BÔCHER'S THEOREM IN AN INFINITE NETWORK

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In this article, we obtain a representations for positive harmonic functions defined outside a finite set in an infinite network X on the lines of the Martin integral representation. This representation takes different forms depending on whether the network is hyperbolic or parabolic.

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1. INTRODUCTION

The Bôcher's theorem in $\mathbb{R}^n$, $n \geq 2$, is: " Let Ω be an open set in $\mathbb{R}^n$ and $a \in \Omega$. If u is harmonic on $\Omega \setminus \{a\}$ and positive near a, then there is a harmonic function v on Ω and a constant $b \geq 0$ such that $u(x) = v(x) + b|x-a|^{2-n}$, $n > 2$ and $u(x) = v(x) + b \log \dfrac{1}{|x-a|}$, $n = 2$, for all $x \in \Omega \setminus \{a\}$." By taking the Kelvin transformation (see Axler *et al.* [3]) in $\mathbb{R}^n$, we have also representations for positive harmonic functions defined outside a compact set in $\mathbb{R}^n$.

In the discrete potential theory, the well known Laurent decomposition theorem for harmonic functions in annular domains in $\mathbb{R}^n$ can also be proved for the harmonic functions in an infinite network (see [5]). As a consequence of this decomposition in infinite networks, Bôcher's theorem for harmonic functions with a finite number of singular points in an infinite network has been proved. We obtain also a representation for positive harmonic functions defined outside a finite set in an infinite network X on the lines of the Martin integral representation [4]. This representation takes different forms depending on whether the network is hyperbolic or parabolic.

2. PRELIMINARIES

Let X be an infinite network with a countable number of vertices, connected and without self loops. There is a collection of numbers $t(x, y) \geq 0$, called conductance such that $t(x, y) > 0$ if and only if $x \sim y$ (that is x and y

are joined by an edge in which case x and y are called neighbours in X). For any vertex $x \in X$, we write $t(x) = \sum_{y \in X} t(x,y)$. Since X is connected, $t(x) > 0$. Note that we have not placed the restriction $t(x,y) = t(y,x)$ for every pair $x, y \in X$. Also we assume the locally finiteness condition, that is any vertex x has only a finite number of neighbours. For any subset E of X, we write $\overset{\circ}{E} = \{x : x \text{ and all its neighbours are in } E\}$ and $\partial E = E \setminus \overset{\circ}{E}$. $\overset{\circ}{E}$ is referred to as the interior of E and ∂E is referred to as the boundary of E.

If u is a real-valued function on E, the Laplacian of u at any $x \in \overset{\circ}{E}$ is defined as $\Delta u(x) = \sum_{y \sim x} t(x,y)[u(y) - u(x)]$. u is said to be *superharmonic* (respectively *harmonic*, *subharmonic*) on E if $\Delta u(x) \leq 0$ (respectively $\Delta u(x) = 0$, $\Delta u(x) \geq 0$) for every $x \in \overset{\circ}{E}$.

Harnack property for superharmonic functions: Let F be a subset of X, and E be a connected subset of $\overset{0}{F}$. Let a and b be two vertices in E. Then, there exist two constants $\alpha > 0$ and $\beta > 0$ such that for any non-negative superharmonic function u on F, $\alpha u(b) \leq u(a) \leq \beta u(b)$. Since E is connected, there exists a path $\{a = a_0, a_1, ..., a_n = b\}$ connecting a and b in E. Take any non-negative superharmonic function u on F. Then, $t(a)u(a) \geq \sum_{x \sim a} t(a,x)u(x)$. In particular, $t(a)u(a) \geq t(a, a_1)u(a_1)$. Again $t(a_1)u(a_1) \geq \sum_{x \sim a_1} t(a_1, x)u(x)$, so that $t(a_1)u(a_1) \geq t(a_1, a_2)u(a_2)$. Hence,

$$u(a) \geq \frac{t(a, a_1)}{t(a)} \times \frac{t(a_1, a_2)}{t(a_1)} u(a_2).$$

Proceeding further, we arrive at the inequality

$$u(a) \geq \frac{t(a, a_1)}{t(a)} \times \frac{t(a_1, a_2)}{t(a_1)} \times ... \times \frac{t(a_{n-1}, a_n)}{t(a_{n-1})} u(b),$$

which is of the form $u(a) \geq \alpha u(b)$. The other inequality $u(a) \leq \beta u(b)$ is proved similarly.

A positive superharmonic function $u \geq 0$ on E is called a potential if and only if for any subharmonic function v on E such that $v \leq u$, we have $v \leq 0$. If there exists a nonconstant positive superharmonic function on X, then X is said to be a hyperbolic network. If it is not hyperbolic, then X is referred to as a parabolic network. In the classical case $\mathbb{R}^2$ is parabolic and $\mathbb{R}^n$, $n \geq 3$ is hyperbolic.

We say that a superharmonic function u on X has the harmonic support in E if $\Delta u(x) = 0$ for every $x \in X \setminus E$. If E is a finite set and if $\Delta u(x) = 0$ for every $x \in X \setminus E$, then we say that u has finite harmonic support.

3. HARMONIC SINGULARITY AT A VERTEX

The classical Bôcher Theorem gives a representation for positive harmonic functions with point singularity in $\mathbb{R}^n$, $n \geq 2$. In $\mathbb{R}$, interpreting as usual linearity as harmonicity, we have a representation analogous to the Bôcher Theorem: If u is a continuous function in $(-1, 1)$ and harmonic outside 0, then there exist a constant α and a harmonic function v in $(-1, 1)$ such that $u(x) = v(x) + \alpha|x|$ for all x. This latter result has the following counter part in an infinite network. Recall that if X is an infinite network and e is a vertex in X, then there exists a unique function $q_e(x)$ in X such that $\Delta q_e(x) = \delta_e(x)$ for all x in X ($q_e(x)$ is a potential if X is hyperbolic and is a pseudo-potential [1] if X is parabolic).

THEOREM 3.1. *Let E be a subset in an infinite network, $e \in \overset{\circ}{E}$. Let u be a real-valued function on E, harmonic at every vertex in $\overset{\circ}{E} \setminus \{e\}$. Then there exists a unique constant α such that $u(x) = v(x) + \alpha q_e(x)$ for $x \in E$, where v is harmonic in E.*

Proof. Let $\Delta u(e) = \alpha$. Write $v(x) = u(x) - \alpha q_e(x)$. Then for every $x \in \overset{\circ}{E} \setminus \{e\}$, $\Delta v(x) = 0$ and $\Delta v(e) = \alpha - \alpha(1)$. Thus, $\Delta v(x) = 0$ for every $x \in \overset{\circ}{E}$, that is v is harmonic in E. □

Remark:

i) The above representation is not so straight forward if the singular vertex happens to be the vertex at infinity.

ii) The above theorem can also be regarded as a special case of the Laurent Decomposition Theorem in an infinite network ([5, Theorem 3.4]) .

4. POSITIVE SUPERHARMONIC FUNCTIONS DEFINED OUTSIDE A FINITE SET IN A HYPERBOLIC NETWORK

In this section, we obtain a Riesz-Martin representation for nonnegative superharmonic functions in a neighbourhood of the vertex at infinity, that is outside a finite set in a hyperbolic (infinite) network. This is a generalisation of the Bôcher representation of nonnegative harmonic functions defined outside a compact set in a Riemannian manifold.

Let X be a hyperbolic network. Fix a vertex e in X. Let $H_0^+(e)$ denote the set of all non-negative functions h on X such that $h(e) = 0$ and $\Delta h(x) = 0$ for each $x \neq e$. H^+ denotes the set of all non-negative harmonic functions on X. Clearly H^+ is a convex cone that is if $u, v \in H^+$ then $\alpha u + \beta v \in H^+$ where α, β are any non-negative numbers. Let H_1^+ denote the class of positive harmonic functions on X taking the value 1 at the fixed vertex e. Let $u \in H_1^+(e)$. u is said to be minimal if and only if any $v \in H^+$, $0 \leq v \leq u$ is of the form $v = \lambda u$ for some constant λ. Let Λ_1 be the set of minimal points in H_1^+. A function u is said to be extremal if u is of the form $u = \lambda u_1 + (1-\lambda)u_2$ for some $0 \leq \lambda \leq 1$, then $u = u_1 = u_2$. If u is extremal then it is minimal. From the following propositions we can claim that H_1^+ is compact (vertexwise convergence topology).

PROPOSITION 4.1 ([1] V. Anandam). *Suppose $\{h_n\}$ is a sequence in H_1^+ converging to h at each vertex in X. Then $h \in H_1^+$.*

PROPOSITION 4.2 ([1] V. Anandam). *Suppose $\{h_n\}$ is a sequence of harmonic functions in H_1^+. Then there exists a subsequence of $\{h_n\}$ converging to a function h in H_1^+.*

In $\mathbb{R}^2$, *i.e.*, in a Riemmanian manifold, suppose $h(x)$ is a harmonic function outside a disc, then $h(x)$ can be written as $h(x) = \alpha \log(x) + H(x) + b(x)$ outside a disc where $H(x)$ harmonic in $\mathbb{R}^2$ and $b(x)$ is bounded. In particular, if $h(x) \geq 0$, then $h(x) = \alpha \log(x) + b(x)$.

In $\mathbb{R}^3$, suppose $h(x)$ is harmonic outside a ball, then $h(x)$ can be written as $h(x) = H(x) + b(x)$ where $H(x)$ is harmonic in $\mathbb{R}^3$ and b is bounded. In particular, if $h(x) \geq 0$ outside, then $H(x) \geq -b(x) \Rightarrow H(x)$ is constant.

V. Anandam proved that (see [2]), in a hyperbolic Riemmannian manifold, suppose $h(x)$ is harmonic outside a compact set, then there always exists a harmonic function H and a potential p such that $|h - H| \leq p$ outside a compact set. This representation is unique. For, if $|h - H_1| \leq p_1$ is another such representation, then $|H - H_1| \leq p + p_1$ outside a compact set. Since $p + p_1$ is potential and $|H - H_1|$ is subharmonic and also majorized by the potential $p + p_1$ outside a compact set, hence $|H - H_1| \equiv 0$, that is $H = H_1$. Suppose $h \geq 0$, then $h - p \leq H \leq h + p$ outside a compact set $\Rightarrow -p \leq h - p \leq H \Rightarrow -H \leq p$ $\Rightarrow -H \leq 0$ *i.e.*, $H \geq 0$. Then $h(x) = H(x) + q(x)$ outside a compact set such that $|q(x)| = |h - H| \leq$ potential, $h(x) = \int_\xi v(x)\mathrm{d}\mu(x) + q(x)$. Thus, in a hyperbolic Riemannian manifold R, if $h \geq 0$ harmonic outside a compact set then there exists a unique Radon measure μ on R with support ξ such that

$$\left| h(x) - \int_\xi v(x)\mathrm{d}\mu(x) \right| \leq p(x) \text{ where } p \text{ is a potential in } R.$$

The following theorem can be regarded as a generalisation of the Bôcher's theorem for positive harmonic functions outside a compact set in a Riemannian manifold. It is actually a Riesz-Martin representation for positive superharmonic functions defined outside a finite set in a hyperbolic network X.

THEOREM 4.3. *Let X be a hyperbolic network. If $u \geq 0$ is a superharmonic function outside a finite set in X, then $u(x)$ can be written as $u(x) = \int_{\Lambda_1} h(x)\mathrm{d}\mu(h) + \sum_{y\in X} G_y(x)\nu(y)$ where μ is a measure with support in Λ_1, ν is a real-valued function on X and $G_y(x)$ is the Green function in X.*

Proof. Let u be a superharmonic function outside a finite set. Then u can be written as $u = v + p_1 - p_2$ outside a finite set, where p_1 and p_2 are potentials in X with point harmonic support and v is a superharmonic on X ([5]). If $u \geq 0$ outside a finite set then $v + p_1 - p_2 \geq 0$ outside a finite set. Hence, we get $v \geq p_2 - p_1 \geq -p_1$ outside a finite set that is $-v \leq p_1$ outside a finite set. As p_1 is a potential we have $v \geq 0$ on X. Then by the Riesz representation, v can be written as the sum of a potential q and a non-negative harmonic function H, that is $v = q + H$. Hence, $u = H + q + p_1 - p_2$.

We know that, in a hyperbolic network, if f is a potential then there exists a real-valued function $\lambda(x) \geq 0$ in X such that $f(x) = \sum_y \lambda(y)G_y(x)$ for all $x \in X$ (see [1, p. 68]). Therefore $q(x) + p_1(x) - p_2(x) = \sum_y G_y(x)\nu(y)$, where $\nu(y)$ is a real-valued function on X. Also H can be represented as a Martin integral (see [1, p. 62]) $H = \int_{\Lambda_1} h\mathrm{d}\mu(h)$. Hence, $u = \int_{\Lambda_1} h\mathrm{d}\mu(h) + \sum_{y\in X} G_y(x)\nu(x)$. □

5. POSITIVE HARMONIC FUNCTIONS DEFINED OUTSIDE A FINITE SET IN A PARABOLIC NETWORK

Let X be a parabolic network. Let e be a fixed vertex. For each $e_i \sim e$, we shall refer to the subset $[e, e_i] = \{x :$ there exists a path joining x to e which passes though $e_i\}$ as the section determined by e and e_i. $[e, e_i]$ is referred to as an infinite section if there is an infinite number of vertices in it. Otherwise it is a finite section. Since X is locally finite, note that there should be at least one infinite section determined by e and one of its neighbours. If $\omega_i = [e, e_i]$, then each $x \in \omega_i \setminus \{e\}$ is an interior vertex of ω_i. e is the only common vertex for all ω_i's. We assume that e and e_i belong to ω_i. Then $X = \bigcup_1^n \omega_i$.

THEOREM 5.1. *Let X be a parabolic network. If $h \geq 0$ is a harmonic function defined outside a finite set in X, then $h(x)$ can be written as $h(x) = \int_{\Lambda_1} u(x)\mathrm{d}\nu(u) +$ a bounded harmonic function outside a finite set, where ν is a measure on Λ_1.*

Proof. Let $\Im$ be the family of non-negative harmonic functions h on $X\setminus\{e\}$ such that $h(e)=0$. We know that $\Im$ is non-empty. For there should be at least one infinite section $\omega_i=[e,e_i]$ determined by e and there exists a function $h\geq 0$ on ω_i such that $h(e)=0$, $h(x)>0$ if $x\in\omega_i\setminus\{e\}$ and $\Delta h(x)=0$ at every vertex $x\in\omega_i\setminus\{e\}$ [1, Theorem 3.2.2]. Suppose ω is a finite section determined by e. If $h\in\Im$, suppose $h\neq 0$ on ω. If possible, let $\max\limits_{x\in\omega} h(x)=M>0$. Then $h(y)=M$ for $y\in\omega$. As $\Delta h(x)=0$ for each $x\in\omega\setminus\{e\}$, $h(z)=M$ at all neighbours z of y. Since the vertex y is connected to the vertex e, we should have $h(e)=M>0$, which is a contradiction to the fact that $h(e)=0$.

Let $\omega_1,\omega_2,\dots,\omega_k$ be the infinite sections determined by e, corresponding to the neighbours $e_1,e_2,\dots,e_k$ of e. Let $\Im_1$ be the subfamily of $\Im$ such that $u\in\Im_1$ if and only if $u\in\Im$ and $u(e_i)=1$ or 0 for each i, $1\leq i\leq k$, and $u=0$ on each finite section. Clearly $\Im_1$ is compact (by Proposition 4.2). If $u\in\Im_1$, $u\not\equiv 0$ is minimal then $u=0$ except in one infinite section ω_i. Hence, if $v\in\Im_1$, then $v=\sum\limits_{i=1}^{k}\int\limits_{\Lambda_{1i}} u\,\mathrm{d}\mu_i(u)$ where Λ_{1i} denotes the set of all minimal functions in ω_i. Let $\Lambda_1=\bigcup\limits_{1}^{k}\Lambda_{1i}$. Let μ be measure on Λ_1 such that $\mu=\mu_i$ on each Λ_{1i}. Then we can write $v=\int\limits_{\Lambda_1} u\,\mathrm{d}\mu(u)$.

We know that [1, Lemma 3.4.2], if h is a harmonic function defined outside a finite set in X, then there exists a real-valued function h_0 on X such that $\Delta h_0(x)=0$ if $x\neq e$ and (h_0-h) is bounded outside a finite set in X. Hence, if h is a lower bounded harmonic function defined outside a finite set, then there exists h_0 on X such that $h_0=h+$ bounded function outside a finite set. That is h_0 is lower bounded on X. Consequently, by the Minimum Principle in a parabolic network [1, Theorem 3.4.1], $h_0(x)\geq h_0(e)$ for all $x\in X$. Then replacing h_0 by $h_0-h_0(e)$, we can suppose $h_0\in\Im$, so that $h_0(x)=\sum\limits_{i=1}^{k} v_i(x)$ where $v_i(x)=\dfrac{h_0^{(i)}(x)}{h_0^{(i)}(e_i)}h_0^{(i)}(e_i)$, $h_0^{(i)}(x)=\begin{cases} h_0(x), & \text{if } x\in\omega_i\\ 0, & \text{if } x\notin\omega_i.\end{cases}$ Note $v_i\in\Im_1$ so that $v_i(x)=\int\limits_{\Lambda_{1i}} u\,\mathrm{d}\mu_i(u)$. Write $\nu=\sum\limits_{i=1}^{k} h_0^{(i)}(e_i)\mu_i$, then $h_0(x)=\sum\limits_{i=1}^{k} h_0^{i}(x)=\sum\limits_{i=1}^{k}\int\limits_{\Lambda_{1i}} u(x)h_0^{(i)}(e_i)\mathrm{d}\mu_i(u)=\int\limits_{\Lambda_1} ud(x)\mathrm{d}\nu(u)$. Hence, $h(x)=\int\limits_{\Lambda_1} u(x)\mathrm{d}\nu(u)+$ a bounded harmonic function outside a finite set. $\square$

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