

ON TREES WITH EQUAL TOTAL DOMINATION AND 2-OUTER-INDEPENDENT DOMINATION NUMBERS

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For a graph $G = (V, E)$, a subset $D \subseteq V(G)$ is a total dominating set if every vertex of G has a neighbor in D . The total domination number of G is the minimum cardinality of a total dominating set of G . A subset $D \subseteq V(G)$ is a 2-dominating set of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D , while it is a 2-outer-independent dominating set of G if additionally the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of G is the minimum cardinality of a 2-outer-independent dominating set of G . We characterize all trees with equal total domination and 2-outer-independent domination numbers.

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1. INTRODUCTION

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v , denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set. The path on n vertices we denote by P_n . Let T be a tree, and let v be a vertex of T . We say that v is adjacent to a tree H if there is a neighbor of v , say x , such that the tree resulting from T by removing the edge vx , and which contains the vertex x , is a tree H . We say that v is adjacent to a path P_n if there is a neighbor of v , say x , such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n . By a star we mean a connected graph in which exactly one vertex has degree greater than one.

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A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) \setminus D$ has a neighbor in D , while it is a total dominating set, abbreviated TDS, of G if every vertex of G has a neighbor in D . The domination (total domination, respectively) number of G , denoted by $\gamma(G)$ ($\gamma_t(G)$, respectively), is the minimum cardinality of a dominating (total dominating, respectively) set of G . A total dominating set of G of minimum cardinality is called a $\gamma_t(G)$ -set. Total domination in graphs was introduced by Cockayne, Dawes, and Hedetniemi [6], and further studied for example in [1–3, 6–8, 10, 15, 19, 20]. For a comprehensive survey of domination in graphs, see [14].

A subset $D \subseteq V(G)$ is a 2-dominating set, abbreviated 2DS, of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D . The 2-domination number of G , denoted by $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set of G . Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least k times for a fixed positive integer k . Multiple domination in graphs was introduced by Fink and Jacobson [9], and further studied for example in [4, 5, 9, 11, 12, 17, 18].

A subset $D \subseteq V(G)$ is a 2-outer-independent dominating set, abbreviated 2OIDS, of G if every vertex of $V(G) \setminus D$ has at least two neighbors in D and the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of G , denoted by $\gamma_2^{oi}(G)$, is the minimum cardinality of a 2-outer-independent dominating set of G . A 2-outer-independent dominating set of G of minimum cardinality is called a $\gamma_2^{oi}(G)$ -set. The study of 2-outer-independent domination in graphs was initiated in [16].

We characterize all trees with equal total domination and 2-outer-independent domination numbers.

2. RESULTS

Since the one-vertex graph does not have a total dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following three straightforward observations.

OBSERVATION 1. *Every support vertex of a graph G is in every TDS of G .*

OBSERVATION 2. *For every connected graph G of diameter at least three there exists a $\gamma_t(G)$ -set that contains no leaf.*

OBSERVATION 3. *Every leaf of a graph G is in every 2OIDS of G .*

We now prove that the 2-outer-independent domination number of any tree is greater than or equal to its total domination number.

LEMMA 4. *For every tree T we have $\gamma_2^{oi}(T) \geq \gamma_t(T)$.*

Proof. Since every 2OIDS of a tree T is a 2DS of this tree, we have $\gamma_2^{oi}(T) \geq \gamma_2(T)$. In [13] it is proved that for every tree T we have $\gamma_2(T) \geq \gamma_t(T)$. We now get $\gamma_2^{oi}(T) \geq \gamma_2(T) \geq \gamma_t(T)$. \square

We characterize all trees with equal total domination and 2-outer-independent domination numbers. For this purpose, we introduce a family \mathcal{T} of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_2, P_3\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation \mathcal{O}_1 : Attach a path P_2 by joining one of its vertices to a vertex of T_k adjacent to a path P_2 .
- Operation \mathcal{O}_2 : Attach a path P_3 by joining one of its leaves to a vertex of T_k , which is not a leaf and is adjacent to a path P_3 or to a support vertex.
- Operation \mathcal{O}_3 : Attach a path P_5 by joining one of its support vertices to a vertex of T_k , which is adjacent to a path P_2 or to a path P_5 through a support vertex.
- Operation \mathcal{O}_4 : Attach a path P_4 by joining one of its leaves to a vertex of T_k , which is a leaf, or is adjacent to a path P_2 or P_4 or to a path P_5 through a support vertex.
- Operation \mathcal{O}_5 : Let x be a vertex of T_k adjacent to a leaf, say y , and to a path P_4 , say $abcd$. Let a and x be adjacent. Remove the leaf y . Then either attach a path P_3 by joining one of its leaves to the vertex c , or attach a path P_3 by joining one of its leaves to the vertex b and a path P_2 by joining one of its vertices to the vertex d .
- Operation \mathcal{O}_6 : Let x be a support vertex of $T_k \neq P_5$ adjacent to a path P_3 . Remove the path and a leaf adjacent to x , and attach a path P_6 by joining one of its support vertices to the vertex x .

We now prove that for every tree of the family \mathcal{T} , the total domination and the 2-outer-independent domination numbers are equal.

LEMMA 5. *If $T \in \mathcal{T}$, then $\gamma_t(T) = \gamma_2^{oi}(T)$.*

Proof. We use the induction on the number k of operations performed to construct the tree T . If $T = P_2$, then obviously $\gamma_t(T) = 2 = \gamma_2^{oi}(T)$. If $T = P_3$, then also $\gamma_t(T) = 2 = \gamma_2^{oi}(T)$. Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family \mathcal{T} constructed by $k - 1$ operations. Let $T = T_{k+1}$ be a tree of the family \mathcal{T} constructed by k operations.

First assume that T is obtained from T' by operation \mathcal{O}_1 . The vertex to which is attached P_2 we denote by x . Let v_1v_2 be the attached path. Let v_1 be joined to x . Let yz be a path P_2 adjacent to x and different from v_1v_2 . Let x and y be adjacent. Let us observe that there exists a $\gamma_2^{oi}(T')$ -set that contains the vertex x . Let D' be such a set. It is easy to see that $D' \cup \{v_2\}$ is a 2OIDS of the tree T . Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. Now let D be a $\gamma_t(T)$ -set that contains no leaf. By Observation 1 we have $v_1, y \in D$. Let us observe that $D \setminus \{v_1\}$ is a TDS of the tree T' as the vertex x has a neighbor in $D \setminus \{v_1\}$. Therefore $\gamma_t(T') \leq \gamma_t(T) - 1$. We now get $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1 = \gamma_t(T') + 1 \leq \gamma_t(T)$. On the other hand, by Lemma 4 we have $\gamma_2^{oi}(T) \geq \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$.

Now assume that T is obtained from T' by operation \mathcal{O}_2 . The vertex to which is attached P_3 we denote by x . Let $v_1v_2v_3$ be the attached path. Let v_1 be joined to x . Let D' be a $\gamma_2^{oi}(T')$ -set. It is easy to observe that $D' \cup \{v_1, v_3\}$ is a 2OIDS of the tree T . Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 2$. First assume that x is adjacent to a path P_3 , say abc . Let a and x be adjacent. Let D be a $\gamma_t(T)$ -set that contains no leaf. By Observation 1 we have $v_2 \in D$. Each one of the vertices v_2 and b has to be dominated, thus $v_1, a \in D$. Let us observe that $D \setminus \{v_1, v_2\}$ is a TDS of the tree T' . Now assume that x is adjacent to a support vertex, say y . Let D be a $\gamma_t(T)$ -set that contains no leaf. By Observation 1 we have $v_2, y \in D$. The vertex v_2 has to be dominated, thus $v_1 \in D$. Let us observe that now also $D \setminus \{v_1, v_2\}$ is a TDS of the tree T' . We conclude that $\gamma_t(T') \leq \gamma_t(T) - 2$. We now get $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 2 = \gamma_t(T') + 2 \leq \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$.

Now assume that T is obtained from T' by operation \mathcal{O}_3 . The vertex to which is attached P_5 we denote by x . Let $v_1v_2v_3v_4v_5$ be the attached path. Let v_2 be joined to x . Let y be a support vertex adjacent to x and different from v_2 . Let us observe that there exists a $\gamma_2^{oi}(T')$ -set that contains the vertex x . Let D' be such a set. It is easy to observe that $D' \cup \{v_1, v_3, v_5\}$ is a 2OIDS of the tree T . Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 3$. Now let D be a $\gamma_t(T)$ -set that contains no leaf. By Observation 1 we have $v_4, v_2, y \in D$. The vertex v_4 has to be dominated, thus $v_3 \in D$. Let us observe that $D \setminus \{v_2, v_3, v_4\}$ is a TDS of the tree T' . Therefore $\gamma_t(T') \leq \gamma_t(T) - 3$. We now get $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 3 = \gamma_t(T') + 3 \leq \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$.

Now assume that T is obtained from T' by operation \mathcal{O}_4 . The vertex to which is attached P_4 we denote by x . Let $v_1v_2v_3v_4$ be the attached path. Let v_1 be joined to x . Let us observe that there exists a $\gamma_2^{oi}(T')$ -set that contains the vertex x . Let D' be such a set. It is easy to observe that $D' \cup \{v_2, v_4\}$ is a 2OIDS of the tree T . Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 2$. Now let us observe that there exists a $\gamma_t(T)$ -set that does not contain the vertices v_4 and v_1 . Let D be such

a set. By Observation 1 we have $v_3 \in D$. The vertex v_3 has to be dominated, thus $v_2 \in D$. Observe that $D \setminus \{v_2, v_3\}$ is a TDS of the tree T' . Therefore $\gamma_t(T') \leq \gamma_t(T) - 2$. We now get $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 2 = \gamma_t(T') + 2 \leq \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$.

Now assume that T is obtained from T' by operation \mathcal{O}_5 . The attached path P_3 we denote by $v_1v_2v_3$. Let v_1 be joined to b or c . If we also attach a path P_2 , then we denote it by v_4v_5 . Let v_4 be joined to d . Let D' be a $\gamma_2^{oi}(T')$ -set that contains the vertices b and x . By Observation 3 we have $d, y \in D'$. If we only attach a path P_3 , then let us observe that $D' \setminus \{y\} \cup \{v_1, v_3\}$ is a 2OIDS of the tree T . If we also attach a path P_2 , then let us observe that $D' \setminus \{y\} \cup \{v_1, v_3, v_5\}$ is a 2OIDS of the tree T . If we only attach a path P_3 , then let us observe that there exists a $\gamma_t(T)$ -set that does not contain the vertices v_3, d, b and a . Let D be such a set. By Observation 1 we have $v_2, c \in D$. Each one of the vertices v_2 and a has to be dominated, thus $v_1, x \in D$. Let us observe that $D \cup \{b\} \setminus \{v_1, v_2\}$ is a TDS of the tree T' . If we also attach a path P_2 , then let us observe that there exists a $\gamma_t(T)$ -set that does not contain the vertices v_5, v_3, c, b and a . Let D be such a set. By Observation 1 we have $v_2, v_4 \in D$. Each one of the vertices v_2, v_4 , and a has to be dominated, thus $v_1, d, x \in D$. Let us observe that $D \cup \{b, c\} \setminus \{v_1, v_2, d, v_4\}$ is a TDS of the tree T' . We now conclude that $\gamma_2^{oi}(T) + \gamma_t(T') \leq \gamma_2^{oi}(T') + \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$.

Now assume that T is obtained from T' by operation \mathcal{O}_6 . The attached path we denote by $v_1v_2v_3v_4v_5v_6$. Let v_2 be joined to x . Let y be a leaf adjacent to x , and which is being removed. Let abc denote a path P_3 adjacent to x , and which is being removed. Let a and x be adjacent. Let D' be a $\gamma_2^{oi}(T')$ -set that contains the vertex a . By Observation 3 we have $c, y \in D'$. The set D' is minimal, thus $b \notin D'$. If $x \in D'$, then it is easy to observe that $D' \setminus \{a, c, y\} \cup \{v_1, v_2, v_4, v_6\}$ is a 2OIDS of the tree T . Now assume that $x \notin D'$. Since $T_k \neq P_5$, the vertex x has at least three neighbors in the tree T' . Let z be a neighbor of x other than a and y . We have $z \in D'$ as the set $V(T') \setminus D'$ is independent. Let us observe that now also $D' \setminus \{a, c, y\} \cup \{v_1, v_2, v_4, v_6\}$ is a 2OIDS of the tree T . Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1$. Now let D be a $\gamma_t(T)$ -set that does not contain the vertices v_6, v_3 and v_1 . Let D be such a set. By Observation 3 we have $v_5, v_2 \in D$. Each one of the vertices v_5 and v_2 has to be dominated, thus $v_4, x \in D$. Let us observe that $D \cup \{a, b\} \setminus \{v_2, v_4, v_5\}$ is a TDS of the tree T' . Therefore $\gamma_t(T') \leq \gamma_t(T) - 1$. We now get $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + 1 = \gamma_t(T') + 1 \leq \gamma_t(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_t(T)$. \square

We now prove that if the total domination and the 2-outer-independent domination numbers of a tree are equal, then the tree belongs to the family \mathcal{T} .

LEMMA 6. *Let T be a tree. If $\gamma_t(T) = \gamma_2^{oi}(T)$, then $T \in \mathcal{T}$.*

Proof. If $\text{diam}(T) = 1$, then $T = P_2 \in \mathcal{T}$. Now assume that $\text{diam}(T) = 2$. Thus T is a star. If $T = P_3$, then $T \in \mathcal{T}$. Now assume that T is a star different from P_3 . We have $\gamma_t(T) = 2 < n - 1 = \gamma_2^{oi}(T)$.

Now assume that $\text{diam}(T) \geq 3$. Thus the order n of the tree T is at least four. We obtain the result by the induction on the number n . Assume that the lemma is true for every tree T' of order $n' < n$.

First assume that some support vertex of T , say x , is strong. Let y and z be leaves adjacent to x . Let $T' = T - y$. Let D' be a $\gamma_t(T')$ -set. By Observation 1 we have $x \in D'$. It is easy to see that D' is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T')$. Now let D be a $\gamma_2^{oi}(T)$ -set. By Observation 3 we have $y, z \in D$. If $x \in D$, then it is easy to observe that $D \setminus \{y\}$ is a 2OIDS of the tree T' . Now assume that $x \notin D$. Let k be a neighbor of x other than y and z . The set $V(T) \setminus D$ is independent, thus $k \in D$. Let us observe that now also $D \setminus \{y\}$ is a 2OIDS of the tree T' as the vertex x has at least two neighbors in $D \setminus \{y\}$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. We now get $\gamma_t(T') \geq \gamma_t(T) = \gamma_2^{oi}(T) \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$. This is a contradiction as by Lemma 4 we have $\gamma_t(T') \leq \gamma_2^{oi}(T')$. Therefore every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r , v be the parent of t , and u be the parent of v in the rooted tree. If $\text{diam}(T) \geq 4$, then let w be the parent of u . If $\text{diam}(T) \geq 5$, then let d be the parent of w . If $\text{diam}(T) \geq 6$, then let e be the parent of d . If $\text{diam}(T) \geq 7$, then let f be the parent of e . By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree T .

Assume that $d_T(u) \geq 3$. Let x be a child of u other than v . First assume that x is a leaf. Let $T' = T - x$. Let D' be a $\gamma_t(T')$ -set that contains no leaf. The vertex v has to be dominated, thus $u \in D'$. It is easy to see that D' is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T')$. Now let us observe that there exists a $\gamma_2^{oi}(T)$ -set that contains the vertex u . Let D be such a set. By Observation 3 we have $x \in D$. It is easy to observe that $D \setminus \{x\}$ is a 2OIDS of the tree T' . Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. We now get $\gamma_t(T') \geq \gamma_t(T) = \gamma_2^{oi}(T) \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$, a contradiction.

Thus x is a support vertex of degree two. Let $T' = T - T_v$. Let D' be a $\gamma_t(T')$ -set that contains no leaf. The vertex x has to be dominated, thus $u \in D'$. It is easy to see that $D' \cup \{v\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let us observe that there exists a $\gamma_2^{oi}(T)$ -set that does not contain the vertex v . Let D be such a set. By Observation 3 we have $t \in D$. Observe that $D \setminus \{t\}$ is a 2OIDS of the tree T' . Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. We now get $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2^{oi}(T) - 1 \geq \gamma_2^{oi}(T')$. On the other hand, by Lemma 4 we have $\gamma_t(T') \leq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive

hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_1 . Thus $T \in \mathcal{T}$.

Now assume that $d_T(u) = 2$. First assume that there is a child of w other than u , say k , such that the distance of w to the most distant vertex of T_k is three or two. It suffices to consider only the possibilities when T_k is a path P_3 or P_2 . Let $T' = T - T_u$. Let D' be a $\gamma_t(T')$ -set. It is easy to observe that $D' \cup \{u, v\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let D be a $\gamma_2^{oi}(T)$ -set that contains the vertex u . By Observation 3 we have $t \in D$. The set D is minimal, thus $v \notin D$. If $w \in D$, then it is easy to observe that $D \setminus \{u, t\}$ is a 2OIDS of the tree T' . Now assume that $w \notin D$. The set $V(T) \setminus D$ is independent, thus $k, d \in D$. Let us observe that now also $D \setminus \{u, t\}$ is a 2OIDS of the tree T' as the vertex w has at least two neighbors in $D \setminus \{u, t\}$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_2 . Thus $T \in \mathcal{T}$.

Now assume that some child of w , say x , is a leaf. We can assume that $d_T(w) = 3$. First assume that there is a child of d other than w , say k , such that the distance of d to the most distant vertex of T_k is four or two. It suffices to consider the possibilities when T_k is isomorphic to T_w , or T_k is a path P_4 or P_2 . First assume that T_k is isomorphic to T_w , or T_k is a path P_2 . Let $T' = T - T_w$. Let D' be a $\gamma_t(T')$ -set. It is easy to observe that $D' \cup \{w, u, v\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 3$. Now let D be a $\gamma_2^{oi}(T)$ -set that does not contain the vertices v and w . By Observation 3 we have $t, x \in D$. The vertex u has no neighbor in D , thus $u \in D$. Observe that $D \setminus \{u, t, x\}$ is a 2OIDS of the tree T' . Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 3$. We now get $\gamma_t(T') \geq \gamma_t(T) - 3 = \gamma_2^{oi}(T) - 3 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_3 . Thus $T \in \mathcal{T}$.

Now assume that T_k is a path P_4 , say $klmp$. Let $T' = T - T_k$. Let D' be a $\gamma_t(T')$ -set. It is easy to observe that $D' \cup \{l, m\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let D be a $\gamma_2^{oi}(T)$ -set that does not contain the vertices m and k . By Observation 3 we have $p \in D$. The vertex l has no neighbor in D , thus $l \in D$. Observe that $D \setminus \{l, p\}$ is a 2OIDS of the tree T' . Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_4 . Thus $T \in \mathcal{T}$.

Now assume that there is a child of d , say k , such that the distance of d to the most distant vertex of T_k is three. It suffices to consider only the

possibility when T_k is a path P_3 , say klm . Let $T' = T - T_k$. Similarly as earlier we conclude that $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let D be a $\gamma_2^{oi}(T)$ -set that contains the vertices u and d . By Observation 3 we have $m \in D$. Without loss of generality we assume that $k \in D$ and $l \notin D$. It is easy to observe that $D \setminus \{k, m\}$ is a 2OIDS of the tree T' . Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_2 . Thus $T \in \mathcal{T}$.

Now assume that some child of d , say k , is a leaf. Let $T' = T - T_u - k$. Let D' be a $\gamma_t(T')$ -set that contains no leaf. By Observation 1 we have $w \in D'$. The vertex w has to be dominated, thus $d \in D'$. It is easy to observe that $D' \cup \{u, v\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let D be a $\gamma_2^{oi}(T)$ -set that contains the vertices u and d . By Observation 3 we have $t, x, k \in D$. The set D is minimal, thus $v \notin D$. Let us observe that $D \setminus \{u, t, k\}$ is a 2OIDS of the tree T' . Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 3$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(d) = 2$. Assume that $d_T(e) \geq 3$. Let $T' = T - T_d$. Let D' be a $\gamma_t(T')$ -set. It is easy to observe that $D' \cup \{w, u, v\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 3$. Now let D be a $\gamma_2^{oi}(T)$ -set that contains the vertices u and d . By Observation 3 we have $t, x \in D$. The set D is minimal, thus $v, w \notin D$. If $e \in D$, then it is easy to observe that $D \setminus \{d, u, t, x\}$ is a 2OIDS of the tree T' . Now assume that $e \notin D$. Let k be a neighbor of e other than d and f . The set $V(T) \setminus D$ is independent, thus $f, k \in D$. Let us observe that now also $D \setminus \{d, u, t, x\}$ is a 2OIDS of the tree T' as the vertex e has at least two neighbors in $D \setminus \{d, u, t, x\}$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 4$. We now get $\gamma_t(T') \geq \gamma_t(T) - 3 = \gamma_2^{oi}(T) - 3 \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$, a contradiction.

If $d_T(e) = 1$, then it is easy to verify that $\gamma_t(T) = 4 < 5 = \gamma_2^{oi}(T)$, a contradiction. Now assume that $d_T(e) = 2$. Let T' be a tree obtained from $T - T_u$ by attaching a vertex, say y , by joining it to the vertex f . Let D' be a $\gamma_t(T')$ -set that contains no leaf. By Observation 1 we have $w, f \in D'$. The vertex w has to be dominated, thus $d \in D'$. Let us observe that $D' \setminus \{d\} \cup \{u, v\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let D be a $\gamma_2^{oi}(T)$ -set that contains the vertices u and d . By Observation 3 we have $t, x \in D$. The set D is minimal, thus $v \notin D$. Let us observe that $D \cup \{y\} \setminus \{u, t\}$ is a 2OIDS of the tree T' . Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. We now get $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2^{oi}(T) - 1 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_5 . Thus $T \in \mathcal{T}$.

If $d_T(w) = 1$, then $T = P_4$. We have $\gamma_t(T) = 2 < 3 = \gamma_2^{oi}(T)$. Now assume that $d_T(w) = 2$. If $d_T(d) = 1$, then $T = P_5$. Let $T' = P_3 \in \mathcal{T}$. The tree

T can be obtained from T' by operation \mathcal{O}_1 . Thus $T \in \mathcal{T}$. Now assume that $d_T(d) = 2$, or $d_T(d) \geq 3$ and there is a child of d other than w , say k , such that the distance of d to the most distant vertex of T_k is four or two (then it suffices to consider only the possibilities when T_k is a path P_4 or P_2). Let $T' = T - T_w$. Similarly as earlier we conclude that $\gamma_t(T) \leq \gamma_t(T') + 2$ and $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_4 . Thus $T \in \mathcal{T}$.

Now assume that some child of d , say x , is a leaf. Assume that there is also another child of d , say k , such that the distance of d to the most distant vertex of T_k is three. It suffices to consider only the possibility when T_k is a path P_3 , say klm . Let $T' = T - x - m$. Let D' be a $\gamma_t(T')$ -set that contains no leaf. By Observation 1 we have $k \in D'$. The vertex k has to be dominated, thus $d \in D'$. It is easy to observe that $D' \cup \{l\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let D be a $\gamma_2^{oi}(T)$ -set that contains the vertices u and d . By Observation 3 we have $x, m \in D$. Without loss of generality we assume that $l \in D$ and $k \notin D$. Let us observe that $D \setminus \{x, m\}$ is a 2OIDS of the tree T' . Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2^{oi}(T) - 1 \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(d) = 3$. Let T' be a tree obtained from $T - T_v - x$ by attaching a vertex, say y , by joining it to the vertex e . Let D' be a $\gamma_t(T')$ -set that contains no leaf. By Observation 1 we have $w, e \in D'$. The vertex w has to be dominated, thus $d \in D'$. Let us observe that $D' \setminus \{w\} \cup \{u, v\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Now let D be a $\gamma_2^{oi}(T)$ -set that contains the vertices u and d . By Observation 3 we have $t, x \in D$. The set D is minimal, thus $v \notin D$. Let us observe that $D \cup \{y\} \setminus \{t, x\}$ is a 2OIDS of the tree T' . Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 1$. We now get $\gamma_t(T') \geq \gamma_t(T) - 1 = \gamma_2^{oi}(T) - 1 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_6 . Thus $T \in \mathcal{T}$.

Now assume that for every child of d other than w , say k , the distance of d to the most distant vertex of T_k is three, and consequently, T_k is a path P_3 . Due to the earlier analysis of the children of the vertex w , we may assume that $d_T(d) = 3$. Assume that $d_T(e) \geq 3$. Let $T' = T - T_d$. Let D' be a $\gamma_t(T')$ -set. It is easy to observe that $D' \cup \{u, v, k, l\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 4$. Now let D be a $\gamma_2^{oi}(T)$ -set that contains the vertices u and d . By Observation 3 we have $t, m \in D$. The set D is minimal, thus $v, w \notin D$. Without loss of generality we assume that $k \in D$ and $l \notin D$. If $e \in D$, then it is easy to observe that $D \setminus \{d, u, t, k, m\}$ is a 2OIDS of the tree T' . Now assume that $e \notin D$. Let x be a neighbor of e other than d and f .

The set $V(T) \setminus D$ is independent, thus $f, x \in D$. Let us observe that now also $D \setminus \{d, u, t, k, m\}$ is a 2OIDS of the tree T' . Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 5$. We now get $\gamma_t(T') \geq \gamma_t(T) - 4 = \gamma_2^{oi}(T) - 4 \geq \gamma_2^{oi}(T') + 1 > \gamma_2^{oi}(T')$, a contradiction.

Now assume that $d_T(e) = 2$. Let T' be a tree obtained from $T - T_v - T_k$ by attaching a vertex, say y , by joining it to the vertex f . Let D' be a $\gamma_t(T')$ -set that contains no leaf. By Observation 1 we have $w, f \in D'$. The vertex w has to be dominated, thus $d \in D'$. Let us observe that $D' \setminus \{d, w\} \cup \{u, v, k, l\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Now let D be a $\gamma_2^{oi}(T)$ -set that contains the vertices u and d . By Observation 3 we have $t, m \in D$. The set D is minimal, thus $v \notin D$. Without loss of generality we assume that $k \in D$ and $l \notin D$. Let us observe that $D \cup \{y\} \setminus \{t, k, m\}$ is a 2OIDS of the tree T' . Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - 2$. We now get $\gamma_t(T') \geq \gamma_t(T) - 2 = \gamma_2^{oi}(T) - 2 \geq \gamma_2^{oi}(T')$. This implies that $\gamma_t(T') = \gamma_2^{oi}(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation \mathcal{O}_5 . Thus $T \in \mathcal{T}$. \square

As an immediate consequence of Lemmas 5 and 6, we have the following characterization of the trees with total domination number equal to 2-outer-independent domination number.

THEOREM 7. *Let T be a tree. Then $\gamma_t(T) = \gamma_2^{oi}(T)$ if and only if $T \in \mathcal{T}$.*

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