ON PATH-SUNFLOWER RAMSEY NUMBERS

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For given graphs G and H, the *Ramsey number* $R(G, H)$ is the least natural number n such that for every graph F of order n the following condition holds: either F contains G or the complement of F contains H. In this paper, we determine the Ramsey number of path P_n versus sunflower graph SF_m when n grows at least as a quadratic function of m. In this case $R(P_n, SF_m) = 3n - 2$ if m is odd and $2n + \frac{m}{2} - 2$ otherwise.

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1. INTRODUCTION

Let $G(V, E)$ be a graph with the vertex-set $V(G)$ and edge-set $E(G)$. If $xy \in E(G)$ then x is called *adjacent* to y, and y is a *neighbor* of x and vice versa. For any $A \subseteq V(G)$, we use $N_A(x)$ to denote the set of all neighbors of x in A, namely $N_A(x) = \{y \in A | xy \in E(G)\}$. Let P_n be a path with n vertices, C_n be a cycle with n vertices, W_k be a wheel with $k+1$ vertices, *i.e.*, a graph consisting of a cycle C_k with one additional vertex adjacent to all vertices of C_k. For $m \geq 3$, the *sunflower graph* SF_m is a graph on $2m+1$ vertices obtained by taking a wheel W_m with hub x, an m-cycle $v_1, v_2, \ldots, v_m$, and additional m vertices $w_1, w_2, \ldots, w_m$, where w_i is joined by edges to v_i, v_{i+1} for $i = 1, 2, \ldots, m$, where $i+1$ is taken modulo m. The hub of W_m is also called the hub of SF_m.

Baskoro and Surahmat [4] determined the Ramsey number of a combination of P_n versus a wheel W_k, as follows.

Theorem ([4]). *We have*

$$R(P_n, W_k) = \begin{cases} 2n-1 & \text{if } k \geq 4 \text{ is even and } n \geq \frac{k}{2}(k-2), \\ 3n-2 & \text{if } k \geq 5 \text{ is odd and } n \geq \frac{k-1}{2}(k-3). \end{cases}$$

Other papers concerning Ramsey numbers of paths versus wheel related graphs are [2–4, 7]; a nice survey paper on Ramsey numbers is [6].

In this paper, we determine the Ramsey numbers involving path and sunflower graph SF_m as follows.

THEOREM. *If* $m \geq 4$ *is even and* $n \geq 4m^2 - 7m + 4$ *then* $R(P_n, SF_m) = 2n + \frac{m}{2} - 2$.

Proof. Consider the graph $F_1 = 2K_{n-1} \cup K_{\frac{m}{2}-1}$. It is obvious that $F_1 \not\supset P_n$. We have also $\overline{F_1} \cong K_{n-1,n-1,\frac{m}{2}-1}$. Let π be a 3-coloring of SF_m with three colors. We deduce that the vertices of C_m are alternately colored with two colors and the third color must be assigned to the hub of SF_m and to vertices $w_1, \ldots, w_m$. This implies that the color classes of π are $A = \{v_1, v_3, \ldots, v_{m-1}\}$, $B = \{v_2, v_4, \ldots, v_m\}$ and $C = \{x, w_1, w_2, \ldots, w_m\}$; we have $|A| = |B| = \frac{m}{2}$ and $|C| = m + 1$. Since no monochromatic color class of π cannot be included in the part having $\frac{m}{2} - 1$ vertices of the complete 3-partite graph $K_{n-1,n-1,\frac{m}{2}-1}$, we deduce that $SF_m \not\subset \overline{F_1}$, which implies $R(P_n, SF_m) \geq 2n + \frac{m}{2} - 2$.

For the reverse inequality, let F be a graph on $2n+\frac{m}{2}-2$ vertices containing no P_n. Let $L_1 = l_{1,1}, l_{1,2}, \ldots, l_{1,k}$ be a longest path in F and so $k \leq n - 1$. If $k = 1$ we have $\overline{F} \cong K_{2n+\frac{m}{2}-2}$ which contains SF_m. Suppose that $k \geq 2$. We shall prove that $\overline{F}$ contains SF_m. Obviously, for each $z \in V_1$, where $V_1 = V(F)\backslash V(L_1)$, $zl_{1,1}, zl_{1,k} \notin E(F)$. Let $L_2 = l_{2,1}, l_{2,2}, \ldots, l_{2,t}$ be a longest path in $F[V_1]$. It is clear that $1 \leq t \leq k$. Let $V_2 = V(F)\backslash(V(L_1) \cup V(L_2))$. Since $|V(F)| = 2n + \frac{m}{2} - 2$, there exist at least $\frac{m}{2}$ vertices in V_2, which are not adjacent to any endpoint $l_{1,1}, l_{1,k}, l_{2,1}, l_{2,t}$. We distinguish three cases.

Case a1: $k < 4m - 2$. If $t = 1$ then the vertices in V_1 induce a subgraph having only isolated vertices. In this case, we shall add an edge uv to F, where $u, v \in V_1$ and denote $L_2 = u, v$. In this way we can define inductively a system of paths $L_1, L_2, \ldots, L_m$ such that L_i is a longest path in $F[V_{i-1}]$, where $V_{i-1} = V(F) \setminus \bigcup_{j=1}^{i-1} V(L_j)$ or an edge added to F as above. If F_1 denotes the graph F or the graph F plus some edges added in the process of defining the system of paths, it follows that endpoints of these L_j $(1 \leq j \leq m)$ induce in $\overline{F_1}$ a complete graph K_{2m} minus a matching having at most m edges if some of the endpoints of the same L_j are adjacent in F_1. If Y denotes the set of the remaining vertices, we have $|V(Y)| \geq 2n + \frac{m}{2} - 2 - m(4m - 3) > \frac{m}{2} \geq 2$. Let x be one vertex which is not adjacent to any endpoint of these L_j for $1 \leq j \leq m$. It is easy to see that x together with all endpoints of paths L_j contains a $SF_m \subset \overline{F_1} \subset \overline{F}$ having the hub x.

Case a2: $k \geq 4m - 2$ and $t \geq 4m - 2$. In this case we define $m - 1$ quadruples A_i in path L_1 as follows:

$$\begin{aligned} A_1 &= \{l_{1,2}, l_{1,3}, l_{1,4}, l_{1,5}\}, \\ A_2 &= \{l_{1,6}, l_{1,7}, l_{1,8}, l_{1,9}\}, \\ &\vdots \\ A_{m-1} &= \{l_{1,4m-6}, l_{1,4m-5}, l_{1,4m-4}, l_{1,4m-3}\}. \end{aligned}$$

In a similar way let

$$\begin{aligned} B_1 &= \{l_{2,2}, l_{2,3}, l_{2,4}, l_{2,5}\},\\ B_2 &= \{l_{2,6}, l_{2,7}, l_{2,8}, l_{2,9}\},\\ &\vdots\\ B_{m-1} &= \{l_{2,4m-6}, l_{2,4m-5}, l_{2,4m-4}, l_{2,4m-3}\}. \end{aligned}$$

for the path L_2.

Since $V_2 = V(F) \setminus (V(L_1) \cup V(L_2))$, we have $|V_2| \geq \frac{m}{2}$ since $t, k \leq n-1$. Hence, we can consider $\frac{m}{2}$ distinct elements in V_2: $y_1, y_2, \ldots, y_{\frac{m}{2}}$ and $\frac{m}{2} - 1$ pairs of elements $Y_i = \{y_i, y_{i+1}\}$ for $i = 1, 2, \ldots, \frac{m}{2} - 1$. By the maximality of L_2 it follows that for each $i = 1, 2, \ldots, \frac{m}{2} - 1$, at least one vertex in B_i is not adjacent to any vertex in Y_i. Denote by b_i vertices in B_i which are not adjacent to any vertex in Y_i for $i = 1, 2, \ldots, \frac{m}{2} - 1$. It follows that $l_{2,1}, y_1, b_1, y_2, b_2, \ldots, y_{\frac{m}{2}-1}, b_{\frac{m}{2}-1}, y_{\frac{m}{2}}$ is an m-cycle in $\overline{F}$ and this cycle together with vertex $l_{1,1}$ induces W_m in $\overline{F}$.

By the maximality of L_1 we get that for any $i = 1, \ldots, m-1$ and any two different vertices $z_1, z_2 \in V_1 = V(F)\setminus V(L_1)$, there exists at least one vertex a_i in A_i that is not adjacent to z_1 nor to z_2. Hence, we can choose an additional vertex set $\{a_1, a_2, \ldots, a_{m-1}, l_{1,k}\}$ which together with W_m induces a graph in $\overline{F}$ which contains SF_m, thus $SF_m \subset \overline{F}$.

Case a3: $k \geq 4m - 2$ and $t < 4m - 2$. Since F has no P_n it follows that $k \leq n - 1$, hence V_1 will have at least $n + \frac{m}{2} - 1$ vertices. Then we can define the same process as in case a1. We obtain a system of paths $L_2, \ldots, L_m$, in the subgraph induced by V_1 such that the endpoints of $L_1, \ldots, L_m$, induce in $\overline{F_1}$ a complete graph K_{2m} minus a matching having at most m edges. We get in this case $|V(Y)| \geq n + \frac{m}{2} - 1 - (m-1)(4m-3) \geq 2$ and the proof is similar to the case a1. □

THEOREM. *For all $n \geq 3$, $R(P_n, SF_3) = 3n - 2$.*

Proof. To show the lower bound, consider graph $F_1 = 3K_{n-1}$. We have $\overline{F_1} \cong K_{n-1,n-1,n-1}$, hence its chromatic number $\chi(\overline{F_1}) = 3$, but $\chi(SF_3) = 4$, which implies that $SF_3 \not\subseteq \overline{F_1}$. It follows that $R(P_n, SF_3) \geq 3n - 2$. For the reverse inequality, let us consider a graph F of order $3n - 2$ such that F does not contain path P_n, we will show that $\overline{F}$ contains sunflower graph SF_3. Let P be a longest path in F with endpoints p_1 and p_2. Obviously, $xp_1, xp_2 \notin E(F)$ for each $x \in X = V(F) \setminus V(P)$. Let Q be a longest path in $F[X]$ with q_1 and q_2 as its endpoints. Then $xq_1, xq_2 \notin E(F)$ for each $x \notin V(P) \cup V(Q)$. Let $Y = V(F) \setminus (V(P) \cup V(Q))$ and R be a longest path in $F[Y]$ with r_1 and r_2 as its endpoints. Since $|V(F)| = 3n - 2$ and the longest path in F is of length less than or equal to $n-1$ then there exists a vertex $a \notin V(P) \cup V(Q) \cup V(R)$ such that a is not adjacent to any endpoint p_1, p_2, q_1, q_2, r_1 and r_2. Thus, we give mapping yielding SF_3 in $\overline{F}$ with a as hub. □

THEOREM. *If $m \geq 5$ is odd and $n \geq 2m^2 - 9m + 11$ then $R(P_n, SF_m) = 3n - 2$.*

Proof. By using an argument similar as above we have $R(P_n, SF_m) \geq 3n - 2$. To prove $R(P_n, SF_m) \leq 3n - 2$, let F be a graph on $3n - 2$ vertices containing no P_n. Let $L_1 = l_{1,1}, l_{1,2}, \ldots, l_{1,k}$ be a longest path in F and so $k \leq n - 1$. If $k = 1$ we have $\overline{F} \simeq K_{3n-2}$, which contains SF_m. Suppose that $k \geq 2$ and $\overline{F}$ does not contain SF_m. Obviously, $zl_{1,1}, zl_{1,k}$ are not in $E(F)$ for each $z \in V_1$, where $V_1 = V(F) \setminus V(L_1)$. Let $L_2 = l_{2,1}, l_{2,2}, \ldots, l_{2,t}$ be a longest path in $F[V_1]$. If $t = 1$ we have $\overline{F} \simeq K_{2n-1}$, which contains SF_m, so we may suppose $2 \leq t \leq k$. Let $V_2 = V(F) \setminus (V(L_1) \cup V(L_2))$. Obviously, $yl_{2,1}, yl_{2,t}$ are not in $E(F)$ for each $y \in V_2$. Let $L_3 = l_{3,1}, l_{3,2}, \ldots, l_{3,s}$ be a longest path in $F[V_2]$. Since $|V(F)| = 3n - 2$ and the longest path in F is of length less than or equal to $n - 1$ then there exists a vertex $x \notin V(L_1) \cup V(L_2) \cup V(L_3)$ such that x is not adjacent to any endpoint $l_{1,1}, l_{2,1}, l_{3,1}, l_{1,k}, l_{2,t}$ and $l_{3,s}$. We distinguish four cases.

Case 1: $k < 2m - 4$. It follows that $t < 2m - 4$. If $s = 1$ then the vertices in V_2 induce a subgraph having only isolated vertices. In this case we shall add an edge uv to F, where $u, v \in V_2$ and denote $L_3 = u, v$. In this way we can define inductively as in proof of theorem 2 the system of paths $L_1, L_2, \ldots, L_m$ such that L_i is a longest path in $F[V_{i-1}]$, where $V_{i-1} = V(F) \setminus \bigcup_{j=1}^{i-1} V(L_j)$ or an edge added to F as above. If F_1 denotes the graph F or the graph F plus some edges added in the process of defining the system of paths, it follows that endpoints of these L_j , where $j = 1, 2, \ldots, m$ induce in $\overline{F_1}$ a complete graph K_{2m} minus a matching having at most m edges if some of the endpoints of the same L_j are adjacent in F_1. For $m \geq 5$ there exists at least one vertex x which is not adjacent to any endpoint of these L_j. Thus, it is easy to see that vertex x together with all endpoints of paths L_j form a $SF_m \subset \overline{F_1} \subset \overline{F}$.

Case 2: $k \geq 2m - 4$, $t \geq 2m - 4$ and $s \geq 2m - 4$. For $i = 1, 2, \ldots, m - 3$ define the couples A_i in path L_1 as follows:

$$A_i = \begin{cases} \{l_{1,i+1}, l_{1,i+2}\} & \text{for i odd,} \\ \{l_{1,k-i}, l_{1,k-i+1}\} & \text{for i even.} \end{cases}$$

Similarly, define couples B_i, C_i in paths L_2 and L_3, respectively as follows:

$$B_i = \begin{cases} \{l_{2,i+1}, l_{2,i+2}\} & \text{for i odd,} \\ \{l_{2,t-i}, l_{2,t-i+1}\} & \text{for i even.} \end{cases}$$

$$C_i = \begin{cases} \{l_{3,i+1}, l_{3,i+2}\} & \text{for i odd,} \\ \{l_{3,s-i}, l_{3,s-i+1}\} & \text{for i even.} \end{cases}$$

We have seen that since $s \leq t \leq k \leq n-1$ and $|F| = 3n-2$, there exists at least one vertex x which is not in $L_1 \cup L_2 \cup L_3$. Since L_1 is a longest path in F, there exists one vertex of A_i for each i, say a_i which is not adjacent with x. Similarly, we obtain vertices b_i and c_i in couples B_i and C_i which are not adjacent to x for every $i = 1, \ldots, m-3$. The maximality of the paths L_1 and L_2 also implies that for every $i, j, k = 1, \ldots, m-3$ we have $a_i b_j, a_i c_k, b_j c_k \notin E(F)$. Thus, vertex set $\{l_{1,1}, l_{2,t}, a_1, b_1, a_2, b_2, \ldots, a_{\frac{m-3}{2}}, b_{\frac{m-3}{2}}, l_{3,1}\}$ with vertex x will contain W_m in $\overline{F}$ and additional vertex set $\{l_{3,s}, c_1, c_2, c_3, \ldots, c_{m-3}, l_{1,k}, l_{2,1}\}$ with wheel W_m, gives $SF_m \subset \overline{F}$.

Case 3: $k \geq 2m-4$, $t \geq 2m-4$ and $s < 2m-4$. Since F has no P_n it follows that $t \leq k \leq n-1$. Consequently, V_2 will have at least n vertices. Then we can define the same process as in case 1. We obtain a system of paths $L_3, \ldots, L_m$ in the subgraph induced by V_2 such that the endpoints of $L_1, \ldots, L_m$ induce in $\overline{F_1}$ a complete graph K_{2m} minus a matching having at most m edges. We get in this case $|V(Y)| \geq n-(m-2)(2m-5) \geq 1$ and the proof is similar to the case 1.

Case 4: $k \geq 2m-4$ and $t < 2m-4$. We deduce that $s < 2m-4$. Since F has no P_n it follows that $k \leq n-1$. Consequently, V_1 will have at least $2n-1$ vertices. Then we can define the same process as in case 1. We obtain a system of paths $L_2, \ldots, L_m$ in the subgraph induced by V_1 such that the endpoints of $L_1, \ldots, L_m$ induce in $\overline{F_1}$ a complete graph K_{2m} minus a matching having at most m edges. We get in this case $|V(Y)| \geq 2n-1-(m-1)(2m-5) > 1$ and the proof is similar to the case 1. □

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