BÔCHER’S THEOREM IN AN INFINITE NETWORK

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In this article, we obtain a representation for positive harmonic functions defined outside a finite set in an infinite network $X$ on the lines of the Martin integral representation. This representation takes different forms depending on whether the network is hyperbolic or parabolic.

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1. INTRODUCTION

The Bôcher’s theorem in $\mathbb{R}^n$, $n \geq 2$, is: “Let $\Omega$ be an open set in $\mathbb{R}^n$ and $a \in \Omega$. If $u$ is harmonic on $\Omega \setminus \{a\}$ and positive near $a$, then there is a harmonic function $v$ on $\Omega$ and a constant $b \geq 0$ such that $u(x) = v(x) + b|x-a|^{2-n}$, $n > 2$ and $u(x) = v(x) + b \log \frac{1}{|x-a|}$, $n = 2$, for all $x \in \Omega \setminus \{a\}$.” By taking the Kelvin transformation (see Axler et al. [3]) in $\mathbb{R}^n$, we have also representations for positive harmonic functions defined outside a compact set in $\mathbb{R}^n$.

In the discrete potential theory, the well known Laurent decomposition theorem for harmonic functions in annular domains in $\mathbb{R}^n$ can also be proved for the harmonic functions in an infinite network (see [5]). As a consequence of this decomposition in infinite networks, Bôcher’s theorem for harmonic functions with a finite number of singular points in an infinite network has been proved. We obtain also a representation for positive harmonic functions defined outside a finite set in an infinite network $X$ on the lines of the Martin integral representation [4]. This representation takes different forms depending on whether the network is hyperbolic or parabolic.

2. PRELIMINARIES

Let $X$ be an infinite network with a countable number of vertices, connected and without self loops. There is a collection of numbers $t(x,y) \geq 0$, called conductance such that $t(x,y) > 0$ if and only if $x \sim y$ (that is $x$ and $y$
are joined by an edge in which case $x$ and $y$ are called neighbours in $X$). For any vertex $x \in X$, we write $t(x) = \sum_{y \in X} t(x, y)$. Since $X$ is connected, $t(x) > 0$.

Note that we have not placed the restriction $t(x, y) = t(y, x)$ for every pair $x, y \in X$. Also we assume the locally finiteness condition, that is any vertex $x$ has only a finite number of neighbours. For any subset $E$ of $X$, we write $\partial E = \{x : x \text{ and all its neighbours are in } E\}$ and $\partial E = E \setminus \partial E$. $\partial E$ is referred to as the interior of $E$ and $\partial E$ is referred to as the boundary of $E$.

If $u$ is a real-valued function on $E$, the Laplacian of $u$ at any $x \in \partial E$ is defined as $\Delta u(x) = \sum_{y \sim x} t(x, y)[u(y) - u(x)]$. $u$ is said to be superharmonic (respectively harmonic, subharmonic) on $E$ if $\Delta u(x) \leq 0$ (respectively $\Delta u(x) = 0$, $\Delta u(x) \geq 0$) for every $x \in \partial E$.

Harnack property for superharmonic functions: Let $F$ be a subset of $X$, and $E$ be a connected subset of $\overline{F}$. Let $a$ and $b$ be two vertices in $E$. Then, there exist two constants $\alpha > 0$ and $\beta > 0$ such that for any non-negative superharmonic function $u$ on $F$, $\alpha u(b) \leq u(a) \leq \beta u(b)$. Since $E$ is connected, there exists a path $\{a = a_0, a_1, \ldots, a_n = b\}$ connecting $a$ and $b$ in $E$. Take any non-negative superharmonic function $u$ on $F$. Then, $t(a)u(a) \geq \sum_{x \sim a} t(a, x)u(x)$.

In particular, $t(a)u(a) \geq t(a, a_1)u(a_1)$. Again $t(a_1)u(a_1) \geq \sum_{x \sim a_1} t(a_1, x)u(x)$, so that $t(a_1)u(a_1) \geq t(a_1, a_2)u(a_2)$. Hence,

$$u(a) \geq \frac{t(a, a_1)}{t(a)} \times \frac{t(a_1, a_2)}{t(a_1)} \cdots \frac{t(a_{n-1}, a_n)}{t(a_{n-1})}u(b),$$

which is of the form $u(a) \geq \alpha u(b)$. The other inequality $u(a) \leq \beta u(b)$ is proved similarly.

A positive superharmonic function $u \geq 0$ on $E$ is called a potential if and only if for any subharmonic function $v$ on $E$ such that $v \leq u$, we have $v \leq 0$. If there exists a nonconstant positive superharmonic function on $X$, then $X$ is said to be a hyperbolic network. If it is not hyperbolic, then $X$ is referred to as a parabolic network. In the classical case $\mathbb{R}^2$ is parabolic and $\mathbb{R}^n$, $n \geq 3$ is hyperbolic.

We say that a superharmonic function $u$ on $X$ has the harmonic support in $E$ if $\Delta u(x) = 0$ for every $x \in X \setminus E$. If $E$ is a finite set and if $\Delta u(x) = 0$ for every $x \in X \setminus E$, then we say that $u$ has finite harmonic support.
3. HARMONIC SINGULARITY AT A VERTEX

The classical Bôcher Theorem gives a representation for positive harmonic functions with point singularity in $\mathbb{R}^n$, $n \geq 2$. In $\mathbb{R}$, interpreting as usual linearity as harmonicity, we have a representation analogous to the Bôcher Theorem: If $u$ is a continuous function in $(-1,1)$ and harmonic outside 0, then there exist a constant $\alpha$ and a harmonic function $v$ in $(-1,1)$ such that $u(x) = v(x) + \alpha |x|$ for all $x$. This latter result has the following counterpart in an infinite network. Recall that if $X$ is an infinite network and $e$ is a vertex in $X$, then there exists a unique function $q_e(x)$ in $X$ such that $\Delta q_e(x) = \delta_e(x)$ for all $x$ in $X$ ($q_e(x)$ is a potential if $X$ is hyperbolic and is a pseudo-potential [1] if $X$ is parabolic).

**Theorem 3.1.** Let $E$ be a subset in an infinite network, $e \in \mathring{E}$. Let $u$ be a real-valued function on $E$, harmonic at every vertex in $\mathring{E} \setminus \{e\}$. Then there exists a unique constant $\alpha$ such that $u(x) = v(x) + \alpha q_e(x)$ for $x \in E$, where $v$ is harmonic in $E$.

**Proof.** Let $\Delta u(e) = \alpha$. Write $v(x) = u(x) - \alpha q_e(x)$. Then for every $x \in \mathring{E} \setminus \{e\}$, $\Delta v(x) = 0$ and $\Delta v(e) = \alpha - \alpha(1)$. Thus, $\Delta v(x) = 0$ for every $x \in \mathring{E}$, that is $v$ is harmonic in $E$. \qed

**Remark:**

i) The above representation is not so straightforward if the singular vertex happens to be the vertex at infinity.

ii) The above theorem can also be regarded as a special case of the Laurent Decomposition Theorem in an infinite network ([5, Theorem 3.4]) .

4. POSITIVE SUPERHARMONIC FUNCTIONS DEFINED OUTSIDE A FINITE SET IN A HYPERBOLIC NETWORK

In this section, we obtain a Riesz-Martin representation for nonnegative superharmonic functions in a neighbourhood of the vertex at infinity, that is outside a finite set in a hyperbolic (infinite) network. This is a generalisation of the Bôcher representation of nonnegative harmonic functions defined outside a compact set in a Riemannian manifold.

Let $X$ be a hyperbolic network. Fix a vertex $e$ in $X$. Let $H_0^+(e)$ denote the set of all non-negative functions $h$ on $X$ such that $h(e) = 0$ and $\Delta h(x) = 0$ for each $x \neq e$. $H^+$ denotes the set of all non-negative harmonic functions on $X$. Clearly $H^+$ is a convex cone that is if $u, v \in H^+$ then $\alpha u + \beta v \in H^+$ where $\alpha, \beta$ are any non-negative numbers. Let $H_1^+$ denote the class of positive harmonic
functions on $X$ taking the value 1 at the fixed vertex $e$. Let $u \in H_1^+(e)$. $u$ is said to be minimal if and only if any $v \in H^+$, $0 \leq v \leq u$ is of the form $v = \lambda u$ for some constant $\lambda$. Let $\Lambda_1$ be the set of minimal points in $H_1^+$. A function $u$ is said to be extremal if $u$ is of the form $u = \lambda u_1 + (1 - \lambda)u_2$ for some $0 \leq \lambda \leq 1$, then $u = u_1 = u_2$. If $u$ is extremal then it is minimal. From the following propositions we can claim that $H_1^+$ is compact (vertexwise convergence topology).

**Proposition 4.1 ([1] V. Anandam).** Suppose $\{h_n\}$ is a sequence in $H_1^+$ converging to $h$ at each vertex in $X$. Then $h \in H_1^+$.

**Proposition 4.2 ([1] V. Anandam).** Suppose $\{h_n\}$ is a sequence of harmonic functions in $H_1^+$. Then there exists a subsequence of $\{h_n\}$ converging to a function $h$ in $H_1^+$.

In $\mathbb{R}^2$, i.e., in a Riemannian manifold, suppose $h(x)$ is a harmonic function outside a disc, then $h(x)$ can be written as $h(x) = \alpha \log(x) + H(x) + b(x)$ outside a disc where $H(x)$ harmonic in $\mathbb{R}^2$ and $b(x)$ is bounded. In particular, if $h(x) \geq 0$, then $h(x) = \alpha \log(x) + b(x)$.

In $\mathbb{R}^3$, suppose $h(x)$ is harmonic outside a ball, then $h(x)$ can be written as $h(x) = H(x) + b(x)$ where $H(x)$ is harmonic in $\mathbb{R}^3$ and $b$ is bounded. In particular, if $h(x) \geq 0$ outside, then $H(x) \geq -b(x) \Rightarrow H(x)$ is constant.

V. Anandam proved that (see [2]), in a hyperbolic Riemannian manifold, suppose $h(x)$ is harmonic outside a compact set, then there always exists a harmonic function $H$ and a potential $p$ such that $|h - H| \leq p$ outside a compact set. This representation is unique. For, if $|h - H_1| \leq p_1$ is another such representation, then $|H - H_1| \leq p + p_1$ outside a compact set. Since $p + p_1$ is potential and $|H - H_1|$ is subharmonic and also majorized by the potential $p + p_1$ outside a compact set, hence $|H - H_1| = 0$, that is $H = H_1$. Suppose $h \geq 0$, then $h - p \leq H \leq h + p$ outside a compact set $\Rightarrow -p \leq h - p \leq H \Rightarrow -H \leq p \Rightarrow -H \leq 0$ i.e., $H \geq 0$. Then $h(x) = H(x) + q(x)$ outside a compact set such that $|q(x)| = |h - H| \leq$ potential, $h(x) = \int_\xi v(x) d\mu(x) + q(x)$. Thus, in a hyperbolic Riemannian manifold $R$, if $h \geq 0$ harmonic outside a compact set then there exists a unique Radon measure $\mu$ on $R$ with support $\xi$ such that $|h(x) - \int_\xi v(x) d\mu(x)| \leq p(x)$ where $p$ is a potential in $R$.

The following theorem can be regarded as a generalisation of the Bôcher’s theorem for positive harmonic functions outside a compact set in a Riemannian manifold. It is actually a Riesz-Martin representation for positive superharmonic functions defined outside a finite set in a hyperbolic network $X$. 
THEOREM 4.3. Let $X$ be a hyperbolic network. If $u \geq 0$ is a superharmonic function outside a finite set in $X$, then $u(x)$ can be written as $u(x) = \int_{\Lambda_1} h(x) d\mu(h) + \sum_{y \in X} G_y(x) \nu(y)$ where $\mu$ is a measure with support in $\Lambda_1$, $\nu$ is a real-valued function on $X$ and $G_y(x)$ is the Green function in $X$.

Proof. Let $u$ be a superharmonic function outside a finite set. Then $u$ can be written as $u = v + p_1 - p_2$ outside a finite set, where $p_1$ and $p_2$ are potentials in $X$ with point harmonic support and $v$ is a superharmonic on $X$ ([5]). If $u \geq 0$ outside a finite set then $v + p_1 - p_2 \geq 0$ outside a finite set. Hence, we get $v \geq p_2 - p_1 \geq -p_1$ outside a finite set that is $-v \leq p_1$ outside a finite set. As $p_1$ is a potential we have $v \geq 0$ on $X$. Then by the Riesz representation, $v$ can be written as the sum of a potential $q$ and a non-negative harmonic function $H$, that is $v = q + H$. Hence, $u = H + q + p_1 - p_2$.

We know that, in a hyperbolic network, if $f$ is a potential then there exists a real-valued function $\lambda(x) \geq 0$ in $X$ such that $f(x) = \sum \lambda(y) G_y(x)$ for all $x \in X$ (see [1, p. 68]). Therefore $q(x) + p_1(x) - p_2(x) = \sum_y G_y(x) \nu(y)$, where $\nu(y)$ is a real-valued function on $X$. Also $H$ can be represented as a Martin integral (see [1, p. 62]) $H = \int_{\Lambda_1} h d\mu(h)$. Hence, $u = \int_{\Lambda_1} h d\mu(h) + \sum_{y \in X} G_y(x) \nu(x)$. \hfill \Box

5. POSITIVE HARMONIC FUNCTIONS DEFINED OUTSIDE A FINITE SET IN A PARABOLIC NETWORK

Let $X$ be a parabolic network. Let $e$ be a fixed vertex. For each $e_i \sim e$, we shall refer to the subset $[e, e_i] = \{x : \text{there exists a path joining } x \text{ to } e \text{ which passes though } e_i\}$ as the section determined by $e$ and $e_i$. $[e, e_i]$ is referred to as an infinite section if there is an infinite number of vertices in it. Otherwise it is a finite section. Since $X$ is locally finite, note that there should be at least one infinite section determined by $e$ and one of its neighbours. If $\omega_i = [e, e_i]$, then each $x \in \omega_i \setminus \{e\}$ is an interior vertex of $\omega_i$. $e$ is the only common vertex for all $\omega_i$'s. We assume that $e$ and $e_i$ belong to $\omega_i$. Then $X = \bigcup_{i=1}^n \omega_i$.

THEOREM 5.1. Let $X$ be a parabolic network. If $h \geq 0$ is a harmonic function defined outside a finite set in $X$, then $h(x)$ can be written as $h(x) = \int_{\Lambda_1} u(x) d\nu(u) + \text{a bounded harmonic function outside a finite set}$, where $\nu$ is a measure on $\Lambda_1$. 
Proof. Let $\mathcal{S}$ be the family of non-negative harmonic functions $h$ on $X \setminus \{e\}$ such that $h(e) = 0$. We know that $\mathcal{S}$ is non-empty. For there should be at least one infinite section $\omega_i = [e, e_i]$ determined by $e$ and there exists a function $h \geq 0$ on $\omega_i$ such that $h(e) = 0$, $h(x) > 0$ if $x \in \omega_i \setminus \{e\}$ and $\Delta h(x) = 0$ at every vertex $x \in \omega_i \setminus \{e\}$ [1, Theorem 3.2.2]. Suppose $\omega$ is a finite section determined by $e$. If $h \in \mathcal{S}$, suppose $h \neq 0$ on $\omega$. If possible, let $\max_{x \in \omega} h(x) = M > 0$. Then $h(y) = M$ for $y \in \omega$. As $\Delta h(x) = 0$ for each $x \in \omega \setminus \{e\}$, $h(z) = M$ at all neighbours $z$ of $y$. Since the vertex $y$ is connected to the vertex $e$, we should have $h(e) = M > 0$, which is a contradiction to the fact that $h(e) = 0$.

Let $\omega_1, \omega_2, \ldots, \omega_k$ be the infinite sections determined by $e$, corresponding to the neighbours $e_1, e_2, \ldots, e_k$ of $e$. Let $\mathcal{S}_1$ be the subfamily of $\mathcal{S}$ such that $u \in \mathcal{S}_1$ if and only if $u \in \mathcal{S}$ and $u(e_i) = 1$ or 0 for each $i$, $1 \leq i \leq k$, and $u = 0$ on each finite section. Clearly $\mathcal{S}_1$ is compact (by Proposition 4.2). If $u \in \mathcal{S}_1$, $u \neq 0$ is minimal then $u = 0$ except in one infinite section $\omega_i$. Hence, if $v \in \mathcal{S}_1$, then $v = \sum_{i=1}^k \int_{\Lambda_{1i}} ud\mu_i(u)$ where $\Lambda_{1i}$ denotes the set of all minimal functions in $\omega_i$. Let $\Lambda_1 = \bigcup_{i=1}^k \Lambda_{1i}$. Let $\mu$ be measure on $\Lambda_1$ such that $\mu = \mu_i$ on each $\Lambda_{1i}$. Then we can write $v = \int_{\Lambda_1} ud\mu(u)$.

We know that [1, Lemma 3.4.2], if $h$ is a harmonic function defined outside a finite set in $X$, then there exists a real-valued function $h_0$ on $X$ such that $\Delta h_0(x) = 0$ if $x \neq e$ and $(h_0 - h)$ is bounded outside a finite set in $X$. Hence, if $h$ is a lower bounded harmonic function defined outside a finite set, then there exists $h_0$ on $X$ such that $h_0 = h^+$ bounded function outside a finite set. That is $h_0$ is lower bounded on $X$. Consequently, by the Minimum Principle in a parabolic network [1, Theorem 3.4.1], $h_0(x) \geq h_0(e)$ for all $x \in X$. Then replacing $h_0$ by $h_0 - h_0(e)$, we can suppose $h_0 \in \mathcal{S}$, so that $h_0(x) = \sum_{i=1}^k v_i(x)$ where $v_i(x) = \frac{h_0^{(i)}(x)}{h_0^{(i)}(e_i)} h_0^{(i)}(e_i)$, $h_0^{(i)}(x) = \begin{cases} h_0(x), & \text{if } x \in \omega_i \\ 0, & \text{if } x \not\in \omega_i. \end{cases}$ Note $v_i \in \mathcal{S}_1$ so that $v_i(x) = \int_{\Lambda_{1i}} ud\mu_i(u)$. Write $\nu = \sum_{i=1}^k h_0^{(i)}(e_i) \mu_i$, then $h_0(x) = \sum_{i=1}^k h_0^{i}(x) = \sum_{i=1}^k \int_{\Lambda_{1i}} u(x)h_0^{(i)}(e_i) d\mu_i(u) = \int_{\Lambda_1} u(x) d\nu(u)$. Hence, $h(x) = \int_{\Lambda_1} u(x) d\nu(u) + a$ bounded harmonic function outside a finite set. □
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