretic result in the sense that under which additional circumstances ω_1-separable groups are Crawley groups, and vice versa. So, the aim of this article is to demonstrate explicitly that such realistic conditions can be successfully found by proving two theorems in ZFC concerning this subject.

2. MAIN DEFINITIONS AND RESULTS

We start here with a series of definitions needed for our presentation. The first two of them were originally stated in [5].

Definition 1. The group G is called *a cps group* if it is isomorphic to (and hence is) the direct sum of a countable group and separable group.

Definition 2. The separable group A is said to have *the ω-cps-elongation property* if whenever G is a group with $G/p^\omega G \cong A$ and $p^\omega G$ is countable, then G is cps.

It was established in [5] that all Σ-cyclic groups possess the ω-cps-elongation property as well as assuming V=L all separable groups which have the ω-cps-elongation property are Σ-cyclic. Nevertheless, with MA $+\neg$CH at hand, there exists a separable group satisfying the ω-cps-elongation property that is not Σ-cyclic. These results, being very close to those quoted in the introductory section, suggest to discover how groups possessing the ω-cps-elongation property are situated inside ω_1-separable groups and Crawley groups. This shall be done later in our basic theorem listed below.

Definition 3. The separable group A is said to have *the strong ω-cps-elongation property* if whenever G is a group such that $G/p^\omega G \cong A$ and $p^\omega G$ is countable, then G splits into the direct sum of a countable group with fixed finite Ulm-Kaplansky invariants and a separable group.

Definition 4. The separable group A is said to have *the weak ω-cps-elongation property* if whenever G is a group with $G/p^\omega G \cong A$ and $p^\omega G \cong \mathbb{Z}(p)$, then G is cps.

A more restricted version of Crawley's groups is the following one:

Definition 5. A separable group A is said to be *strongly Crawley* if for any two groups G and H with $p^\omega G \cong p^\omega H$ countable and $G/p^\omega G \cong A \cong H/p^\omega H$, it follows that $G \cong H$.

Note that in [12] was stated another version of a strongly Crawley group: There a separable group A is called *strongly Crawley* if $A \oplus S$ remains Crawley for any Σ-cyclic group S. To avoid a duplication of the terminology we call these groups just *super Crawley*.

We are now ready to prove the following two statements, which all are results in ZFC.

THEOREM 2.1. *If A is an ω_1-separable group which satisfies the ω-cps-elongation property, then A is a strongly Crawley group.*

Before showing it, we first need the following very elementary but useful observation on direct decompositions:

LEMMA 2.2. *Suppose C and L are groups, L is separable and $G = C \oplus L$. Let K be a subgroup of G containing C, and suppose $G/p^\omega G$ is the (internal) direct sum $(K/p^\omega G) \oplus M$ for some subgroup M of $G/p^\omega G$. Then G is the (internal) direct sum $K \oplus N$ for some subgroup N of L, where $N \cong G/K \cong M$.*

Proof. Observing that $p^\omega G = p^\omega C$, we deduce that $G/p^\omega G \cong (C/p^\omega C) \oplus L$. So, let now $\pi : G/p^\omega G \to L$ be the projection associated to this isomorphic decomposition (see, for instance, [7]), and let $N = \pi(M)$. We divide our further considerations into two claims:

Claim 1 – $K \cap N = \{0\}$: In fact, given $x \in K \cap N$, it follows that $x \in \pi(M)$, so let $y \in M$ satisfy $\pi(y) = x$, that is, $y = z + x + p^\omega G$, where $z + p^\omega G \in C/p^\omega G = C/p^\omega C \subseteq K/p^\omega G$. Since $x + p^\omega G \in K/p^\omega G$, we derive that $y \in K/p^\omega G$. But since $y \in M$, we can conclude that $y = 0$, whence $x = 0$, as required.

Claim 2 – $K + N = G$: Note $p^\omega G = p^\omega C \subseteq K$, so that we will be done if we can show $K/p^\omega G + (N + p^\omega G)/p^\omega G = G/p^\omega G$. One sees that if $z \in G/p^\omega G$, then $z = x+y$, where $x \in K/p^\omega G$ and $y \in M$. However, if $w = \pi(y)+p^\omega G$, then $w \in (N+p^\omega G)/p^\omega G$ because $\pi(y) \in N$, and $y-w \in (C+p^\omega G)/p^\omega G \subseteq K/p^\omega G$, and hence $z = x + y = (x + (y - w)) + w \in K/p^\omega G + (N + p^\omega G)/p^\omega G$, as required. □

And so, we begin now with

Proof of Theorem 2.1. Suppose that $p^\omega G \cong p^\omega H$ be countable and $G/p^\omega G \cong A \cong H/p^\omega H$ for some arbitrary couple of groups G, H. Since A satisfies the ω-cps-elongation property, one may write that $G = C_1 \oplus S_1$ and $H = C_2 \oplus S_2$ where C_1, C_2 are countable whereas S_1, S_2 are separable. Thus $p^\omega G = p^\omega C_1$ as well as $p^\omega H = p^\omega C_2$. But since A is also ω_1-separable, let J be its countable summand which contains both C_1 and C_2, and let K_1 and K_2 be subgroups of G and H that contain C_1 and C_2, respectively, such that

$$K_1/p^\omega K_1 \cong J \cong K_2/p^\omega K_2.$$

Notice that such a choice is possible due to the sequence of isomorphisms

$$G/p^\omega G \cong (C_1/p^\omega C_1) \oplus S_1 \cong A/ \cong H/p^\omega H \cong (C_2/p^\omega C_2) \oplus S_2.$$

Moreover, if $A = J \oplus M$, it follows from Lemma 2.2 that $G = K_1 \oplus M_1$ and $H = K_2 \oplus M_2$ with $M_1 \cong M \cong M_2$. Note also that K_1 and K_2 are both countable because $p^\omega K_1 = p^\omega G$ and $p^\omega K_2 = p^\omega H$ are countable, as well as so are $K_1/p^\omega K_1$ and $K_2/p^\omega K_2$. Besides, $p^\omega K_1 = p^\omega C_1 \cong p^\omega C_2 = p^\omega K_2$ plus by what we have obtained above $K_1/p^\omega K_1 \cong K_2/p^\omega K_2$. It therefore follows from [7] that $K_1 \cong K_2$, whence $G \cong H$, as wanted. □

We continue in this light with

PROPOSITION 2.3. *If A is a group satisfying the strong ω-cps-elongation property, then A is a strongly Crawley group.*

Proof. Let $p^\omega G \cong p^\omega H$ be countable and $G/p^\omega G \cong A \cong H/p^\omega H$ for any couple of groups G and H. Since both $p^\omega G$ and $p^\omega H$ are countable, one may write by hypothesis that $G = C \oplus S$ and $H = C' \oplus S'$, where in view of Definition 3 the summands C and C' are countable with equal finite Ulm-Kaplansky invariants, while S and S' are separable. Hence $p^\omega G = p^\omega C$ and $G/p^\omega G \cong (C/p^\omega C) \oplus S$ as well as $p^\omega H = p^\omega C'$ and $H/p^\omega H \cong (C'/p^\omega C') \oplus S'$. In particular, $p^\omega C \cong p^\omega C'$. Furthermore, we infer that

$$(C/p^\omega C) \oplus S \cong (C'/p^\omega C') \oplus S',$$

where $C/p^\omega C$ and $C'/p^\omega C'$ are both countable Σ-cyclic groups (see [7]). Moreover, since $C/p^\omega C$ and $C'/p^\omega C'$ have equal finite Ulm-Kaplansky invariants (in fact, this is guaranteed because in virtue of [7] the quotient $C/p^\omega C$, respectively $C'/p^\omega C'$, has the same n-th Ulm-Kaplansky invariants as these of C, respectively of C', for any $n < \omega$), we obtain that $C/p^\omega C \cong C'/p^\omega C'$, whence $C \cong C'$ (see again [7]). According to [1], it follows that $S \cong S'$. Consequently, we finally conclude that $G \cong H$, as desired. □

Adapting the same method of proof as that of Theorem 2.1, we can also derive that following:

Theorem 2.4. *If A is an ω_1-separable group which satisfies the weak ω-cps-elongation property, then A is a Crawley group.*

Employing the discussion initiated above, one extracts the following consequence:

Corollary 2.5. *Assuming V=L, each ω_1-separable group satisfying the weak ω-cps-elongation property is Σ-cyclic.*

Note once again that it was established in [5] that under the validity of V=L every separable group having the ω-cps-elongation property is Σ-cyclic.

Likewise, a critical point is to investigate when summands of Crawley groups are again Crawley. As a core of some advantage on that could be the idea utilized in the proof of Proposition 2.3. Specifically, the following is valid:

Proposition 2.6. *Suppose that $A = B \oplus C$ is a Crawley group, where C is a Σ-cyclic group with all the Ulm-Kaplansky invariants finite. Then B is a Crawley group.*

Proof. Given $p^\omega G \cong \mathbb{Z}(p) \cong p^\omega H$ and $G/p^\omega G \cong B \cong H/p^\omega H$ for any two groups G and H, we yield that $p^\omega(G \oplus C) \cong \mathbb{Z}(p) \cong p^\omega(H \oplus C)$ and $(G\oplus C)/p^\omega(G\oplus C) \cong (G/p^\omega G)\oplus C \cong A \cong (H/p^\omega H)\oplus C \cong (H\oplus C)/p^\omega(H\oplus C)$. Observe that C has to be necessarily countable. So, by assumption, $G \oplus C \cong H \oplus C$ and therefore [1] applies to get that $G \cong H$. This forces by definition that B is a Crawley group, indeed, as claimed. □

The last thoughts unambiguously show that whether or not a direct summand of a Crawley group is again Crawley entirely depends on the structure of the complementary summand.

3. FINAL REMARKS AND PROBLEMS

By what we have shown in the preceding section, the following relationships hold:

strong ω-cps-elongation property $\Rightarrow$ ω-cps-elongation property $\Rightarrow$ weak ω-cps-elongation property.

Moreover, the following inclusions of group classes are fulfilled too:

{strong ω-cps-elongation property} $\cup$ {ω_1-separable + ω-cps-elongation property} $\subseteq$ {strongly Crawley}

and

{strongly Crawley} $\cup$ {ω_1-separable + weak ω-cps-elongation property} $\subseteq$ {Crawley}.

They allow us to pose in closing the following conjectures, problems and questions, all of some interest and importance.

It was asked in [12] whether each Crawley group satisfies the weak ω-cps-elongation property. If yes, there was indicated that in V=L every Crawley group is Σ-cyclic. Although this is still left-open, in [13] the cited claim was resolved utilizing another technique. So, the first conjecture somewhat generalizes that in double way.

Conjecture 1. *Crawley groups are precisely the groups having the weak ω-cps-elongation property.*

Conjecture 2. *In V=L, the separable groups having the weak ω-cps-elongation property are Σ-cyclic.*

Appealing to Corollary 2.5, it is enough to show that these groups are ω_1-separable.

Conjecture 3. *In MA + $\neg$CH all ω_1-separable groups satisfy the weak ω-cps-elongation property.*

By analogy with Conjecture 1 we state:

Conjecture 4. *Strongly Crawley groups are exactly the groups having the ω-cps-elongation property.*

Similarly, one can ask:

Problem 1. *Does there exist a group with the ω-cps-elongation property (with the weak ω-cps-elongation property, respectively) which is not ω_1-separable?*

Problem 2. *Is each group equipped with the strong ω-cps-elongation property ω_1-separable?*

Problem 3. *Under the truthfulness of MA +$\neg$CH is there a group having the strong ω-cps-elongation property that is not Σ-cyclic?*

If yes, this construction will refine with the aid of Proposition 2.3 the mentioned before existence of Crawley group that is not Σ-cyclic.

PROBLEM 4. *Is it true that Crawley torsion-complete groups are bounded?*

Recall that a separable group A is called *weakly ω_1-separable* if each of its countable subsets can be embedded in its countable pure and nice subgroup. Certainly, ω_1-separable groups are themselves weakly ω_1-separable, but the converse implication is wrong in general. However, it was established in [11] with the assumption of MA $+\neg$CH that any weakly ω_1-separable group of cardinality $\aleph_1$ is ω_1-separable. Moreover, it was proved in Corollary 3.10 of [6] that under the truthfulness of the inequality $2^{\aleph_0} < 2^{\aleph_1}$ every Crawley group of cardinality $\aleph_1$ must be weakly ω_1-separable.

On the other hand, it is worthwhile noticing that in [4] was proved that under the validity of CH every quasi-complete (in particular, torsion-complete) weakly ω_1-separable group is bounded; the crucial argument in any version of set theory is that each unbounded torsion-complete group has a torsion-complete summand of cardinality $2^{\aleph_0}$ with a countable basic subgroup; thus $2^{\aleph_0} = \aleph_1$ provided CH holds. Without assuming CH, *i.e.* in ZFC, in ([3], Proposition 7.1) was established that weakly ω_1-separable torsion-complete groups with countable basic subgroups must always be bounded. So, contrary to Conjecture 7.1 of [3], in ZFC every weakly ω_1-separable torsion-complete group has to be bounded. In fact, the p-adic closure of an arbitrary countable direct summand of a basic subgroup is a direct summand of the full group, and so is torsion-complete weakly ω_1-separable with a countable basic subgroup and hence bounded by the already cited Proposition 7.1. This plainly implies that the basic subgroup is bounded and thus equal to the whole group, as required.

Furthermore, analyzing the properties of ω-elongations, one may state the following queries:

QUESTION 1. *Suppose A is a $p^{\omega+n}$-bounded group. If all groups G with $p^{\omega+n}G$ cyclic of order p and $G/p^{\omega+n}G \cong A$ are mutually isomorphic, is then A a $p^{\omega+n}$-projective group?*

QUESTION 2. *Suppose that A is a p^{ω}-bounded group. If all groups G with $p^{\omega}G$ cyclic of order p^{n+1} and $G/p^{\omega}G \cong A$ are mutually isomorphic, is then A a $p^{\omega+n}$-projective group?*

Notice that Crawley's variant of the problem can now be immediately derived by taking $n = 0$. The next point is a weaker version of the Crawley's problem.

QUESTION 3. *Let A be a p^{ω}-bounded group. If all groups G with $p^{\omega}G$ cyclic of order p and $G/p^{\omega}G \cong A$ are mutually isomorphic, is then A a $p^{\omega+1}$-projective-group?*

We accent that if this holds (eventually with the additional assumption of CH), we can deduce as a consequence the aforementioned result due to [15] that in CH every Crawley group with a countable basic subgroup is Σ-cyclic.

Even more, it might be asked whether or not these groups G are $p^{\omega+1}$-projective as well; in fact, in V=L this is so because A is of necessity Σ-cyclic (see [13]). However, in MA+¬ CH, this also could be true although A need not be Σ-cyclic as already emphasized above.

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REFERENCES

[1] P. Crawley, *The cancellation of torsion abelian groups in direct sums.* J. Algebra **2** (1965), 432–442.

[2] P. Crawley, *Abelian p-groups determined by their Ulm sequences.* Pac. J. Math. **22** (1967), 235–239.

[3] P. Danchev, *Generalized Dieudonné and Hill criteria.* Port. Math. (N.S.) **65** (2008), 121–142.

[4] P. Danchev, *Weakly $\aleph_1$-separable quasi-complete abelain p-groups are bounded.* Vladikavkaz Math. J. **11** (2009), 8–9.

[5] P. Danchev and P. Keef, *Nice elongations of primary abelian groups.* Publ. Mat. **54** (2010), 317–339.

[6] P. Eklof and M. Huber, *On ω-filtered vector spaces and their application to abelian p-groups II.* Rocky Mount. J. Math. **18** (1988), 123–136.

[7] L. Fuchs, *Infinite Abelian Groups.* **I** and **II**, Academic Press, New York and London, 1970 and 1973.

[8] T. Jech, *Set Theory.* **79**, Academic Press, New York and London, 1978; Third Millennium ed., revised and expanded, Springer, Berlin and Heidelberg, 2002.

[9] P. Hill and C. Megibben, *Extending automorphisms and lifting decompositions in abelian groups.* Math. Ann. **175** (1968), 159–168.

[10] C. Megibben, *Crawley's problem on the unique ω-elongation of p-groups is undecidable.* Pac. J. Math. **107** (1983), 205–212.

[11] C. Megibben, *ω_1-separable p-groups.* Abelian Group Theory (Oberwolfach, 1985), New York, Gordon and Breach, 1987, 117–136.

[12] A. Mekler and S. Shelah, *ω-elongations and Crawley's problem.* Pac. J. Math. **121** (1986), 121–132.

[13] A. Mekler and S. Shelah, *The solution to Crawley's problem.* Pac. J. Math. **121** (1986), 133–134.

[14] R. Nunke, *Uniquely elongating modules.* Symposia Math. **13** (1974), 315–330.

[15] R. Warfield, *The uniqueness of elongations of abelian groups.* Pac. J. Math. **52** (1974), 289–304.

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