GENERALIZED PELL SEQUENCES IN SOME PRINCIPAL CONGRUENCE SUBGROUPS OF THE HECKE GROUPS

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In this paper, we consider the Hecke groups $H(\sqrt{m})$ for $m = 1$, 2 and 3. Firstly, we give the generators of the principal congruence subgroups $H_2(\sqrt{m})$ of $H(\sqrt{m})$, respectively. Then, using some of these generators, we define a sequence U_k which is generalized version of the Pell numbers sequence P_k given in [12] for the modular group, in the extended Hecke groups $H(\sqrt{m})$ for $m = 1, 2$ and 3.

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1. INTRODUCTION

In [5], Erich Hecke introduced the groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \qquad \text{and} \qquad S(z) = -\frac{1}{z+\lambda},$$

where λ is a fixed positive real number. E. Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$, q is an integer, $q \geq 3$, or $\lambda \geq 2$. We will focus on the discrete case with $\lambda < 2$. These groups have come to be known as the *Hecke Groups*, and we will denote them $H(\lambda_q)$ for $q \geq 3$. The Hecke group $H(\lambda_q)$ is isomorphic to the free product of two finite cyclic groups of orders 2 and q and it has a presentation

$$H(\lambda_q) =< T, S \mid T^2 = S^q = I >\cong C_2 * C_q. \tag{1}$$

The first several of these groups are $H(\lambda_3) = \Gamma = PSL(2,\mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$. It is clear that $H(\lambda_q) \subset PSL(2,\mathbb{Z}[\lambda_q])$, for $q \geq 4$. The groups $H(\sqrt{2})$ and $H(\sqrt{3})$ are of particular interest, since they are the only Hecke groups, aside from the modular group, whose elements are completely known (see, [11]). Also conjugates of the Hecke groups $H(\sqrt{2})$ and $H(\sqrt{3})$ are commensurable to $H(\lambda_3) = H(1)$. The other $H(\lambda_q)$'s are incommensurable to conjugates of $H(\lambda_3) = H(1)$ and of each other. Thus $H(\sqrt{m})$, $m = 1, 2$ and 3, are called arithmetic as subgroups of $SL(2, \mathbb{R})$. Also these arithmetic Hecke groups have been studied by many authors, for example, see [2], [7] and [8].

Throughout this paper, we identify each matrix A in $SL(2, \mathbb{Z}[\lambda_q])$ with $-A$, so that they each represent the same element of $H(\lambda_q)$. Thus, we can represent the generators of Hecke groups $H(\lambda_q)$ as

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & \lambda_q \end{pmatrix}.$$

The principal congruence subgroups of level p, p prime, of $H(\lambda_q)$ are defined in [6], as

$$\begin{aligned} H_p(\lambda_q) &= \{M \in H(\lambda_q) : M \equiv \pm I \ (mod\ p)\}, \\ &= \left\{ \begin{bmatrix} a & b\lambda_q \\ c\lambda_q & d \end{bmatrix} : a \equiv d \equiv \pm 1, b \equiv c \equiv 0 \ (mod\ p), ad - \lambda_q^2 bc = 1 \right\}. \end{aligned}$$

$H_p(\lambda_q)$ is always a normal subgroup of finite index in $H(\lambda_q)$.

The principal congruence subgroups of Hecke group $H(\sqrt{m})$, $m = 2$ and 3, has been studied by Cangül and Bizim in [3]. They proved that the quotient group of the Hecke group $H(\sqrt{m})$ by its principal congruence subgroup $H_2(\sqrt{m})$ is the dihedral group D_{2m}, *i.e.* :

$$H(\sqrt{m})/H_2(\sqrt{m}) \cong D_{2m}.$$

In the literature, principal congruence subgroups $H_2(\lambda_3)$ of $H(\lambda_3)$ have been extensively studied in many aspects, see [1], [4], [9] and [12]. It is known that principal congruence subgroup $H_2(\lambda_3)$ is generated by

$$a_1 = TSTS = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } a_2 = TS^2TS^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

In [12], they proved that if $A(g)$ is the matrix representing of the element $g = (a_1.a_2)^k = ((TS)^2(TS^{-1})^2)^k$, $k \geq 1$, which is product of the generators of $H_2(\lambda_3)$, and if $g \in H(\lambda_3)$ act on a real quadratic irrational number α, then

$$A(g) = \begin{bmatrix} P_{2k-1} & P_{2k} \\ P_{2k} & P_{2k+1} \end{bmatrix},$$

where P_k is the k^{th} Pell number. It is well-known that the Pell numbers are defined by the recurrence relation $P_0 = 0$, $P_1 = 1$ and $P_k = 2P_{k-1} + P_{k-2}$, for $k \geq 2$. The Pell-Lucas numbers are defined by the recurrence relation $Q_0 = 2$, $Q_1 = 2$ and $Q_k = 2Q_{k-1} + Q_{k-2}$, for $k \geq 2$. The Pell-Lucas number can be also expressed by $Q_k = 2P_{k-1} + 2P_k$.

The aim of this paper is to generalize results given in [12] for the modular group to the Hecke groups $H(\sqrt{m})$ for $m = 1, 2$ and 3. To do these, firstly, we give the generators of the principal congruence subgroups $H_2(\sqrt{m})$ of $H(\sqrt{m})$. Then, using some of these generators, we define a sequence which is generalized version of the Pell numbers sequence given in [12] for the modular group, in Hecke groups $H(\sqrt{m})$ for $m = 1, 2$ and 3. Finally, we investigate the fixed points of the transformations $((TS^{-1})^2(TS)^2)^k$ and $((TS)^2(TS^{-1})^2)^k$ in $Q(\sqrt{d})$.

2. GENERALIZED PELL NUMBERS IN $H_2(\lambda_q)$ FOR $q = 3,4$ AND 6

First, we give the group structure of the principal congruence subgroup $H_2(\lambda_q)$ of Hecke group $H(\lambda_q)$ for $q = 3$, 4 and 6.

THEOREM 1. *If $q = 3, 4$ and 6, then the principal congruence subgroup $H_2(\lambda_q)$ of $H(\lambda_q)$ is the free product of $(q-1)$ infinite cyclic groups.*

Proof. We have

$$H(\lambda_q)/H_2(\lambda_q) \cong \langle T, S \mid T^2 = S^q = (TS)^2 = I \rangle .$$

Hence we obtain

$$H(\lambda_q)/H_2(\lambda_q) \cong D_q, \ ([10])$$

and

$$|H(\lambda_q) : H_2(\lambda_q)| = 2q.$$

If we choose a Schreier transversal for $H_2(\lambda_q)$ as

$$I, T, S, S^2, \cdots, S^{q-1}, TS, TS^2, ..., TS^{q-2}, ST.$$

Then all possible products are

$$\begin{array}{ll}
I.T.(T)^{-1} = I, & I.S.(S)^{-1} = I, \\
T.T.(I)^{-1} = I, & T.S.(TS)^{-1} = I, \\
S.T.(ST)^{-1} = I, & S.S.(S^2)^{-1} = I, \\
S^2.T.(TS^{q-2})^{-1} = S^2TS^2T, & S^2.S.(S^3)^{-1} = I, \\
\vdots & \vdots \\
S^{q-1}.T.(TS)^{-1} = S^{q-1}TS^{q-1}T, & S^{q-1}.S.(I)^{-1} = I, \\
TS.T.(S^{q-1})^{-1} = TSTS, & TS.S.(TS^2)^{-1} = I, \\
TS^2.T.(S^{q-2})^{-1} = TS^2TS^2, & TS^2.S.(TS^3)^{-1} = I, \\
\vdots & \vdots \\
TS^{q-2}.T.(S^2)^{-1} = TS^{q-2}TS^{q-2}, & TS^{q-2}.S.(ST)^{-1} = TS^{q-1}TS^{q-1}, \\
ST.T.(S)^{-1} = I, & ST.S.(T)^{-1} = STST,
\end{array}$$

The generators $H_2(\lambda_q)$ are $TSTS, TS^2TS^2, \cdots, TS^{q-1}TS^{q-1}$. Thus $H_2(\lambda_q)$ has a presentation

$$H_2(\lambda_q) = \langle TSTS \rangle * \langle TS^2TS^2 \rangle * \cdots * \langle TS^{q-1}TS^{q-1} \rangle$$

Here, using the permutation method and Riemann-Hurwitz formula, we also get the signature of $H_2(\lambda_q)$ as $(0;\infty^{(2m)})$. $\square$

Thus the principal congruence subgroup $H_2(\lambda_q)$, $q=3,4$ or 6, of $H(\lambda_q)$ is the free product of $(q-1)$ finite cyclic groups of order infinite and it is generated by

$$a_1 = TSTS, a_2 = TS^2TS^2, ..., a_{q-1} = TS^{q-1}TS^{q-1}.$$

Now, we give some generalizations of the Pell numbers and the Pell-Lucas numbers. To do this, we use the generators $a_1 = TSTS$ and $a_{q-1} = TS^{-1}TS^{-1}$ of $H_2(\lambda_q)$ of $H(\lambda_q)$, $q=3,4$ and 6. Here we replace λ_q, $q=3,4$ or 6 with $\sqrt{m}$, $m=1,2$ and 3, respectively. Then we have the matrix representation of $a_1=(TS)^2$ and $a_{q-1}=(TS^{-1})^2$ as

$$\begin{bmatrix} 1 & 2\sqrt{m} \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 \\ 2\sqrt{m} & 1 \end{bmatrix}.$$

Therefore we obtain the matrix representation of the product $a_{q-1}.a_1=(TS^{-1})^2 .(TS)^2$ as

$$A=\begin{bmatrix} 1 & 2\sqrt{m} \\ 2\sqrt{m} & 1+4m \end{bmatrix}.$$

Then, we can show the following lemma.

Lemma 2. *The k th power of A is*

$$A^k=\begin{bmatrix} U_{2k-1} & U_{2k} \\ U_{2k} & U_{2k+1} \end{bmatrix},$$

where $U_0=0$, $U_1=1$ and $U_k=2\sqrt{m}U_{k-1}+U_{k-2}$, for $k\geq 2$.

Proof. In order to prove its we use induction method on k. Let

$$A=\begin{bmatrix} U_1 & U_2 \\ U_2 & U_3 \end{bmatrix}$$

and

$$A^k=\begin{bmatrix} U_{2k-1} & U_{2k} \\ U_{2k} & U_{2k+1} \end{bmatrix}.$$

Then we have

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 2\sqrt{m} \\ 2\sqrt{m} & 1+4m \end{bmatrix}.\begin{bmatrix} 1 & 2\sqrt{m} \\ 2\sqrt{m} & 1+4m \end{bmatrix} \\ &= \begin{bmatrix} 1+4m & 2\sqrt{m}(1+4m)+2\sqrt{m} \\ 2\sqrt{m}(1+4m)+2\sqrt{m} & 4m+(4m+1)^2 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} U_3 & U_4 \\ U_4 & U_5 \end{bmatrix}.$$

Hence assertion is true for $k = 2$. Now, let us assume that

$$A^{k-1} = \begin{bmatrix} U_{2k-3} & U_{2k-2} \\ U_{2k-2} & U_{2k-1} \end{bmatrix}.$$

Finally A_k is obtained as

$$\begin{aligned} A^k &= \begin{bmatrix} U_{2k-3} & U_{2k-2} \\ U_{2k-2} & U_{2k-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 2\sqrt{m} \\ 2\sqrt{m} & 1+4m \end{bmatrix} \\ &= \begin{bmatrix} U_{2k-3} + 2\sqrt{m}(U_{2k-2}) & 2\sqrt{m}U_{2k-3} + (1+4m)U_{2k-2} \\ U_{2k-2} + 2\sqrt{m}(U_{2k-1}) & 2\sqrt{m}U_{2k-2} + (1+4m)U_{2k-1} \end{bmatrix} \\ &= \begin{bmatrix} U_{2k-1} & U_{2k} \\ U_{2k} & U_{2k+1} \end{bmatrix}. \end{aligned}$$

Therefore we have a real number sequence U_k. The definition and boundary conditions of this sequence are

$$\begin{aligned} U_k &= 2\sqrt{m}U_{k-1} + U_{k-2}, \text{ for } k \geq 2, \\ U_0 &= 0,\, U_1 = 1. \quad \square \end{aligned}$$

Proposition 3. *For all $k \geq 2$,*

$$U_k = \frac{1}{2\sqrt{m+1}} \left[(\sqrt{m} + \sqrt{m+1})^k - (\sqrt{m} - \sqrt{m+1})^k \right].$$

Proof. If U_k is a characteristic polynomial r^k to solve this equation, then we get the following equation

$$r^k = 2\sqrt{m}r^{k-1} + r^{k-2} \Rightarrow r^2 - 2\sqrt{m}r - 1 = 0.$$

Hence we find the roots of this equation as

$$r_{1,2} = \sqrt{m} \pm \sqrt{m+1}.$$

Using r_1 and r_2, we can obtain a general formula of U_k. If we write U_k as combinations of the roots r_1 and r_2, we have

$$U_k = A(\sqrt{m} + \sqrt{m+1})^k + B(\sqrt{m} - \sqrt{m+1})^k.$$

Since

$$\begin{aligned} U_0 &= 0 = A + B \\ U_1 &= 1 = A(\sqrt{m} + \sqrt{m+1}) + B(\sqrt{m} - \sqrt{m+1}) \end{aligned}$$

and so

$$2A\sqrt{m+1} = 1.$$

Hence constants A and B

$$A = \frac{1}{2\sqrt{m+1}} \text{ and } B = -\frac{1}{2\sqrt{m+1}}.$$

Therefore we find the formula of U_k as

$$U_k = \frac{1}{2\sqrt{m+1}} \left[(\sqrt{m} + \sqrt{m+1})^k - (\sqrt{m} - \sqrt{m+1})^k \right]. \quad \square$$

This formula is a generalized Pell number sequence U_k. If $m = 1$, we get $U_k = P_k$ (the k^{th} Pell number) and

$$U_k = \frac{1}{2\sqrt{2}} \left[\left(1 + \sqrt{2}\right)^k - \left(1 - \sqrt{2}\right)^k \right].$$

In general, the trace $tr(A^k)$ of A^k is

$$U_{2k-1} + U_{2k+1} = U_{2k-1} + 2\sqrt{m}U_{2k} + U_{2k-1} = 2\sqrt{m}U_{2k} + 2U_{2k-1}.$$

Now we can define the generalized Pell-Lucas numbers V_k. The generalized Pell-Lucas numbers V_k are defined by the recurrence relation $V_0 = 2$, $V_1 = 2\sqrt{m}$ and $V_k = 2\sqrt{m}V_{k-1} + V_{k-2}$, for $k \geq 2$. The generalized Pell-Lucas number can be also expressed by $V_k = 2\sqrt{m}U_k + 2U_{k-1}$. Then the trace $tr(A^k)$ of A_k is found as V_{2k}. Also the determinant of A_k is 1.

On the other hand, if we take the product $a_1.a_{q-1} = (TS)^2 . \left(TS^{-1}\right)^2$, then we obtain the matrix representation of $a_1.a_{q-1}$ as

$$B = \begin{bmatrix} 1+4m & 2\sqrt{m} \\ 2\sqrt{m} & 1 \end{bmatrix}.$$

Thus for each k we have

$$B^k = \begin{bmatrix} U_{2k+1} & U_{2k} \\ U_{2k} & U_{2k-1} \end{bmatrix}.$$

Here the trace $tr(B^k)$ of B^k is V_{2k} and the determinant of B^k is 1. Additionally, if we consider the matrice representations of A and B, we find that they have same eigenvalues $r_1 = (2m+1)+2\sqrt{m(m+1)}$ and $r_2 = (2m+1)-2\sqrt{m(m+1)}$ of the characteristic equation $r^2 - (4m+2)r + 1 = 0$.

3. FIXED POINTS OF A^k AND B^k IN $Q(\sqrt{d})$

Now we investigate the case when A^k and B^k fix elements of $Q(\sqrt{d})$. If $\alpha \in Q(\sqrt{d})$ and if B^k is to fix α then

$$\frac{U_{2k+1}\alpha + U_{2k}}{U_{2k}\alpha + U_{2k-1}} = \alpha.$$

Hence we obtain $U_{2k}(\alpha^2 - 2\sqrt{m}\alpha - 1) = 0$ for all integers $k \geq 1$. Here $\alpha = \sqrt{m} \pm \sqrt{m+1}$. Now we have three possibilities:

i) if $m = 1$ then please see [12, p. 101].

ii) if $m = 2$ then $\alpha = \sqrt{2} \pm \sqrt{3}$, so $d = 2$ or 3.

iii) if $m = 3$ then $\alpha = \sqrt{3} \pm 2$, so $d = 3$.

If $\alpha \in Q(\sqrt{d})$ and if A^k is to fix α then

$$\frac{U_{2k-1}\alpha + U_{2k}}{U_{2k}\alpha + U_{2k+1}} = \alpha.$$

Thus we find $U_{2k}(\alpha^2 + 2\sqrt{m}\alpha - 1) = 0$ for all integers $k \geq 1$. Here $\alpha = -\sqrt{m} \pm \sqrt{m+1}$. Now we have three possibilities:

i) if $m = 1$ then please see [12, p. 101].

ii) if $m = 2$ then $\alpha = -\sqrt{2} \pm \sqrt{3}$, so $d = 2$ or 3.

iii) if $m = 3$ then $\alpha = -\sqrt{3} \pm 2$, so $d = 3$.

For all cases of m, if we take $\alpha = \tau = \sqrt{m} + \sqrt{m+1}$ then $\tau^{-1} = -\sqrt{m} + \sqrt{m+1}$ and if $\bar{\tau} = \sqrt{m} - \sqrt{m+1}$ then $\bar{\tau}^{-1} = -\sqrt{m} - \sqrt{m+1}$.

Therefore if the generators T and S of $H(\sqrt{m})$ act on $Q(\sqrt{d})$ under the condition that for all $k \geq 1$, $\left((TS^{-1})^2(TS)^2\right)^k$ or $\left((TS)^2(TS^{-1})^2\right)^k$ fixes elements of $Q(\sqrt{d})$, then $d = 2, 2$ or 3 and 3 for $m = 1, 2$ and 3, respectively.

Now we give the following.

Corollary 4. *If α is a real qudratic irrational number and if*

$$\left((TS^{-1})^2(TS)^2\right)^k \in H(\sqrt{m})(k \geq 1)$$

act on α, then the matrix A^k of $\left((TS^{-1})^2(TS)^2\right)^k$ is

$$A^k = \begin{bmatrix} U_{2k-1} & U_{2k} \\ U_{2k} & U_{2k+1} \end{bmatrix}$$

where U_k is the k^{th} generalized Pell number and $tr(A^k)$ is $2\sqrt{m}U_{2k} + 2U_{2k-1}$.

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