

EXTREMAL BETTI NUMBERS OF SOME CLASSES OF BINOMIAL EDGE IDEALS

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Let G be a cycle or a complete bipartite graph. We show that the binomial edge ideal J_G and its initial ideal with respect to the lexicographic order have the same extremal Betti number. This is a partial positive answer to a conjecture proposed in [2].

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INTRODUCTION

Let G be a simple graph on the vertex set $[n]$ with edge set $E(G)$ and let S be the polynomial ring $K[x_1, \dots, x_n, y_1, \dots, y_n]$ in $2n$ variables endowed with the lexicographic order induced by $x_1 > \dots > x_n > y_1 > \dots > y_n$. The binomial edge ideal $J_G \subset S$ associated with G is generated by all the binomials $f_{ij} = x_i y_j - x_j y_i$ with $\{i, j\} \in E(G)$. The binomial edge ideals were introduced in [5] and, independently, in [8]. Meanwhile, many algebraic and homological properties of these ideals have been investigated; see, for instance, [1–3, 5, 7, 9–14].

In [2], the authors conjectured that the extremal Betti numbers of J_G and $\text{in}_{<}(J_G)$ coincide for any graph G . Here, $<$ denotes the lexicographic order in S induced by the natural order of the variables. In this article, we give a positive answer to this conjecture when the graph G is a complete bipartite graph or a cycle. To this aim, we use some results proved in [12] and [14] which completely characterize the resolution of the binomial edge ideal J_G when G is a cycle or a complete bipartite graph. In particular, in this case, it follows that J_G has a unique extremal Betti number. In the first section we recall all the known facts on the resolutions of binomial edge ideals of the complete bipartite graphs and cycles. In Section 2, we study the initial ideal of J_G when G is a bipartite graph or a cycle. We show that $\text{proj dim in}_{<}(J_G) = \text{proj dim } J_G$ and $\text{reg in}_{<}(J_G) = \text{reg } J_G$, and, therefore, $\text{in}_{<}(J_G)$ has a unique extremal Betti number as well. Finally, we show that the extremal Betti number of $\text{in}_{<}(J_G)$ is equal to that of J_G .

To our knowledge, this is the first attempt to prove the conjecture stated in [2] for extremal Betti numbers. In our study, we take advantage of the known results on the resolutions of binomial edge ideals of cycles and complete bipartite graphs and of the fact that their initial ideals have nice properties. For instance, as we show in Section 2, the initial ideal of J_G for a complete bipartite graph has linear quotients and is generated in degrees 2 and 3. Therefore, it is componentwise linear and its Betti numbers may be computed easily (Theorem 2.2). The initial ideal of J_G when G is a cycle does not have linear quotients, but by ordering its generators in a suitable way, we may easily compute its extremal Betti number (Theorem 2.9). It is interesting to remark that even if the admissible paths of the cycle (in the sense of [7, Section 3]) determine the minimal set of monomial generators of $\text{in}_<(J_G)$, the Lyubeznik resolution [6] does not provide a minimal resolution of $\text{in}_<(G)$.

1. PRELIMINARIES

1.1. BINOMIAL EDGE IDEALS OF COMPLETE BIPARTITE GRAPHS

Let $G = K_{m,n}$ be the complete bipartite graph on the vertex set $\{1, \dots, m\} \cup \{m + 1, \dots, m + n\}$ with $m \geq n \geq 1$ and let J_G be its binomial edge ideal. J_G is generated by all the binomials $f_{ij} = x_i y_j - x_j y_i$ where $1 \leq i \leq m$ and $m + 1 \leq j \leq m + n$. In [12, Theorem 5.3] it is shown that the Betti diagram of S/J_G has the form

	0	1	2	\dots	p
0	1	0	0	\dots	0
1	0	mn	0	\dots	0
2	0	0	β_{24}	\dots	$\beta_{p,p+2}$

where $p = \text{proj dim } S/J_G = \begin{cases} m, & \text{if } n = 1, \\ 2m + n - 2, & \text{if } n > 1. \end{cases}$

In particular, from the above Betti diagram we may read that S/J_G has a unique extremal Betti number, namely $\beta_{p,p+2}$.

Moreover, in [12, Theorem 5.4] all the Betti numbers of S/J_G are computed. Since we are interested only in the extremal Betti number, we recall here its value as it was given in [12, Theorem 5.4], namely, $\beta_{p,p+2} = \begin{cases} m - 1, & \text{if } p = m, \\ n - 1, & \text{if } p = 2m + n - 2. \end{cases}$

Since we will study the initial ideal of J_G with respect to the lexicographic order induced by the natural order of the variables, we need to recall the following definition and result of [5].

Definition 1.1. Let $i < j$ be two vertices of an arbitrary graph G . A path $i = i_0, i_1, \dots, i_{r-1}, i_r = j$ from i to j is called **admissible** if the following conditions are fulfilled:

- (i) $i_k \neq i_\ell$ for $k \neq \ell$;
- (ii) for each $k = 1, \dots, r - 1$, one has either $i_k < i$ or $i_k > j$;
- (iii) for any proper subset $\{j_1, \dots, j_s\}$ of $\{i_1, \dots, i_{r-1}\}$, the sequence i, j_1, \dots, j_s, j is not a path in G .

Given an admissible path π of G from i to j , we set

$$u_\pi = \left(\prod_{i_k > j} x_{i_k} \right) \left(\prod_{i_\ell < i} y_{i_\ell} \right).$$

Obviously, any edge of G is an admissible path. In this case, the associated monomial is just 1.

THEOREM 1.2 (HHHKR). *Let G be an arbitrary graph. The set of binomials*

$$\Gamma = \bigcup_{i < j} \{u_\pi f_{ij} : \pi \text{ is an admissible path from } i \text{ to } j\}$$

is the reduced Gröbner basis of J_G with respect to lexicographic order on S induced by the natural order of indeterminates, $x_1 > \dots > x_n > y_1 > \dots > y_n$.

One may easily see that the only admissible paths of the complete graph $G = K_{m,n}$ are the edges of G , the paths of the form $i, m + k, j$ with $1 \leq i < j \leq m, 1 \leq k \leq n$, and $m + i, k, m + j$ with $1 \leq i < j \leq n, 1 \leq k \leq m$. Therefore, we get the following consequence of the above theorem.

COROLLARY 1.3. *Let $G = K_{m,n}$ be the complete bipartite graph on the vertex set $V(G) = \{1, \dots, m\} \cup \{m + 1, \dots, m + n\}$. Then*

$$\text{in}_<(J_G) = \left(\{x_i y_j\}_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq m+n}}, \{x_i x_{m+k} y_j\}_{\substack{1 \leq i < j \leq m, \\ 1 \leq k \leq n}}, \{x_{m+i} y_k y_{m+j}\}_{\substack{1 \leq i < j \leq n, \\ 1 \leq k \leq m}} \right).$$

1.2. BINOMIAL EDGE IDEALS OF CYCLES

In this subsection, G denotes the n -cycle on the vertex set $[n]$ with edges $\{1, 2\}, \{2, 3\}, \dots, \{n - 1, n\}, \{1, n\}$.

In [14] it was shown that the Betti diagram of S/J_G has the form

	0	1	2	3	⋯	n
0	1	0	0	0	⋯	0
1	0	n	0	0	⋯	0
2	0	0	β_{24}	0	⋯	0
3	0	0	0	β_{36}	⋯	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$n - 2$	0	0	$\beta_{2,n}$	$\beta_{3,n+1}$	⋯	$\beta_{n,2n-2}$

and all the Betti numbers were computed. One sees that we have a unique extremal Betti number and, by [14], we have $\beta_{n,2n-2} = \binom{n-1}{2} - 1$.

We now look at the initial ideal of J_G . It is obvious by Definition 1.1 and by the labeling of the vertices of G that the admissible paths are the edges of G and the paths of the form $i, i - 1, \dots, 1, n, n - 1, \dots, j + 1, j$ with $2 \leq j - i \leq n - 2$. Consequently, we get the following system of generators for the initial ideal of J_G .

COROLLARY 1.4. *Let G be the n -cycle with the natural labeling of its vertices. Then*

$$\text{in}_{<}(J_G) = (x_1y_2, \dots, x_{n-1}y_n, x_1y_n, \{x_i x_{j+1} \cdots x_n y_1 \cdots y_{i-1} y_j\}_{2 \leq j-i \leq n-2}).$$

2. EXTREMAL BETTI NUMBERS

2.1. COMPLETE BIPARTITE GRAPHS

Let $G = K_{m,n}$ be the complete bipartite graph on the vertex set $\{1, \dots, m\} \cup \{m + 1, \dots, m + n\}$ with $m \geq n \geq 1$ and let J_G be its binomial edge ideal. The initial ideal $\text{in}_{<}(J_G)$ has a nice property which is stated in the following proposition.

PROPOSITION 2.1. *Let $G = K_{m,n}$ be the complete bipartite graph. Then $\text{in}_{<}(J_G)$ has linear quotients.*

Proof. Let u_1, \dots, u_r be the minimal generators of $\text{in}_{<}(J_G)$ given in Corollary 1.3 where, for $i < j$, either $\text{deg } u_i < \text{deg } u_j$ or $\text{deg } u_i = \text{deg } u_j$ and $u_i > u_j$. We show that with respect to this order of its minimal monomial generators, $\text{in}_{<}(J_G)$ has linear quotients, that is, for any $\ell > 1$, the ideal quotient $(u_1, \dots, u_{\ell-1}) : (u_\ell)$ is generated by variables.

Let $u_\ell = x_i y_j$ for some $1 \leq i \leq m$ and $m + 1 \leq j \leq m + n$. In this case, one may easily check that

$$(1) \quad (u_1, \dots, u_{\ell-1}) : (u_\ell) = (x_1, \dots, x_{i-1}, y_{m+1}, \dots, y_{j-1}).$$

Let $u_\ell = x_i x_{m+k} y_j$ for some $1 \leq i < j \leq m$ and $1 \leq k \leq n$. Then we get

$$(2) \quad (u_1, \dots, u_{\ell-1}) : (u_\ell) = (y_{m+1}, \dots, y_{m+n}, x_1, \dots, x_{i-1}, x_{m+1}, \dots, x_{m+k-1}, y_{i+1}, \dots, y_{j-1}).$$

Finally, if $u_\ell = x_{m+i} y_k y_{m+j}$ for some $1 \leq i < j \leq n$ and $1 \leq k \leq m$, we have

$$(3) \quad (u_1, \dots, u_{\ell-1}) : (u_\ell) = (x_1, \dots, x_m, x_{m+1}, \dots, x_{m+i-1}, y_1, \dots, y_{k-1}, y_{m+i+1}, \dots, y_{m+j-1}). \quad \square$$

THEOREM 2.2. *Let $G = K_{m,n}$ be the complete bipartite graph. Then*

$$\beta_{t,t+2}(\text{in}_{<}(J_G)) = \sum_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq m+n}} \binom{i+j-m-2}{t},$$

$$\beta_{t,t+3}(\text{in}_{<}(J_G)) = \begin{cases} \sum_{\substack{1 \leq i < j \leq m \\ 1 \leq k \leq n}} \binom{n+k+j-3}{t}, & n = 1 \\ \sum_{\substack{1 \leq i < j \leq m \\ 1 \leq k \leq n}} \binom{n+k+j-3}{t} + \sum_{\substack{1 \leq i < j \leq n \\ 1 \leq k \leq m}} \binom{m+k+j-3}{t}, & n > 1 \end{cases}$$

Proof. Since $\text{in}_{<}(J_G)$ has linear quotients, we may apply [4, Exercise 8.8] and get

$$\beta_{t,t+d}(\text{in}_{<}(J_G)) = \sum_{\substack{\ell=1 \\ \deg u_\ell=d}}^m \binom{q_\ell}{t}$$

where q_ℓ is the number of variables which generate $(u_1, \dots, u_{\ell-1}) : (u_\ell)$. Hence, by using equations (1)–(3) for counting the number of variables which generate $(u_1, \dots, u_{\ell-1}) : (u_\ell)$, we get all the graded Betti numbers of $\text{in}_{<}(J_G)$. \square

In particular, by the above theorem, it follows the following corollary which shows that for $G = K_{m,n}$ the extremal Betti numbers of S/J_G and $S/\text{in}_{<}(J_G)$ coincide.

COROLLARY 2.3. *Let $G = K_{m,n}$ be the complete bipartite graph. Then:*

- (a) $\text{proj dim}(S/\text{in}_{<}(J_G)) = \text{proj dim}(\text{in}_{<}(J_G)) + 1 = \begin{cases} m, & n = 1 \\ 2m + n - 2, & n > 1 \end{cases}$
- (b) $S/\text{in}_{<}(J_G)$ has a unique extremal Betti number, namely

$$\beta_{p,p+2}(S/\text{in}_{<}(J_G)) = \beta_{p-1,p+2}(\text{in}_{<}(J_G)) = \begin{cases} m - 1, & n = 1, \\ n - 1, & n > 1. \end{cases}$$

Proof. (a) follows immediately from Betti number formulas of Theorem 2.2.

Let us prove (b). By using again Theorem 2.2, we get

$$\beta_{p-1,p+2}(\text{in}_{<}(J_G)) = \begin{cases} \sum_{1 \leq i < m} \binom{m-1}{p-1} = \sum_{1 \leq i < m} \binom{m-1}{m-1} = m - 1 & \text{if } n = 1, \\ \sum_{1 \leq i < n} \binom{2m+n-3}{p-1} = \sum_{1 \leq i < n} \binom{2m+n-3}{2m+n-3} = n - 1 & \text{if } n > 1. \end{cases} \quad \square$$

2.2. CYCLES

In this subsection, the graph G is an n -cycle. If $n = 3$, then G is a complete graph, therefore the ideals J_G and $\text{in}_{<}(J_G)$ have the same graded Betti numbers. Thus, in the sequel, we may consider $n \geq 4$.

As we have already seen in Corollary 1.4, $\text{in}_{<}(J_G)$ is minimally generated by the initial monomials of the binomials corresponding to the edges of G and by $m = n(n - 3)/2$ monomials of degree ≥ 3 which we denote by v_1, \dots, v_m where we assume that if $i < j$, then either $\text{deg } v_i < \text{deg } v_j$ or $\text{deg } v_i = \text{deg } v_j$ and $v_i > v_j$. Let us observe that if $v_k = x_i x_{j+1} \cdots x_n y_1 \cdots y_{i-1} y_j$, we have $\text{deg } v_k = n - j + i + 1$. Hence, there are two monomials of degree 3, namely, $v_1 = x_1 x_n y_{n-1}$ and $v_2 = x_2 y_1 y_n$, three monomials of degree 4, namely, $v_3 = x_1 x_{n-1} x_n y_{n-2}$, $v_4 = x_2 x_n y_1 y_{n-1}$, $v_5 = x_3 y_1 y_2 y_n$, etc.

We introduce the following notation. We set $J = (x_1 y_2, x_2 y_3, \dots, x_{n-1} y_n)$, $I = J + (x_1 y_n)$, and, for $1 \leq k \leq m$, $I_k = I_{k-1} + (v_k)$, with $I_0 = I$. Therefore, $I_m = \text{in}_{<}(J_G)$.

LEMMA 2.4. *The ideals quotient $J : (x_1 y_n)$ and $I_{k-1} : (v_k)$, for $k \geq 1$, are minimally generated by regular sequences of monomials of length $n - 1$.*

Proof. The statement is obvious for $J : (x_1 y_n)$ since J is minimally generated by a regular sequence. Now let $k \geq 1$ and let $v_k = x_i x_{j+1} \cdots x_n y_1 \cdots y_{i-1} y_j$ for some i, j with $2 \leq j - i \leq n - 2$. Then $I_{k-1} : (v_k) = I : (v_k) + (v_1, \dots, v_{k-1}) : (v_k)$. One easily observes that $(v_1, \dots, v_{k-1}) : (v_k) = (x_1, \dots, x_{i-1}, y_{j+1}, \dots, y_n)$. Hence,

$$\begin{aligned} I_{k-1} : (v_k) &= (x_1, \dots, x_{i-1}, x_i y_{i+1}, x_{i+1} y_{i+2}, \dots, x_{j-2} y_{j-1}, x_{j-1} y_j, \\ & y_{j+1}, \dots, y_n) : (v_k) = \\ & (x_1, \dots, x_{i-1}, y_{i+1}, x_{i+1} y_{i+2}, \dots, x_{j-2} y_{j-1}, x_{j-1}, y_{j+1}, \dots, y_n). \quad \square \end{aligned}$$

Remark 2.5. From the above proof we also note that if $v_k = x_i x_{j+1} \cdots x_n y_1 \cdots y_{i-1} y_j$, then the regular sequence of monomials which generates $I_{k-1} : (v_k)$ contains $j - i - 2$ monomials of degree 2 and $n - j + i + 1$ variables.

In the following lemma, we compute the projective dimension and the regularity of S/I . This will be useful for the inductive study of the invariants of S/I_k .

LEMMA 2.6. *We have $\text{proj dim } S/I = n - 1$ and $\text{reg } S/I = n - 2$.*

Proof. In the exact sequence

$$(4) \quad 0 \rightarrow \frac{S}{J : (x_1 y_n)} (-2) \xrightarrow{x_1 y_n} \frac{S}{J} \rightarrow \frac{S}{I} \rightarrow 0,$$

we have $\text{proj dim } S/J : (x_1 y_n) = \text{proj dim } S/J = n - 1$ since both ideals are generated by regular sequences of length $n - 1$. Moreover, since J is generated by a regular sequence of monomials of degree 2, by using the Koszul complex, we get

$$\beta_{ij}(S/J) = \begin{cases} 0, & j \neq 2i, \\ \binom{n-1}{i}, & j = 2i. \end{cases}$$

In particular, it follows that $\text{Tor}_{n-1}(S/J, K) \cong K(-2n + 2)$.

Analogously, since $J : (x_1y_n)$ is generated by a regular sequence of $n - 3$ monomials of degree 2 and two variables, it follows that $\text{Tor}_{n-1}(S/J : (x_1y_n), K) \cong K(-2n + 4)$, which implies that $\text{Tor}_{n-1}(S/J : (x_1y_n)(-2), K) \cong K(-2n + 2)$. By the long exact sequence of Tor's derived from sequence (4), we get $\text{Tor}_n(S/I, K) = 0$, hence $\text{proj dim } S/I \leq n - 1$.

On the other hand, we have

$$\beta_{n-2,j}(S/J : (x_1y_n)) = \begin{cases} 2, & j = 2n - 5, \\ n - 3, & j = 2n - 6, \\ 0, & \text{otherwise.} \end{cases}$$

From the exact sequence of Tor's applied to (4), as

$$\text{Tor}_{n-1}(S/J, K)_{2n-3} = \text{Tor}_{n-2}(S/J, K)_{2n-3} = 0,$$

we get the following exact sequence

$$0 \rightarrow \text{Tor}_{n-1}(S/I, K)_{2n-3} \rightarrow \text{Tor}_{n-2}(S/J : (x_1y_n), K)_{2n-5} \rightarrow 0.$$

Thus, $\text{Tor}_{n-1}(S/I, K) \neq 0$ which implies that $\text{proj dim } S/I = n - 1$.

For the regularity, we first observe that since the Koszul complex of the minimal generators gives the minimal graded free resolution of S/J and, respectively, $S/J : (x_1y_n)$, we have $\text{reg } S/J = \text{reg}(S/J : (x_1y_n)(-2)) = n - 1$. Then sequence (4) implies that $\text{reg } S/I \leq n - 1$. We have observed above that $\text{Tor}_{n-1}(S/I, K)_{2n-3} \neq 0$, thus $\text{reg } S/I \geq n - 2$. In order to derive the equality $\text{reg } S/I = n - 2$ we have to show that $\beta_{i,i+n-1}(S/I) = 0$ for all i .

For $i = n - 1$, as $\text{Tor}_{n-2}(S/J : (x_1y_n), K)_{2n-4} = 0$, we get

$$0 \rightarrow \text{Tor}_{n-1}(S/J : (x_1y_n), K)_{2n-4} \rightarrow \text{Tor}_{n-1}(S/J, K)_{2n-2} \rightarrow \text{Tor}_{n-1}(S/I, K)_{2n-2} \rightarrow 0.$$

But $\dim_K \text{Tor}_{n-1}(S/J : (x_1y_n), K)_{2n-4} = \dim_K \text{Tor}_{n-1}(S/J, K)_{2n-2} = 1$, which implies that $\text{Tor}_{n-1}(S/I, K)_{2n-2} = 0$. For $i < n - 1$, $\text{Tor}_i(S/I, K)_{i+n-1} = 0$ since $\text{Tor}_i(S/J, K)_{i+n-1} = 0$ and $\text{Tor}_{i-1}(S/J : (x_1y_n), K)_{i+n-3} = 0$. The latter equality holds since, as we have already observed, $\text{reg } S/J : (x_1y_n) = n - 3$. \square

LEMMA 2.7. *For $1 \leq k \leq m$, we have $\text{proj dim } S/I_k \leq n$ and $\text{reg } S/I_k \leq n - 2$.*

Proof. We proceed by induction on k , by using the following exact sequence and Lemma 2.6 for the initial step,

$$(5) \quad 0 \rightarrow \frac{S}{I_{k-1} : (v_k)}(-\deg v_k) \rightarrow \frac{S}{I_{k-1}} \rightarrow \frac{S}{I_k} \rightarrow 0.$$

Indeed, as $I_{k-1} : (v_k)$ is generated by a regular sequence of length $n - 1$, it follows that $\text{proj dim } S/I_{k-1} : (v_k) = n - 1$. Thus, if $\text{proj dim } S/I_{k-1} \leq n$, by (5), it follows that $\text{proj dim } S/I_k \leq n$.

In addition, we have $\text{reg}(S/I_{k-1} : (v_k))(-\deg v_k) = \text{reg}(S/I_{k-1} : (v_k)) + \deg v_k$. If $v_k = x_i x_{j+1} \cdots x_n y_1 \cdots y_{i-1} y_j$, by using Remark 2.5, we obtain

$\text{reg}(S/I_{k-1} : (v_k)) = j - i - 2$. As $\deg v_k = n - j + i + 1$, we get $\text{reg}(S/I_{k-1} : (v_k))(-\deg v_k) = n - 1$. Let us assume, by induction, that $\text{reg} S/I_{k-1} \leq n - 2$. Then, by using the sequence (5), we obtain

$$\text{reg} S/I_k \leq \max\{\text{reg}(S/I_{k-1} : (v_k))(-\deg v_k) - 1, \text{reg} S/I_{k-1}\} = n - 2. \quad \square$$

PROPOSITION 2.8. *We have $\text{proj dim } S/\text{in}_{<}(J_G) = n$ and $\text{reg } S/\text{in}_{<}(J_G) = n - 2$.*

Proof. As $\text{in}_{<}(J_G) = I_m$, by Lemma 2.7, we get $\text{proj dim } S/\text{in}_{<}(J_G) \leq n$ and $\text{reg } S/\text{in}_{<}(J_G) \leq n - 2$. For the other inequalities, we use [4, Theorem 3.3.4] which gives the inequalities $n = \text{proj dim } S/J_G \leq \text{proj dim } S/\text{in}_{<}(J_G)$ and $n - 2 = \text{reg } S/J_G \leq \text{reg } S/\text{in}_{<}(J_G)$. \square

THEOREM 2.9. *Let G be the n -cycle. Then $S/\text{in}_{<}(J_G)$ and S/J_G have the same extremal Betti number, namely $\beta_{n,2n-2}(S/J_G) = \beta_{n,2n-2}(S/\text{in}_{<}(J_G)) = \binom{n-1}{2} - 1$.*

Proof. We only need to show that $\beta_{n,2n-2}(S/\text{in}_{<}(J_G)) = m$ since $m = (n^2 - 3n)/2 = \binom{n-1}{2} - 1$.

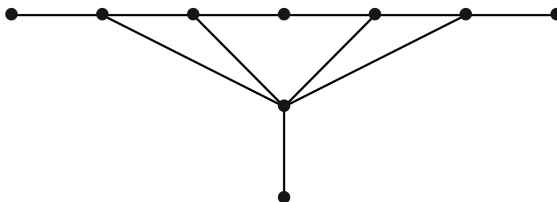
We use again the sequence (5). By considering its long exact sequence of Tor's and using the equality $\text{Tor}_{n-1}(S/I_{k-1}, K)_{2n-2} = 0$, we get

$$\begin{aligned} 0 &\rightarrow \text{Tor}_n(S/I_{k-1}, K)_{2n-2} \rightarrow \text{Tor}_n(S/I_k, K)_{2n-2} \\ &\rightarrow \text{Tor}_{n-1}(S/I_{k-1} : (v_k), K)_{2n-2-\deg v_k} \rightarrow 0, \text{ for } 1 \leq k \leq m. \end{aligned}$$

By Remark 2.5, if $v_k = x_i x_{j+1} \cdots x_n y_1 \cdots y_{i-1} y_j$, then $I_{k-1} : (v_k)$ is generated by a regular sequence of monomials which contains $j - i - 2$ elements of degree 2 and $n - j + i + 1$ variables. This implies that $\dim_K \text{Tor}_{n-1}(S/I_{k-1} : (v_k), K)_{2n-2-\deg v_k} = 1$. Therefore, we get $\dim_K \text{Tor}_n(S/I_k, K)_{2n-2} = \dim_K \text{Tor}_n(S/I_{k-1}, K)_{2n-2} + 1$ for $1 \leq k \leq m$. By summing up all these equalities, it follows that $\beta_{n,2n-2}(S/\text{in}_{<}(J_G)) = \dim_K \text{Tor}_n(S/I_m, K)_{2n-2} = \dim_K \text{Tor}_n(S/I, K)_{2n-2} + m = m$.

The last equality is due to Lemma 2.6. \square

Remark 2.10. *There are examples of graphs whose binomial edge ideal have several extremal Betti numbers. For instance, the graph G displayed below*



has two extremal Betti numbers which are equal to the extremal Betti numbers of $\text{in}_<(J_G)$.

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