ON PATH-SUNFLOWER RAMSEY NUMBERS

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Communicated by Vasile Brînzănescu

For given graphs $G$ and $H$, the Ramsey number $R(G, H)$ is the least natural number $n$ such that for every graph $F$ of order $n$ the following condition holds: either $F$ contains $G$ or the complement of $F$ contains $H$. In this paper, we determine the Ramsey number of path $P_n$ versus sunflower graph $SF_m$ when $n$ grows at least as a quadratic function of $m$. In this case $R(P_n, SF_m) = 3n - 2$ if $m$ is odd and $2n + \frac{m}{2} - 2$ otherwise.

AMS 2010 Subject Classification: 05C55.

Key words: Ramsey number, path, sunflower graph.

1. INTRODUCTION

Let $G(V, E)$ be a graph with the vertex-set $V(G)$ and edge-set $E(G)$. If $xy \in E(G)$ then $x$ is called adjacent to $y$, and $y$ is a neighbor of $x$ and vice versa. For any $A \subseteq V(G)$, we use $N_A(x)$ to denote the set of all neighbors of $x$ in $A$, namely $N_A(x) = \{y \in A : xy \in E(G)\}$. Let $P_n$ be a path with $n$ vertices, $C_n$ be a cycle with $n$ vertices, $W_k$ be a wheel with $k + 1$ vertices, i.e., a graph consisting of a cycle $C_k$ with one additional vertex adjacent to all vertices of $C_k$. For $m \geq 3$, the sunflower graph $SF_m$ is a graph on $2m + 1$ vertices obtained by taking a wheel $W_m$ with hub $x$, an $m$-cycle $v_1, v_2, \ldots, v_m$, and additional $m$ vertices $w_1, w_2, \ldots, w_m$, where $w_i$ is joined by edges to $v_i, v_{i+1}$ for $i = 1, 2, \ldots, m$, where $i + 1$ is taken modulo $m$. The hub of $W_m$ is also called the hub of $SF_m$.

Baskoro and Surahmat [4] determined the Ramsey number of a combination of $P_n$ versus a wheel $W_k$, as follows.

**Theorem ([4]).** We have

$$R(P_n, W_k) = \begin{cases} 2n - 1 & \text{if } k \geq 4 \text{ is even and } n \geq \frac{k}{2}(k - 2), \\
3n - 2 & \text{if } k \geq 5 \text{ is odd and } n \geq \frac{k - 1}{2}(k - 3). \end{cases}$$

Other papers concerning Ramsey numbers of paths versus wheel related graphs are [2–4, 7]; a nice survey paper on Ramsey numbers is [6].

In this paper, we determine the Ramsey numbers involving path and sunflower graph $SF_m$ as follows.
THEOREM. If \( m \geq 4 \) is even and \( n \geq 4m^2 - 7m + 4 \) then \( R(P_n, SF_m) = 2n + \frac{m}{2} - 2 \).

Proof. Consider the graph \( F_1 = 2K_{n-1} \cup K_{\frac{m}{2}-1} \). It is obvious that \( F_1 \not\subseteq P_n \). We have also \( \overline{F_1} \cong K_{n-1,n-1,\frac{m}{2}-1} \). Let \( \pi \) be a 3-coloring of \( SF_m \) with three colors. We deduce that the vertices of \( C_m \) are alternately colored with two colors and the third color must be assigned to the hub of \( SF_m \) and to vertices \( w_1, \ldots, w_m \). This implies that the color classes of \( \pi \) are \( A = \{v_1, v_3, \ldots, v_{m-1}\}, B = \{v_2, v_4, \ldots, v_m\} \) and \( C = \{x, w_1, w_2, \ldots, w_m\} \); we have \( |A| = |B| = \frac{m}{2} \) and \( |C| = m + 1 \). Since no monochromatic color class of \( \pi \) cannot be included in the part having \( \frac{m}{2} - 1 \) vertices of the complete 3-partite graph \( K_{n-1,n-1,\frac{m}{2}-1} \), we deduce that \( SF_m \not\subseteq \overline{F_1} \), which implies \( R(P_n, SF_m) \geq 2n + \frac{m}{2} - 2 \).

For the reverse inequality, let \( F \) be a graph on \( 2n + \frac{m}{2} - 2 \) vertices containing no \( P_n \). Let \( L_1 = l_{1,1}, l_{1,2}, \ldots, l_{1,k} \) be a longest path in \( F \) and so \( k \leq n - 1 \). If \( k = 1 \) we have \( \overline{F} \cong K_{2n+\frac{m}{2}-2} \) which contains \( SF_m \). Suppose that \( k \geq 2 \). We shall prove that \( \overline{F} \) contains \( SF_m \). Obviously, for each \( z \in V_1 \), where \( V_1 = V(F) \setminus V(L_1) \), \( z_{l_{1,1}}, z_{l_{1,k}} \not\in E(F) \). Let \( L_2 = l_{2,1}, l_{2,2}, \ldots, l_{2,t} \) be a longest path in \( F[V_1] \). It is clear that \( 1 \leq t \leq k \). Let \( V_2 = V(F) \setminus (V(L_1) \cup V(L_2)) \).

Since \( |V(F)| = 2n + \frac{m}{2} - 2 \), there exist at least \( \frac{m}{2} \) vertices in \( V_2 \), which are not adjacent to any endpoint \( l_{1,1}, l_{1,k}, l_{2,1}, l_{2,t} \). We distinguish three cases.

Case a1: \( k < 4m - 2 \). If \( t = 1 \) then the vertices in \( V_1 \) induce a subgraph having only isolated vertices. In this case, we shall add an edge \( uv \) to \( F \), where \( u, v \in V_1 \) and denote \( L_2 = u, v \). In this way we can define inductively a system of paths \( L_1, L_2, \ldots, L_m \) such that \( L_i \) is a longest path in \( F[V_{i-1}] \), where \( V_{i-1} = V(F) \setminus \cup_{j=1}^{i-1} V(L_j) \) or an edge added to \( F \) as above. If \( F_1 \) denotes the graph \( F \) or the graph \( F \) plus some edges added in the process of defining the system of paths, it follows that endpoints of these \( L_j \) (\( 1 \leq j \leq m \)) induce in \( \overline{F_1} \) a complete graph \( K_{2m} \) minus a matching having at most \( m \) edges if some of the endpoints of the same \( L_j \) are adjacent in \( F_1 \). If \( Y \) denotes the set of the remaining vertices, we have \( |V(Y)| \geq 2n + \frac{m}{2} - 2 - m(4m - 3) > \frac{m}{2} \geq 2 \).

Let \( x \) be one vertex which is not adjacent to any endpoint of these \( L_j \) for \( 1 \leq j \leq m \). It is easy to see that \( x \) together with all endpoints of paths \( L_j \) contains a \( SF_m \subset F_1 \subset \overline{F} \) having the hub \( x \).

Case a2: \( k \geq 4m - 2 \) and \( t \geq 4m - 2 \). In this case we define \( m - 1 \) quadruples \( A_i \) in path \( L_1 \) as follows:

\[
A_1 = \{l_{1,2}, l_{1,3}, l_{1,4}, l_{1,5}\},
A_2 = \{l_{1,6}, l_{1,7}, l_{1,8}, l_{1,9}\},
\vdots
A_{m-1} = \{l_{1,4m-6}, l_{1,4m-5}, l_{1,4m-4}, l_{1,4m-3}\}.
\]
In a similar way let

\[ B_1 = \{l_{2,2}, l_{2,3}, l_{2,4}, l_{2,5}\}, \]
\[ B_2 = \{l_{2,6}, l_{2,7}, l_{2,8}, l_{2,9}\}, \]
\[ \vdots \]
\[ B_{m-1} = \{l_{2,4m-6}, l_{2,4m-5}, l_{2,4m-4}, l_{2,4m-3}\}. \]

for the path \( L_2 \).

Since \( V_2 = V(F) \setminus (V(L_1) \cup V(L_2)) \), we have \( |V_2| \geq \frac{m}{2} \) since \( t, k \leq n - 1 \). Hence, we can consider \( \frac{m}{2} \) distinct elements in \( V_2 \): \( y_1, y_2, \ldots, y_{\frac{m}{2}} \) and \( \frac{m}{2} - 1 \) pairs of elements \( Y_i = \{y_i, y_{i+1}\} \) for \( i = 1, 2, \ldots, \frac{m}{2} - 1 \). By the maximality of \( L_2 \) it follows that for each \( i = 1, 2, \ldots, \frac{m}{2} - 1 \), at least one vertex in \( B_i \) is not adjacent to any vertex in \( Y_i \). Denote by \( b_i \) vertices in \( B_i \) which are not adjacent to any vertex in \( Y_i \) for \( i = 1, 2, \ldots, \frac{m}{2} - 1 \). It follows that \( l_{2,1}, y_1, b_1, y_2, b_2, \ldots, y_{\frac{m}{2}-1}, b_{\frac{m}{2}-1}, y_{\frac{m}{2}} \) is an \( m \)-cycle in \( \overline{F} \) and this cycle together with vertex \( l_{1,1} \) induces \( W_m \) in \( \overline{F} \).

By the maximality of \( L_1 \) we get that for any \( i = 1, \ldots, m - 1 \) and any two different vertices \( z_1, z_2 \in V_1 = V(F) \setminus V(L_1) \), there exists at least one vertex \( a_i \) in \( A_i \) that is not adjacent to \( z_1 \) nor to \( z_2 \). Hence, we can choose an additional vertex set \( \{a_1, a_2, \ldots, a_{m-1}, l_{1,1}\} \) which together with \( W_m \) induces a graph in \( \overline{F} \) which contains \( SF_m \), thus \( SF_m \subset \overline{F} \).

Case a3: \( k \geq 4m - 2 \) and \( t < 4m - 2 \). Since \( F \) has no \( P_n \) it follows that \( k \leq n - 1 \), hence \( V_1 \) will have at least \( n + \frac{m}{2} - 1 \) vertices. Then we can define the same process as in case a1. We obtain a system of paths \( L_2, \ldots, L_m \), in the subgraph induced by \( V_1 \) such that the endpoints of \( L_1, \ldots, L_m \), induce in \( \overline{F}_1 \) a complete graph \( K_{2m} \) minus a matching having at most \( m \) edges. We get in this case \( |V(Y)| \geq n + \frac{m}{2} - 1 - (m - 1)(4m - 3) \geq 2 \) and the proof is similar to the case a1. \( \square \)

**Theorem.** For all \( n \geq 3 \), \( R(P_n, SF_3) = 3n - 2 \).

**Proof.** To show the lower bound, consider graph \( F_1 = 3K_{n-1} \). We have \( \overline{F}_1 \cong K_{n-1, n-1, n-1} \), hence its chromatic number \( \chi(\overline{F}_1) = 3 \), but \( \chi(SF_3) = 4 \), which implies that \( SF_3 \not\subseteq \overline{F}_1 \). It follows that \( R(P_n, SF_3) \geq 3n - 2 \). For the reverse inequality, let us consider a graph \( F \) of order \( 3n - 2 \) such that \( F \) does not contain path \( P_n \), we will show that \( \overline{F} \) contains sunflower graph \( SF_3 \). Let \( P \) be a longest path in \( F \) with endpoints \( p_1 \) and \( p_2 \). Obviously, \( xp_1, xp_2 \not\in E(F) \) for each \( x \in X = V(F) \setminus V(P) \). Let \( Q \) be a longest path in \( F[X] \) with \( q_1 \) and \( q_2 \) as its endpoints. Then \( xq_1, xq_2 \not\in E(F) \) for each \( x \not\in V(P) \cup V(Q) \). Let \( Y = V(F) \setminus (V(P) \cup V(Q)) \) and \( R \) be a longest path in \( F[Y] \) with \( r_1 \) and \( r_2 \) as its endpoints. Since \( |V(F)| = 3n - 2 \) and the longest path in \( F \) is of length
less than or equal to \( n - 1 \) then there exists a vertex \( a \notin V(P) \cup V(Q) \cup V(R) \) such that \( a \) is not adjacent to any endpoint \( p_1, p_2, q_1, q_2, r_1 \) and \( r_2 \). Thus, we give mapping yielding \( SF_3 \) in \( \overline{F} \) with \( a \) as hub. \( \Box \)

**Theorem.** If \( m \geq 5 \) is odd and \( n \geq 2m^2 - 9m + 11 \) then \( R(P_n, SF_m) = 3n - 2 \).

**Proof.** By using an argument similar as above we have \( R(P_n, SF_m) \geq 3n - 2 \). To prove \( R(P_n, SF_m) \leq 3n - 2 \), let \( F \) be a graph on \( 3n - 2 \) vertices containing no \( P_n \). Let \( L_1 = l_{1,1}, l_{1,2}, \ldots, l_{1,k} \) be a longest path in \( F \) and so \( k \leq n - 1 \). If \( k = 1 \) we have \( \overline{F} \cong K_{3n-2} \), which contains \( SF_m \). Suppose that \( k \geq 2 \) and \( \overline{F} \) does not contain \( SF_m \). Obviously, \( zl_{1,1}, zl_{1,k} \) are not in \( E(F) \) for each \( z \in V_1 \), where \( V_1 = V(F) \setminus V(L_1) \). Let \( L_2 = l_{2,1}, l_{2,2}, \ldots, l_{2,t} \) be a longest path in \( F[V_1] \). If \( t = 1 \) we have \( \overline{F} \cong K_{2n-1} \), which contains \( SF_m \), so we may suppose \( 2 \leq t \leq k \). Let \( V_2 = V(F) \setminus (V(L_1) \cup V(L_2)) \). Obviously, \( yl_{2,1}, yl_{2,t} \) are not in \( E(F) \) for each \( y \in V_2 \). Let \( L_3 = l_{3,1}, l_{3,2}, \ldots, l_{3,s} \) be a longest path in \( F[V_2] \). Since \( |V(F)| = 3n - 2 \) and the longest path in \( F \) is of length less than or equal to \( n - 1 \) then there exists a vertex \( x \notin V(L_1) \cup V(L_2) \cup V(L_3) \) such that \( x \) is not adjacent to any endpoint \( l_{1,1}, l_{2,1}, l_{3,1}, l_{1,k}, l_{2,t} \) and \( l_{3,s} \). We distinguish four cases.

Case 1: \( k < 2m - 4 \). It follows that \( t < 2m - 4 \). If \( s = 1 \) then the vertices in \( V_2 \) induce a subgraph having only isolated vertices. In this case we shall add an edge \( uv \) to \( F \), where \( u, v \in V_2 \) and denote \( L_3 = u, v \). In this way we can define inductively as in proof of theorem 2 the system of paths \( L_1, L_2, \ldots, L_m \) such that \( L_i \) is a longest path in \( F[V_{i-1}] \), where \( V_{i-1} = V(F) \setminus \bigcup_{j=1}^{i-1} V(L_j) \) or an edge added to \( F \) as above. If \( F_1 \) denotes the graph \( F \) or the graph \( F \) plus some edges added in the process of defining the system of paths, it follows that endpoints of these \( L_j \), where \( j = 1, 2, \ldots, m \) induce in \( \overline{F_1} \) a complete graph \( K_{2m} \) minus a matching having at most \( m \) edges if some of the endpoints of the same \( L_j \) are adjacent in \( F_1 \). For \( m \geq 5 \) there exists at least one vertex \( x \) which is not adjacent to any endpoint of these \( L_j \). Thus, it is easy to see that vertex \( x \) together with all endpoints of paths \( L_j \) form a \( SF_m \subset F_1 \subset \overline{F} \).

Case 2: \( k \geq 2m - 4, t \geq 2m - 4 \) and \( s \geq 2m - 4 \). For \( i = 1, 2, \ldots, m - 3 \) define the couples \( A_i \) in path \( L_1 \) as follows:

\[
A_i = \begin{cases} 
\{l_{1,i+1}, l_{1,i+2}\} & \text{for } i \text{ odd}, \\
\{l_{1,k-i}, l_{1,k-i+1}\} & \text{for } i \text{ even}.
\end{cases}
\]

Similarly, define couples \( B_i, C_i \) in paths \( L_2 \) and \( L_3 \), respectively as follows:

\[
B_i = \begin{cases} 
\{l_{2,i+1}, l_{2,i+2}\} & \text{for } i \text{ odd}, \\
\{l_{2,t-i}, l_{2,t-i+1}\} & \text{for } i \text{ even}.
\end{cases}
\]

\[
C_i = \begin{cases} 
\{l_{3,i+1}, l_{3,i+2}\} & \text{for } i \text{ odd}, \\
\{l_{3,t-i}, l_{3,t-i+1}\} & \text{for } i \text{ even}.
\end{cases}
\]
\[ C_i = \begin{cases} 
\{l_{3,i+1}, l_{3,i+2}\} & \text{for i odd,} \\
\{l_{3,s-i}, l_{3,s-i+1}\} & \text{for i even.} 
\end{cases} \]

We have seen that since \( s \leq t \leq k \leq n - 1 \) and \(|F| = 3n - 2\), there exists at least one vertex \( x \) which is not in \( L_1 \cup L_2 \cup L_3 \). Since \( L_1 \) is a longest path in \( F \), there exists one vertex of \( A_i \) for each \( i \), say \( a_i \) which is not adjacent with \( x \). Similarly, we obtain vertices \( b_i \) and \( c_i \) in couples \( B_i \) and \( C_i \) which are not adjacent to \( x \) for every \( i = 1, \ldots, m-3 \). The maximality of the paths \( L_1 \) and \( L_2 \) also implies that for every \( i, j, k = 1, \ldots, m-3 \) we have \( a_ib_j, a_ic_k, bjc_k \notin E(F) \). Thus, vertex set \( \{l_{1,1}, l_{2,t}, a_1, b_1, a_2, b_2, \ldots, a_{m-3}, b_{m-3}, l_{3,1}\} \) with vertex \( x \) will contain \( W_m \) in \( \overline{F} \) and additional vertex set \( \{l_{3,s}, c_1, c_2, c_3, \ldots, c_{m-3}, l_{1,k}, l_{2,1}\} \) with wheel \( W_m \), gives \( SF_m \subset \overline{F} \).

Case 3: \( k \geq 2m - 4 \), \( t \geq 2m - 4 \) and \( s < 2m - 4 \). Since \( F \) has no \( P_n \) it follows that \( t \leq k \leq n - 1 \). Consequently, \( V_2 \) will have at least \( n \) vertices. Then we can define the same process as in case 1. We obtain a system of paths \( L_3, \ldots, L_m \) in the subgraph induced by \( V_2 \) such that the endpoints of \( L_1, \ldots, L_m \) induce in \( \overline{F_1} \) a complete graph \( K_{2m} \) minus a matching having at most \( m \) edges. We get in this case \(|V(Y)| \geq n - (m - 2)(2m - 5) \geq 1 \) and the proof is similar to the case 1.

Case 4: \( k \geq 2m - 4 \) and \( t < 2m - 4 \). We deduce that \( s < 2m - 4 \). Since \( F \) has no \( P_n \) it follows that \( k \leq n - 1 \). Consequently, \( V_1 \) will have at least \( 2n - 1 \) vertices. Then we can define the same process as in case 1. We obtain a system of paths \( L_2, \ldots, L_m \) in the subgraph induced by \( V_1 \) such that the endpoints of \( L_1, \ldots, L_m \) induce in \( \overline{F_1} \) a complete graph \( K_{2m} \) minus a matching having at most \( m \) edges. We get in this case \(|V(Y)| \geq 2n - 1 - (m - 1)(2m - 5) > 1 \) and the proof is similar to the case 1. \( \square \)

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Received 29 November 2013

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