CHARACTERIZATION OF SOME $L_2(q)$ BY THE LARGEST ELEMENT ORDERS

QINHUI JIANG and CHANGGUO SHAO*

Communicated by Vasile Brînzănescu

We characterize linear groups $L_2(q)$ by group order and the largest element order, where $q = p^n < 125$. This generalizes some results of [3].

AMS 2010 Subject Classification: 20D60.

Key words: finite groups, group order, the largest element orders.

1. INTRODUCTION

Throughout this paper, all groups are finite and G is always a group. We denote by $\pi(G)$ the set of prime divisors of $|G|$, by $\pi_e(G)$ the set of element orders of G. Besides, $k(G)$ denotes the largest element in $\pi_e(G)$. G is called a simple K_n-group if G is simple with $|\pi(G)| = n$. All further unexplained notation is standard, readers may refer to [2].

It is an interesting topic to characterize simple groups by using the group order and the set $\pi_e(G)$ of element orders. In 1987, Professor W.J. Shi posed the following conjecture:

CONJECTURE. *Let G be a group and M a simple group. Then $G \cong M$ if and only if $|G| = |M|$ and $\pi_e(G) = \pi_e(M)$.*

It is worth mentioning that this conjecture has been proved by A.V. Vasil'ev, M.A. Grechkoseeva, and V.D. Mazurov in [8]. To continue this work, some authors tried to characterize simple groups by using less conditions. For instance, in [4], L.G. He and G.Y. Chen characterized simple K_3-groups by using the group orders and the largest and the second largest element orders. Further, Q.L. Zhang and W.J. Shi ([9]) characterized all simple K_3-groups and some linear groups $L_2(p)$ by using the group order and the largest element order, where p is a prime with $p = 8n \pm 3 > 3$. Recently, in [6] we gave a new characterization of simple linear groups $L_2(q)$ by both the group order and the largest element order, where either q is a prime or $q = 2^a$ such that $2^a + 1$ or $2^a - 1$ is a prime.

*Corresponding author.

Note that not all simple linear groups $L_2(q)$ with $q < 125$ a prime power order can be characterized by its group order and its largest element order. For instance, in [9, Theorem 3.3], Q.L. Zhang and W.J. Shi claimed that if the group G is satisfying $|G| = |L_2(7)|$ and $k(G) = k(L_2(7))$, then G is either isomorphic to $L_2(7)$ or a 2-Frobenius group. Further, in [6, Theorem B], $L_2(p)$ can not be determined by the group order and the largest element order if p is a prime such that $p+1$ is a power of 2. The authors proved that if the group G is satisfying $|G| = |L_2(p)|$ and $k(G) = k(L_2(p))$, then G is either isomorphic to $L_2(p)$ or a 2-Frobenius group.

Recall that L.G. He and G.Y. Chen showed in [3] that linear simple groups $L_2(q)$ with $q = p^n < 125$ may be characterized by group order, the largest, the second largest and the third largest element orders. In this paper, we generalize their result and prove that simple linear groups $L_2(q)$ can be determined exactly by their group order and their largest element order, where $q < 125$ is a prime power with $q \neq 7, 31$. Our main result is:

Theorem A. *Let G be a group. Then $G \cong L_2(q)$ if and only if $|G| = |L_2(q)|$ and $k(G) = k(L_2(q))$, where $q \neq 7$, 31 and $q < 125$ is a prime power.*

Recall that L.G. He and G.Y. Chen proved that $L_2(q)$ can be determined by its group order and its largest element order if $q < 125$ and $q \neq 7, 31, 49$ and 64. Hence, our task is to prove that:

Theorem B. *Let G be a group. Then $G \cong L_2(49)$ if and only if $|G| = |L_2(49)|$ and $k(G) = k(L_2(49))$.*

Theorem C. *Let G be a group. Then $G \cong L_2(64)$ if and only if $|G| = |L_2(64)|$ and $k(G) = k(L_2(64))$.*

We remark here that one may use the method employed in this paper, to simplify the proofs of the main results in [3].

2. PRELIMINARIES

Before proceeding with the proof of our results, we first give some useful results.

Lemma 2.1 ([5, Theorem 2]). *Let G be a simple K_3-group. Then G is isomorphic to one of the following groups: $A_5, A_6, L_2(7), L_2(8), L_2(17), L_3(3), U_3(3)$ or $U_4(2)$.*

Lemma 2.2. *Let G be a simple K_4-group. If $|G| \mid 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$, then $G \cong L_2(49)$; if $|G| \mid 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ and $13 \mid |G|$, then $G \cong Sz(8)$ or $L_2(13)$.*

Proof. It follows immediately by [7, Theorem 2]. □

LEMMA 2.3. *Let G be a simple K_5-group. If $|G| \mid 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, then $G \cong L_2(64)$.*

Proof. It is clear from [1, Theorem]. □

3. PROOF OF THEOREM B

Proof. It is obvious that the necessity holds. We only prove the sufficiency. Suppose that $|G| = |L_2(49)| = 2^4 \cdot 3 \cdot 5^2 \cdot 7^2$ and $k(G) = k(L_2(49)) = 25$. Let N be a minimal normal subgroup of G.

Assume first that $7 \mid |N|$. We show that $5 \mid |N|$. Otherwise, we consider the action of P_5 on N by conjugation, where P_5 is a Sylow 5-subgroup of G. As the action is coprime, there is some P_5-invariant Sylow 7-subgroup N_7 of N, yielding that $N_7P_5 \leq G$. Note that $|P_5| = 5^2$ and $|N_7| \mid 7^2$. It follows by Sylow's Theorem that $N_7P_5 = N_7 \times P_5$, indicating that $35 \in \pi_e(G)$, contrary to the fact that $k(G) = 25$. Hence, $5 \mid |N|$. Further, by the minimality of N, we see that N is unsolvable with $\{2, 5, 7\} \subseteq \pi(N)$. If $|\pi(N)| = 3$, then N is either a simple K_3-group or a direct product of two isomorphic simple K_3-groups by considering its order. However, according to Lemma 2.1, there is no such group whose order is divisible by 35. Hence, $3 \in \pi(N)$. Moreover, $3 \parallel |N|$ implies that N is a simple K_4-group. Hence, it follows by Lemma 2.2 that $N \cong L_2(49)$. Consequently, $G = N \cong L_2(49)$, as required.

Now we consider the case $7 \nmid |N|$. We will work to get a contradiction. Let $P_7 \in \mathrm{Syl}_7(G)$. Then the action of P_7 on N by conjugation is coprime. We prove that $5 \nmid |N|$. Otherwise, there is a P_7-invariant Sylow 5-subgroup N_5 of N. Hence, $N_5P_7 \leq G$. Moreover, by Sylow's Theorem, we obtain that $N_5P_7 = N_5 \times P_7$ as $|N_5| \mid 5^2$ and $|P_7| = 7^2$, leading to $35 \in \pi_e(G)$, a contradiction. As a result, $5 \nmid |N|$, yielding that $\pi(N) \subseteq \{2, 3\}$. Furthermore, N is either a 2-group or 3-group since it is a minimal normal subgroup of G.

If G/N is solvable, then G/N has a Hall $\{5, 7\}$-subgroup H/N. By Sylow's Theorem, we have that H/N is nilpotent, which shows that $175 \in \pi_e(G)$. This contradiction shows that G/N is unsolvable. In particular, $4 \mid |G/N|$. If N is a 3-group, then $|N| = 3$. In this case, $G/C_G(N) \leq C_2$. Hence, $5^2 \mid |C_G(N)|$. Since $k(G) = 25$, we obtain that G has a cyclic Sylow 5-subgroup of order 25, yielding that $75 \in \pi_e(G)$, a contradiction. Thus, N is a 2-group. Moreover, $|N| \mid 4$ since G/N is unsolvable. Hence, either $|N| = 2$ or $|N| = 4$. If the former holds, then $G/C_G(N) \leq \mathrm{Aut}\ (N)$, yielding that $N \leq Z(G)$. As $25 \in \pi_e(G)$, we obtain that $50 \in \pi_e(G)$, a contradiction to the fact $k(G) = 25$. Thus, $|N| = 4$. Arguing as above, we derive a contradiction. This completes the proof. □

4. PROOF OF THEOREM C

Proof. Assume that $|G|=|L_2(64)|=2^6\cdot 3^2\cdot 5\cdot 7\cdot 13$ and $k(G)=k(L_2(64))=65$. Let N be a minimal normal subgroup of G. Write $D:=C_G(N)$.

Assume that $13 \mid |N|$. Then we show that $7 \mid |N|$. Otherwise, let P_7 be a Sylow 7-subgroup of G. It is clear that the action of P_7 on N by conjugation is coprime. Then there exists a P_7-invariant Sylow 13-subgroup N_{13} of N. Hence, $N_{13}P_7 \leq G$. Moreover, by Sylow's Theorem, we obtain that $N_{13}P_7 = N_{13}\times P_7$ as $|N_{13}|=13$ and $|P_7|=7$, leading to the fact that $91 \in \pi_e(G)$. This contradiction implies that $13 \mid |N|$ and $7 \mid |N|$, leading to the fact that N is an unsolvable simple group.

Moreover, Lemma 2.1 gives that N is not a simple K_3-group since $7\cdot 13 \mid |N|$. Hence, $4 \leq |\pi(N)| \leq 5$. If $|\pi(N)|=4$, then $N \cong Sz(8)$ or $L_2(13)$ according to Lemma 2.2. However, $N \ncong Sz(8)$ since, otherwise, we see that $65 \in \pi_e(Sz(8))$, a contradiction to [2]. Hence, $N \cong L_2(13)$.

Since $G/D \leq \mathrm{Aut}(N)$, we see that $|G|/—\mathrm{Aut}(N)|$ divides $|D|$, that is $2^3\cdot 3\cdot 5 \mid |D|$. On the other hand, $N\cap D=1$ and $ND \leq G$. This shows that $|D|$ divides $|G/N|$. Thus, $|D| \mid 2^4\cdot 3\cdot 5$. As a consequence, $|D|=2^3\cdot 3\cdot 5$ or $|D|=2^4\cdot 3\cdot 5$. If D is solvable, then D has a Hall $\{3,5\}$-subgroup T. By Sylow's Theorem, we have that T is nilpotent and thus $15 \in \pi_e(D)$. Therefore, $15\cdot 13 \in \pi_e(G)$, a contradiction to the fact that $k(G)=65$.

Hence, D is unsolvable. According to Lemma 2.1, by considering its order, it is clear that D is not a simple K_3-group. Hence, there exists some minimal normal subgroup D_1 of G which is contained in D. If D_1 is a non-abelian simple group, then $D_1 \cong A_5$ by Lemma 2.1. Note that $D/C_D(D_1) \leq S_5$ and $|D| = 2^3\cdot 3\cdot 5$ or $|D|=2^4\cdot 3\cdot 5$. We obtain that either $C_D(D_1)\cong C_2$ by the simplicity of D_1. Thus, $10 \in \pi_e(G)$ and $10\cdot 13 \in \pi_e(G)$, a contradiction. This shows that D_1 is an elementary abelian p-group with $p \in \{2,3,5\}$. Furthermore, $|D_1|=2$ or 4 as D/D_1 is unsolvable. If $|D_1|=2$, then $D_1 \leq Z(G)$, yielding that $130 \in \pi_e(G)$ since $65 \in \pi_e(G)$, a contradiction. If $|D_1|=4$, then $G/C_G(D_1) \leq \mathrm{Aut}(D_1)$. Note that $—\mathrm{Aut}(D_1)| \mid 6$, we have that $65 \mid |C_G(D_1)|$. Hence, $130 \in \pi_e(G)$, again a contradiction. Therefore, N is a simple K_5-group. By Lemma 2.3, we have that $N \cong L_2(64)$. Consequently, $G=N\cong L_2(64)$, as needed.

Now we assume that $13 \nmid |N|$. If we apply an argument similar to that in the second paragraph, we will obtain that $7 \nmid |N|$. Hence, $\pi(N) \subseteq \{2,3,5\}$. If N is unsolvable, then in view of Lemma 2.3 we deduce that $N \cong A_5$ or $N \cong A_6$.

Suppose first that $N \cong A_5$. Then it is clear that $|G/N| = 2^4\cdot 3\cdot 7\cdot 13$. Suppose that G/N is solvable, then G/N has a Hall $\{7,13\}$-subgroup F. By Sylow's Theorem, F is nilpotent, which leads to that $91 \in \pi_e(G)$, a contradiction. As a result, G/N is unsolvable.

Notice that $G/D \leq S_5$. It follows that $2^3 \cdot 3 \cdot 7 \cdot 13 \mid |D|$. On the other hand, $D \cap N = 1$ and $DN = D \times N \leq G$, which implies that $|D| \mid 2^4 \cdot 3 \cdot 7 \cdot 13$. As a consequence, $|D| = 2^3 \cdot 3 \cdot 7 \cdot 13$ or $2^4 \cdot 3 \cdot 7 \cdot 13$. It follows easily that D is unsolvable, or otherwise, we would obtain $91 \in \pi_e(G)$, a contradiction.

Let D_1 be a minimal normal subgroup of G, which is contained in D. Then D_1 is either a simple group or an elementary abelian p-group with $p \in \{2, 3, 7, 13\}$. If the former holds, then $D_1 \cong L_2(7)$ or $L_2(13)$ by [2]. Assume first that $D_1 \cong L_2(13)$. Then $D/C_D(D_1) \leq \mathrm{Aut}(D_1)$. As $|D| = 2^3 \cdot 3 \cdot 7 \cdot 13$ or $2^4 \cdot 3 \cdot 7 \cdot 13$, and —Aut$(D_1)$|=—Aut$(L_2(13))| = 2^3 \cdot 3 \cdot 7 \cdot 13$, we obtain that either $C_D(D_1) = 1$ or $C_D(D_1) \cong C_2$. If the latter holds, then $26 \in \pi_e(D)$ and thus $26 \cdot 5 \in \pi_e(G)$. This contradiction shows that $D = \mathrm{Aut}(L_2(13))$ which shows that $14 \in \pi_e(D)$ and thus $14 \cdot 5 \in \pi_e(G)$, also contrary to the fact that $k(G) = 65$. Hence, $D_1 \cong L_2(7)$. Moreover, $D/C_D(D_1) \leq \mathrm{Aut}(D_1)$ implies that $13 \mid |C_D(D_1)|$ and thus $13 \cdot 7 \in \pi_e(G)$, again a contradiction. Consequently, D_1 is an elementary abelian p-group with $p \in \{2, 3, 7, 13\}$. If D_1 is a 7-group, then we consider the action of P_{13} on D_1 by conjugation, where P_{13} is a Sylow 13-subgroup of G. By Sylow's Theorem, we obtain that $91 \in \pi_e(G)$, a contradiction. Similarly, if D_1 is a 13-group, then we consider the action of P_7 on D_1 by conjugation, where P_7 is a Sylow 7-subgroup of G. We also obtain that $91 \in \pi_e(G)$, the same contradiction. Hence, D_1 is either a 2-group or a 3-group. Suppose that D_1 is a 3-group. As $|D| = 2^3 \cdot 3 \cdot 7 \cdot 13$ or $2^4 \cdot 3 \cdot 7 \cdot 13$, we see that $|D_1| = 3$. Further, $D/C_D(D_1) \leq C_2$ implies that $3 \cdot 65 \in \pi_e(G)$. This contradiction shows that D_1 is an elementary abelian 2-group. Note that D is unsolvable. We obtain that the order of D_1 is either 2 or 4. If $|D_1| = 2$, then $D_1 \leq Z(G)$, yielding that $130 \in \pi_e(G)$ since $65 \in \pi_e(G)$, a contradiction. If $|D_1| = 4$, then $G/C_G(D_1) \leq \mathrm{Aut}(D_1)$. Note that —Aut$(D_1)| \mid 6$, we have that $65 \mid |C_G(D_1)|$. Hence, $130 \in \pi_e(G)$, again a contradiction.

Now we consider the case $N \cong A_6$. Since $G/D \leq S_6$, we see that $2^2 \cdot 7 \cdot 13 \mid |D|$. On the other hand, $N \cap D = 1$ and $ND \leq G$, we get $|D| \mid 2^3 \cdot 7 \cdot 13$. Thus, $|D| = 2^2 \cdot 7 \cdot 13$ or $2^3 \cdot 7 \cdot 13$. We claim that D is unsolvable, since, otherwise, $91 \in \pi_e(G)$, a contradiction. Let D_1 be a minimal normal subgroup of G contained in D. Further, by Lemma 2.1 D_1 is an elementary abelian p-group with $p \in \{2, 7, 13\}$. Assume first that D_1 is a 7-group. Then $D/C_D(D_1) \leq C_2$, which implies that $91 \in \pi_e(G)$, a contradiction. On the other hand, if D_1 is a 13-group, then $D/C_D(D_1) \leq C_{12}$, which also indicates that $91 \in \pi_e(G)$, again a contradiction. Therefore, D_1 is an elementary abelian 2-group. Note that D is unsolvable. We obtain that either $|D_1| = 2$ or $|D_1| = 4$, which by an argument similar to that in the previous paragraph, is impossible. This completes the proof of the theorem. □

Acknowledgments. The authors would like to thank the referee for his/her valuable suggestions. It should be said that we could not have polished the final version of this paper well without their outstanding efforts. The authors are also grateful to the research Project NNSF of China (Grant No. 11301218) and the Nature Science Fund of Shandong Province (No. ZR2014AM020) and University of Jinan Research Funds for Doctors (XBS1335, XBS1336).

REFERENCES

[1] Z.F. Cao, *On simple groups of order* $2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4} \cdot p^{\alpha_5}$. Chin. Ann. Math. **16A(2)** (1995), 244–250.

[2] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups.* Oxford Univ. Press, London, 1985.

[3] L.G. He and G.Y. Chen, *A new characterization of simple* $L_2(q)$ *where* $q = p^n < 125$. Ital. J. Pure Appl. Math, **28** (2011), 127–136.

[4] L.G. He and G.Y. Chen, *A new characterization of simple* K_3*-groups.* Comm. Algebra. **40** (2012), 3903–3911.

[5] M. Herzog, *On finite simple groups of order divisible by three primes only.* J. Algebra. **120(10)** (1968), 383–388.

[6] Q.H. Jiang, C.G. Shao, W.J. Shi and Q.L. Zhang, *A new characterization of* $L_2(q)$ *by largest element orders.* To appear in Bull. Iranian Math. Soc.

[7] W.J. Shi, *On simple* K_4*-groups.*(in Chinese) Chinese Sci. Bull. **36(17)** (1991), 1281–1283.

[8] A.V. Vasil′ev, M.A. Grechkoseeva and V.D. Mazurov, *Characterization of the finite simple groups by spectrum and order.* Algebra and Logic. **48(6)** (2009), 385–409.

[9] Q.L. Zhang and W.J. Shi, *A new characterization of simple* K_3*-groups and some* $L_2(p)$. Algebra Colloq. **20(3)** (2013), 361–368.

Received 24 October 2013

University of Jinan,
School of Mathematical Sciences,
Jinan, 250022, Shandong, China
syjqh2001@163.com

University of Jinan,
School of Mathematical Sciences,
Jinan, 250022, Shandong, China
shaoguozi@163.com,
chgshao@gmail.com