A CHARACTERIZATION OF $L_4(3)$ BY NSE

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Let $\omega(G)$ be the set of element orders of G. For $k \in \omega(G)$ let s_k be the number of elements of order k in G. Let $\mathrm{nse}(G) = \{s_k | k \in \omega(G)\}$. The group $L_4(2) \cong A_8$ is uniquely determined by nse. In this paper, we prove that if G is a group such that nse(G)=nse($L_4(3)$), then $G \cong L_4(3)$.

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1. INTRODUCTION

A finite group G is called a *simple K_n-group*, if G is a simple group with $|\pi(G)| = n$. Here $\pi(G)$ denotes the set of prime divisors of $|G|$.

In 1987, J.G. Thompson posed a very interesting problem related to algebraic number fields as follows (see [18]).

Thompson's Problem. Let $T(G) = \{(n, s_n) | n \in \omega(G) \text{ and } s_n \in \mathrm{nse}(G)\}$, where s_n is the number of elements with order n. Suppose that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

It is easy to see that if G and H are of the same order type, then

$$\mathrm{nse}(G) = \mathrm{nse}(H), \qquad |G| = |H|.$$

Some groups may be characterized by their order and their nse. In this respect we have the following result.

THEOREM 1.1. *Let G be a group and H be one of the following groups. Then $|G| = |H|$ and nse(G)=nse(H) if and only if $G \cong H$.*

(1) M is a simple K_i-group, where $i = 3, 4$ (see [15] and [14] respectively).

(2) A_{12} [11]; A_{13} [5].

(3) Sporadic simple groups [2].

(4) $L_2(2^m)$ where $2^m + 1$ is a prime or $2^m - 1$ is a prime (see [13]).

Not all groups can be determined by nse(G) and $|G|$. Let A, B be two finite groups, let $G := A \rtimes B$ denote the semidirect of A, B with $A \lhd G$. For instance J.G. Thompson in 1987 gave the following example. Let

$$G_1 = C_2 \times C_2 \times C_2 \times C_2 \rtimes A_7, \quad G_2 = L_3(4) \rtimes C_2,$$

where both G_1 and G_2 are maximal subgroups of M_{23}. Then nse(G_1)=nse(G_2)={1, 435, 2240, 6300, 8064, 6720, 5040, 5760}, but $G_1 \ncong G_2$.

In connection with Thompson's problem, one may ask whether finite simple groups can be characterized by their nse. In this respect, we have the following result.

THEOREM 1.2. *Let G be a group and H be one of the following groups. Then nse(G)=nse(H) if and only if $G \cong H$.*

(1) $L_2(5) \cong A_5$ [16]; $L_3(4), L_3(5)$ [8, 9]; and $L_5(2)$ [10].

(2) A_7, A_8 (see [1]).

In this paper, it is shown that $L_4(3)$ also can be characterized by nse.

We will introduce some notations that will be used in this paper. For a prime number r we will denote the number of the Sylow r-subgroups P_r of a finite group G by $n_r(G)$ or by n_r. $L_n(q)$ denotes the projective special linear group of degree n over finite fields of order q. $U_n(q)$ denotes the projective special unitary group of degree n over finite fields of order q. The other notations are standard (see [3], for instance).

2. SOME LEMMAS

LEMMA 2.1 ([4]). *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \big| g^m = 1\}$, then $m \big| |L_m(G)|$.*

LEMMA 2.2 ([16]). *Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2-1)$.*

LEMMA 2.3 ([12]). *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

To prove that $L_4(3)$ can be determined by nse, we will also need the structure of simple K_4-groups.

LEMMA 2.4 ([17]). *Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*

(1) A_7, A_8, A_9 or A_{10}.

(2) M_{11}, M_{12} or J_2.

(3) One of the following:

(a) $L_2(r)$, *where* r *is a prime and* $r^2-1=2^a\cdot 3^b\cdot v^c$ *with* $a\geq 1$, $b\geq 1$, $c\geq 1$, *and* v *is a prime greater than 3.*

(b) $L_2(2^m)$, *where* $2^m-1=u$, $2^m+1=3t^b$ *with* $m\geq 2$, u, t *are primes,* $t>3$, $b\geq 1$.

(c) $L_2(3^m)$, *where* $3^m+1=4t$, $3^m-1=2u^c$ *or* $3^m+1=4t^b$, $3^m-1=2u$, *with* $m\geq 2$, u, t *are odd primes,* $b\geq 1$, $c\geq 1$.

(4) One of the following 28 simple groups: $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^2D_4(2)$ *or* ${}^2F_4(2)'$.

LEMMA 2.5 ([6], Theorem 9.3.1). *Let* G *be a finite solvable group and* $|G|=mn$, *where* $m=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, $(m,n)=1$. *Let* $\pi=\{p_1,\cdots,p_r\}$ *and* h_m *be the number of Hall* π*-subgroups of* G. *Then* $h_m=q_1^{\beta_1}\cdots q_s^{\beta_s}$ *satisfies the following conditions for all* $i\in\{1,2,\cdots,s\}$:

(1) $q_i^{\beta_i}\equiv 1$ *(mod* p_j*) for some* p_j.

(2) The order of some chief factor of G *is divided by* $q_i^{\beta_i}$.

The following lemma is a key factor in the proof of our main theorem.

LEMMA 2.6. *Let* G *be a simple* K_4*-group with* $5\mid |G|\mid 2^7\cdot 3^6\cdot 5\cdot 13$. *Then* $G\cong L_4(3)$.

Proof. Order consideration rules out the cases (1)(2) of Lemma 2.4.

So we consider Lemma 2.4(3). We distinguish the following cases.

- Case 1. $G\cong L_2(r)$, where $r\in\{3,5,13\}$.
  - For $r=3$, we have $|\pi(r^2-1)|=1$, which contradicts the fact that $|\pi(r^2-1)|=3$.
  - For $r=5$, we have $|\pi(r^2-1)|=2$, which contradicts again the fact that $|\pi(r^2-1)|=3$.
  - We can not have $r=13$, since $7\mid |L_2(13)|$, while $7\nmid |G|$.
- Case 2. $G\cong L_2(2^m)$, where $u\in\{3,5,13\}$.
  - We can not have $u=3$, since this would imply $m=2$, which further gives $5=3t^b$, an equation that has no solution in $\mathbf{N}$.
  - For $u=5$, we obtain $2^m-1=5$, which has no solution in $\mathbb{N}$.
  - For $u=13$, the equation $2^m-1=13$ has no solution in $\mathbb{N}$.
- Case 3. $G\cong L_2(3^m)$

  We will consider the following two sub-cases.
  - Subcase 3.1. $3^m+1=4t$ and $3^m-1=2u^c$.

    We can suppose that $t\in\{3,5,13\}$.

    For $t=3,5,13$, the equation $3^m+1=4t$ has no solution in $\mathbb{N}$.

* Subcase 3.2. $3^m + 1 = 4t^b$ and $3^m - 1 = 2u$.
  We can suppose that $u \in \{3, 5, 13\}$.
  For $u = 3, 5, 13$, the equation $3^m - 1 = 2u$ has no solution in $\mathbb{N}$.

In review of Lemma 2.4(4), we see that $G \cong L_4(3)$.
This completes the proof of the Lemma. □

3. MAIN THEOREM AND ITS PROOF

In this section, we will give the proof of the main theorem.

Let G be a group such that nse(G)=nse($L_4(3)$), and s_n be the number of elements of order n. By Lemma 2.2 we have that G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 2.1 and the above discussion, we have

$$(1) \qquad \begin{cases} \phi(m) \mid s_m \\ m \mid \sum_{d \mid m} s_d \end{cases}$$

Theorem 3.1. *Let G be a group. Then $G \cong L_4(3)$ if and only if nse(G) = nse($L_4(3)$) = {1, 7371, 82160, 256932, 303264, 449280, 589680, 606528, 758160, 842400, 1866240}.*

Proof. If $G \cong L_4(3)$, then from [3], nse(G)=nse($L_4(3)$).

So we assume that nse(G)=nse($L_4(3)$).

By (1), $\{2, 3, 5, 7, 13, 17, 53, 589681, 758161\} \subseteq \pi(G)$ and $s_2 = 7371$, $2 \in \pi(G)$ since 7371 is the only odd number greater than 1 that appears in $nse(G)$. If $17 \in \pi(G)$, $s_{17} \in \{82160, 842400\}$. If $2 \cdot 17 \in \omega(G)$, since $\phi(2 \cdot 17) = \phi(17)$, we deduce that $s_{17} = s_{2 \cdot 17}$. But by Lemma 2.1, $2 \cdot 17 \mid 1 + s_2 + s_{17} + s_{2 \cdot 17}$(=171692, 1692172), a contradiction. Therefore $2 \cdot 17 \notin \omega(G)$. It follows that the Sylow 17-subgroup of G acts fixed point freely on the set of elements of order 2 and $|P_{17}| \mid s_2$, a contradiction. So $17 \notin \pi(G)$. Similarly we can rule out the primes 53, 589681, 758161, which can not belong to $\pi(G)$.

Therefore $\{2, 3, 5, 7, 13\} \subseteq \pi(G)$. If $3, 5, 7, 13 \in \pi(G)$, then by (1), $s_3 = 82160$, $s_5 = 303264$, $s_7 \in \{449280, 606528, 842400\}$ and $s_{13} = 1866240$.

If $2^a \in \omega(G)$, then $0 \le a \le 10$. By Lemma 2.1, $|P_2| \mid 1 + s_2 + ... + s_{2^{10}}$ and so $|P_2| \mid 2^{10}$.

If $3^a \in \omega(G)$, then $0 \le a \le 7$. By Lemma 2.1, $|P_3| \mid 1 + s_3 + ... + s_{3^i}$ with $i = 3, 4, 5, 6$, and $|P_3| \mid 3^8$ (in this case, $s_3 = 82160$, $s_{3^2} = 256932$, $s_{3^3} = 449280$, $s_{3^4} = 303264$, $s_{3^5} = 842400$, and $s_{3^6} = 303264$). We also see that $3^7 \notin \omega(G)$.

If $2^2 \cdot 3 \in \omega(G)$, then by Lemma 2.3 of [13], $s_{2^2 \cdot 3} = 2 \cdot s_{2^2} \cdot t$ for some integer t and $s_{2^2 \cdot 3} = 606528$ (when $s_{2^2} = 303264$). Similarly, if $2^3 \cdot 3 \in \omega(G)$, then $s_{2^3\cdot 3} = 606528$ (when $s_{2^3} = 303264$). But by (1), we have a contradiction. Therefore $2^3 \cdot 3 \notin \omega(G)$.

If $3^2 \in \omega(G)$, then by (1), $s_{3^2} \in \{256932, 303264, 449280, 589680, 606528, 758160, 842400, 1866240\}$.

Let $2 \cdot 3^2 \in \omega(G)$.

- If $s_{3^2} \in \{256932,\ 449280,\ 589680,\ 606528,\ 758160,\ 842400,\ 1866240\}$, then $s_{2.3^2} = s_{3^2}$.
- If s_{3^2} =303264, then then $s_{2.3^2} \in \{303264, 606528\}$.

By (1), we then obtain a contradiction, so $2 \cdot 3^2 \notin \omega(G)$.

If $5^a \in \omega(G)$, then $0 \le a \le 3$. If $a = 3$, then by Lemma 2.1, $5^3 \mid 1 + s_5 + s_{5^2} + s_{5^3}$, and $s_{5^3} \notin \omega(G)$. Therefore $|P_5| \mid 1 + s_5 + s_{5^2}$ and $|P_5| \mid 5^2$.

If $2 \cdot 5 \in \omega(G)$, then by (1), $s_{2\cdot 5} = 303264$.

If $2\cdot 5^2 \in \omega(G)$, then by Lemma 2.3 of [13], $s_{2\cdot 5^2} = s_{5^2} \cdot t$ for some integer t and $s_{2\cdot 5^2} = s_{5^2}$. But by Lemma 2.1, $2\cdot 5^2 \mid 1+s_2+s_5+s_{5^2}+s_{2\cdot 5}+s_{2\cdot 5^2}$(=778220, 2130220), a contradiction. So $2 \cdot 5^2 \notin \omega(G)$.

If $2^3 \cdot 5 \in \omega(G)$, then $s_{2^3 .5} = 4 \cdot s_{2^3} \cdot t$ for some integer t but the equation has no solution since $s_{2^3 .5} \in$nse(G). Therefore $2^3 \cdot 5 \notin \omega(G)$.

If $3 \cdot 5 \in \omega(G)$, then $s_{3\cdot 5} = 2 \cdot s_5 \cdot t$ for some integer t and $s_{3\cdot 5} = 606528$. But by Lemma 2.1, $3 \cdot 5 \mid 1 + s_3 + s_5 + s_{3\cdot 5}$(=991953), a contradiction. Hence $3 \cdot 5 \notin \omega(G)$.

If $7^a \in \omega(G)$, then $0 \le a \le 2$. If $\exp(P_7) = 7^2$, then by Lemma 2.1, $7^2 \mid 1 + s_7 + s_{7^2}$. But the equation has no solution in $\mathbb{N}$ since $s_{7^2} \in$nse(G). So $\exp(P_7) = 7$.

- Let s_7=449280. Then by Lemma 2.1, $|P_7| \mid 1 + s_7$(=449281) and so $|P_7| \mid 7^2$.
- Let s_7=606528. Then by Lemma 2.1, $|P_7| \mid 1 + s_7$(=606529) and so $|P_7| \mid 7$.
- Let s_7=842400. Then by Lemma 2.1, $|P_7| \mid 1 + s_7$(=842401) and so $|P_7| \mid 7$.

If $2 \cdot 7 \in \omega(G)$, then by Lemma 1, $s_{2\cdot 7} = s_7 \cdot t$ for some integer t and so $s_{2\cdot 7} = s_7$. But by Lemma 1, $2 \cdot 7 \mid 1 + s_2 + s_7 + s_{2\cdot 7}$ ($\in$ {905932, 1220428, 1692172}), a contradiction. Similarly $3 \cdot 7, 5 \cdot 7 \notin \omega(G)$.

If $13^a \in \omega(G)$, then $0 \le a \le 2$. If $a = 1$, then $|P_{13}| \mid 1 + s_{13}$ and so $|P_{13}| \mid 13$. If $13^2 \in \omega(G)$, then $s_{13^2} = 758160$ and $|P_{13}| \mid 13^2$.

If $2 \cdot 13 \in \omega(G)$, then $s_{2\cdot 13} = s_{13} \cdot t$ for some integer t and $s_{2\cdot 13} = s_{13}$. By Lemma 2.1, $2 \cdot 13 \mid 1 + s_2 + s_{13} + s_{2\cdot 13}$(=3814641), a contradiction. Hence $2 \cdot 13 \notin \omega(G)$. Similarly $3 \cdot 13, 5 \cdot 13, 7 \cdot 13 \notin \omega(G)$.

To remove the prime 7, we must show that $13 \in \pi(G)$.

Assume that $13 \notin \pi(G)$. If $3, 5, 7 \notin \pi(G)$, then G is a 2-group and so

$6065280+82160k_1+256932k_2+303264k_3+449280k_4+589680k_5+606528k_6+758160k_7+842400k_8+1866240k_9=2^a$, where k_1, k_2, ..., k_9 and a are non-negative integers. Since $\omega(G)\subseteq\{1,2,2^2,...,2^{10}\}$ and $|\mathrm{nse}(G)|=11$, then the equation has no solution in $\mathbb{N}$.

Let us assume that $7\in\pi(G)$.

- Let s_7=449280.
  - Let $|P_7|=7$. Then $n_7=s_7/\phi(7)=2^7\cdot3^2\cdot5\cdot13$ and $13\in\pi(G)$, a contradiction.
  - Let $|P_7|=49$. Then $6065280+82160k_1+256932k_2+303264k_3+449280k_4+589680k_5+606528k_6+758160k_7+842400k_8+1866240k_9=2^a\cdot3^b\cdot5^c\cdot7^2$, where k_1, k_2, ..., k_9, a,b,c are non-negative integers and $0\leq\sum\limits_{i=1}^{9}k_i\leq13$. But the equation has no solution in $\mathbb{N}$.
- Let s_7=606528. Then $n_7=s_7/\phi(7)=2^5\cdot3^5\cdot13$ and $13\in\pi(G)$, a contradiction.
- Let s_7=842400. Then $n_7=s_7/\phi(7)=2^4\cdot3^3\cdot5^2\cdot13$ and $13\in\pi(G)$, again a contradiction.

Let us assume that $5\in\pi(G)$. We know that $\exp(P_5)=5,5^2$.

- Let $\exp(P_5)=5$. Then by Lemma 2.1, $|P_5|\mid1+s_5$ and so $|P_5|=5$. Since $n_5=s_5/\phi(5)=2^3\cdot3^6\cdot13$, we see that $13\in\pi(G)$, which is a contradiction.
- Let $\exp(P_5)=5^2$. Then by Lemma 2.1, $|P_5|\mid1+s_5+s_{5^2}$ for $s_{5^2}\in\{82160,758160\}$, and so $|P_5|=5^2$.
  - If s_{5^2}=82160 and $n_5=s_{5^2}/\phi(5^2)=2^2\cdot13\cdot79$, then $13\in\pi(G)$, a contradiction.
  - If s_{5^2}=758160 and $n_5=s_{5^2}/\phi(5^2)=2^2\cdot3^6\cdot13$, then $13\in\pi(G)$, also a contradiction.

Let us assume that $3\in\pi(G)$. We know that $\exp(P_3)=3,3^2,...,3^6$.

- Let $\exp(P_3)=3$. Then by Lemma 2.1, $|P_3|\mid1+s_3$ and $|P_3|\mid3^3$. If $|P_3|=3$, then $n_3=s_3/\phi(2)=2^3\cdot5\cdot13\cdot79$ and so $13\in\pi(G)$, a contradiction. If $|P_3|=3^2$, then $6065280+82160k_1+256932k_2+303264k_3+449280k_4+589680k_5+606528k_6+758160k_7+842400k_8+1866240k_9=2^a\cdot3^2$, where k_1, k_2, ..., k_9, a are non-negative integers and $0\leq\sum\limits_{i=0}^{9}k_i\leq2$. Since $6065280\leq|G|=2^a\cdot9\leq6065280+2.1866240$, then the equation has no solution since a is at most ten. Also we can rule out the case $|P_3|=3^3$ as $|P_3|=3^2$.
- Let $\exp(P_3)=3^2$. Then by (1), $s_{3^2}\in\{256932, 303264, 449280, 589680, 606528, 758160, 842400, 1866240\}$.

If $|P_3| = 3^2$, then $5, 13 \in \pi(G)$, a contradiction.
If $|P_3| > 3^2$, then $6065280 + 82160k_1 + 256932k_2 + 303264k_3 + 449280k_4 + 589680k_5 + 606528k_6 + 758160k_7 + 842400k_8 + 1866240k_9 = 2^a \cdot 3^b$, where $k_1, k_2, ..., k_9, a$ and $b > 2$ are non-negative integers and $0 \leq \sum_{k=0}^{9} k_i \leq 3$, and the equation has no solution since a is at most ten. Similarly we can rule out this case as the case "$\exp(P_3) = 3$ and $|P_3| = 3^2$".

- The cases when $\exp(P_3) \in \{3^3, 3^4, 3^5, 3^6\}$ may be ruled out in a manner similar to that corresponding to the case $\exp(P_3) = 3$.

Therefore $13 \in \pi(G)$.

Let $7 \in \pi(G)$. If $7 \cdot 13 \in \omega(G)$, then by Lemma 2.1, $7 \cdot 13 \mid 1 + s_7 + s_{13} + s_{7 \cdot 13}$ and so $s_{7 \cdot 13} \notin \text{nse}(G)$. It follows that the Sylow 7-subgroup of G acts fixed point freely on the set of elements of order 13 and $|P_7| \mid s_{13}(=1866240)$, a contradiction. Hence $7 \notin \pi(G)$.

Therefore $\{2, 3, 5, 13\} \subseteq \pi(G)$.

If $13 \in \pi(G)$ and $\exp(P_{13}) = 13$, then $|P_{13}| = 13$ and $n_{13} = s_{13}/\phi(13)$, $3, 5 \in \pi(G)$.

If $13 \in \pi(G)$ and $\exp(P_{13}) = 13^2$, then $|P_{13}| = 13^2$ and $n_{13} = s_{13^2}/\phi(13^2)$, $3, 5 \in \pi(G)$.

So we consider the following cases: $\pi(G) = \{2, 13\}$ and $\pi(G) = \{2, 3, 5, 13\}$.

Case a. $\pi(G) = \{2, 13\}$.

If $\exp(P_{13}) = 13$, then $|P_{13}| = 13$ and $n_{13} = s_{13}/\phi(13)$, $3, 5 \in \pi(G)$, a contradiction.

If $\exp(P_{13}) = 13^2$, then $|P_{13}| = 13^2$ and $n_{13} = s_{13^2}/\phi(13^2)$, $3, 5 \in \pi(G)$, a contradiction.

Case b. $\pi(G) = \{2, 3, 5, 13\}$.

Since $5 \cdot 13 \notin \omega(G)$, it follows that the Sylow 5-subgroup P_5 of G acts fixed point freely on the set of elements of order 13, $|P_5| \mid s_{13}$ and so $|P_5| = 5$. We also have $|P_{13}| = 13$. Similarly we have that $2 \cdot 13, 3 \cdot 13 \notin \omega(G)$ and $|P_2| \mid 2^9$, $|P_3| \mid 3^6$.

Therefore we can assume that $|G| = 2^a \cdot 3^b \cdot 5 \cdot 13$. Since $6065280 = 2^7 \cdot 3^6 \cdot 5 \cdot 13 \leq |G|$, then $(a, b) \in \{(7, 6), (8, 6), (9, 6)\}$. So we have $|G| = 2^7 \cdot 3^6 \cdot 5 \cdot 13$, $|G| = 2^8 \cdot 3^6 \cdot 5 \cdot 13$, or $|G| = 2^9 \cdot 3^6 \cdot 5 \cdot 13$.

In the following we prove that there is no group such that $|G| = 2^8 \cdot 3^6 \cdot 5 \cdot 13$ and $\text{nse}(G) = \text{nse}(L_4(3))$.

We will first prove that G is insoluble. Assume that G is soluble. Since $s_{13} = 1866240$ and $|P_{13}| = 13$, then $n_{13} = 1866240/12 = 2^7 \cdot 3^5 \cdot 5$. By Lemma 2.5, we then obtain $5 \equiv 1 \pmod{13}$, a contradiction. Hence G is insoluble.

Therefore there is a normal series $1 \trianglelefteq K \trianglelefteq L \trianglelefteq G$ such that L/K is a simple K_i-group with $i = 3, 4$.

If L/K is isomorphic to a simple K_3-group, we deduce from [7] that L/K is isomorphic to one of the groups $A_5, A_6, U_4(2)$. Let $L/K \cong A_5$. Then $|G/L| \mid 2^6 \cdot 3^5 \cdot 13$. Let $A/K := C_{G/K}(L/K)$. Then $A/K \cap L/K = 1$. It is easy to see that $(G/K)/(A/K) \lesssim \mathrm{Aut}(L/K) = S_5$ and so $G/A \lesssim S_5$. Since $A/K, L/K \lhd G/K$, $A/K \times L/K \leq G/K$. Therefore $|L/K| \mid |G/A|$ and so $G/K \cong A_5$ or S_5. i.e., $|A| = 2^6 \cdot 3^5 \cdot 13$ or $2^5 \cdot 3^5 \cdot 13$. By Sylow's theorem, $n_{13}(A) \in \{1, 27, 144, 3888\}$. Since $A \lhd G$, we have that $n_{13}(A) = n_{13}(G)$, and so $s_{13}(G) \in \{12, 351, 1872, 50544\}$, which contradicts $s_{13}(G) \in \mathrm{nse}(G)$. Similarly we can rule out the other cases "$L/K \cong A_6$ and $L/K \cong U_4(2)$" as the case "$L/K \cong A_5$".

If L/K is isomorphic to a simple K_4-group, then by Lemma 2.6, $L/K \cong L_4(3)$. So $G/A \cong L_4(3)$, $G/A \cong Z_2.L_4(3)$ or $G/A \cong Z_{2^2}.L_4(3)$.

If $G/A \cong L_4(3)$, then $|A| = 4$ and $A = Z(G)$. It follows that there exists an element of order $4 \cdot 13$, a contradiction (Similarly as the argument above, $s_{4 \cdot 13} = 12 \cdot s_4 \cdot t$ for some integer t, but the equation has no solution since $s_{4 \cdot 13} \in \mathrm{nse}(G)$).

If $G/A \cong Z_2.L_4(3)$, then $|A| = 2$. It follows that A is a normal subgroup generated by a 2-central element and so there is an element of order $2 \cdot 13$, a contradiction since $2 \cdot 13 \notin \omega(G)$.

If $G/A \cong Z_{2^2}.L_4(3)$, then $A = 1$. But $\mathrm{nse}(G) \neq \mathrm{nse}(Z_{2^2}.L_4(3))$.

Similarly, we can rule out the case "$|G| = 2^9 \cdot 3^6 \cdot 5 \cdot 13$ and $\mathrm{nse}(G) = \mathrm{nse}(L_4(3))$".

So we have that $|G| = 2^7 \cdot 3^6 \cdot 5 \cdot 13 = |L_4(3)|$. By assumption, $\mathrm{nse}(G) = \mathrm{nse}(L_4(3))$. Then by [14], $G \cong L_4(3)$.

This completes the proof. □

4. SOME APPLICATIONS

On Thompson's conjecture, if G and H are of the same order type, then $\mathrm{nse}(G) = \mathrm{nse}(H)$ and $|G| = |H|$. It is easy to get the following result.

COROLLARY 4.1. *Let G is a group and $p \geq 5$ is a prime. Then $G \cong L_4(3)$ if and only if $nse(G) = nse(L_4(3))$ and $|G| = |L_4(3)|$.*

Shi gave the following conjecture.

CONJECTURE ([18]). *Let G be a group and H a finite simple group. Then $G \cong H$ if and only if (a) $\omega(G) = \omega(H)$ and (b) $|G| = |H|$.*

Then we have the following corollary.

COROLLARY 4.2. *Let G is a group and $p \geq 5$ is a prime. Then $G \cong L_4(3)$ if and only if $\omega(G) = \omega(L_4(3))$ and $|G| = |L_4(3)|$.*

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