APPROXIMATE WEAK AMENABILITY
OF CERTAIN BANACH ALGEBRAS

BEHROUZ SHOJAEE and ABASALT BODAGHI

It is shown that for a locally compact group G, if $L^1(G)^{**}$ is approximately weakly amenable, then $M(G)$ is approximately weakly amenable. Then, new notions of approximate weak amenability and approximate cyclic amenability for Banach algebras are introduced. Bounded ω^*-approximately weakly [cyclic] amenable ℓ^1-Munn algebras are characterized.

1. **INTRODUCTION**

The notion of weak amenability was introduced by Bade, Curtis and Dales in [1] for commutative Banach algebras. Later, Johnson defined weak amenability for arbitrary Banach algebras [17] and showed that for a locally compact G, the group algebra $L^1(G)$ is weakly amenable (for shorter proof see [6]). It is shown in [13] that if $L^1(G)^{**}$ is weakly amenable, then $M(G)$, the measure algebra of G is weakly amenable. It is also proved in [6] that $M(G)$ is amenable if and only if the group G is discrete and amenable. The notion of cyclic amenability for Banach algebras was introduced by Grønbæk in [16]. Then the approximate version of mentioned notions are studied in [12] and [18] for Banach algebras.

In [7], Esslamzadeh introduced ℓ^1-Munn algebras which are a class of Banach algebras. He investigated some basic facts about the structure of ℓ^1-Munn algebras and characterized those with bounded approximate identities. The characterizing of amenable ℓ^1-Munn algebras by explicit construction of approximate diagonals is also given there. In [19], Shojaee et al. studied weak and cyclic amenability of ℓ^1-Munn algebras and showed that under certain condition, cyclic [resp. weakly] amenability of a ℓ^1-Munn algebra is equivalent to the cyclic [resp. weakly] amenability of the underlying Banach algebra $\mathcal{A}$.

In Section 2 of this paper, we show that if $\mathcal{A}^{**}$, the second dual of a Banach algebra $\mathcal{A}$ is approximately weakly amenable then $\mathcal{A}$ is essential. This could

be regarded as the approximate version of a result of Ghahramani and Laali [9, Proposition 2.1]. We investigate some relationships between approximate weak amenability of Banach algebras $\mathcal{A}$, $\mathcal{B}$ and the tensor product $\mathcal{A}\widehat{\otimes}\mathcal{B}$. The main result of this section is Theorem 2.6 which asserts that for a locally compact G, approximate weak amenability of $L^1(G)^{**}$ implies approximate weak amenability of $M(G)$. In Section 3, we introduce the concepts of bounded ω^*-approximate weak [cyclic] amenability for Banach algebras. By means of some examples, we show that these concepts are weaker than the weak and cyclic amenability. We also indicate some properties of such Banach algebras. Finally, we characterize ℓ^1-Munn algebras that are bounded ω^*-approximately weakly [cyclic] amenable.

2. **APPROXIMATE WEAK AMENABILITY**

We first recall some definitions in the Banach algebras setting. Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \longrightarrow X$ is called a derivation if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in \mathcal{A}).$$

For each $x \in X$, we define a map $\mathrm{ad}_x : \mathcal{A} \longrightarrow X$ by $\mathrm{ad}_x(a) = a\cdot x - x\cdot a$ ($a \in \mathcal{A}$). It is easily seen that ad_x is a derivation. Derivations of this form are called inner derivations. A derivation $D : \mathcal{A} \longrightarrow X$ is said to be approximately inner if there exists a net $(x_i) \subseteq X$ such that

$$D(a) = \lim_i (a \cdot x_i - x_i \cdot a) \qquad (a \in \mathcal{A}).$$

Hence D is approximately inner if it is in the closure of the set of inner derivations with respect to the strong operator topology on $B(\mathcal{A})$, the space of bounded linear operators on $\mathcal{A}$. The Banach algebra $\mathcal{A}$ is approximately amenable if every bounded derivation $D : \mathcal{A} \longrightarrow X^*$ is approximately inner, for each Banach $\mathcal{A}$-bimodule X [12], where X^* denotes the first dual space of X which is a Banach $\mathcal{A}$-bimodule in the canonical way. A Banach algebra $\mathcal{A}$ is called weakly amenable if every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ is inner and it is called approximately weakly amenable, if any such derivation is approximately inner. $\mathcal{A}$ is called cyclic amenable if every cyclic derivation from $\mathcal{A}$ into $\mathcal{A}^*$ (i.e., $\langle D(a), b\rangle + \langle D(b), a\rangle = 0$, for all $a, b \in \mathcal{A}$) is inner (see [19].

Let $\square$ and $\lozenge$ be the first and second Arens products on the second dual space $\mathcal{A}^{**}$, then $\mathcal{A}^{**}$ is a Banach algebra with respect to both of these products. Let $Z_1(\mathcal{A}^{**})$ denote the first topological center of $\mathcal{A}^{**}$, that is

$$Z_1(\mathcal{A}^{**}) = \{a^{**} \in \mathcal{A}^{**} : b^{**} \mapsto a^{**}\square b^{**} \text{ is } \sigma(\mathcal{A}^{**}, \mathcal{A}^*)\text{-continuous}\}.$$

The second topological centre is defined by

$$Z_2(\mathcal{A}^{**}) = \{a^{**} \in \mathcal{A}^{**} : b^{**} \mapsto b^{**} \Diamond a^{**} \text{ is } \sigma(\mathcal{A}^{**}, \mathcal{A}^*)\text{-continuous}\}.$$

THEOREM 2.1. *Let $\mathcal{A}$ be a Banach algebra.*

(i) *Suppose that $\mathbf{B}$ is a closed subalgebra of $(\mathcal{A}^{**}, \Box)$ such that $\mathcal{A} \subseteq \mathbf{B}$. If $\mathbf{B}$ is approximately amenable, then $\mathcal{A}$ is approximately amenable;*

(ii) *Suppose that $Z_1(A^{**})$ (or $Z_2(A^{**})$) is approximately amenable. Then $\mathcal{A}$ is approximately amenable.*

Proof. (i) Assume that $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is a continuous derivation. By [2, Proposition 2.7.17(i)] the map $D^{**} : (\mathcal{A}^{**}, \Box) \longrightarrow \mathcal{X}^{***}$ is a continuous derivation, and so $D^{**}|_{\mathbf{B}}$ is a derivation. Thus there exists a net $(x_\alpha^{***}) \subseteq \mathcal{X}^{***}$ such that

$$D^{**}(b) = \lim_\alpha b \cdot x_\alpha^{***} - x_\alpha^{***} \cdot b \qquad (b \in \mathbf{B}).$$

Consider the projection map $P : \mathcal{X}^{***} \longrightarrow \mathcal{X}^*$ which is an $\mathcal{A}$-module. Then

$$D(a) = P(D^{**}(a)) = \lim_\alpha a \cdot P(x_\alpha^{***}) - P(x_\alpha^{***}) \cdot a \quad (a \in \mathcal{A}).$$

Therefore $\mathcal{A}$ is approximately amenable.

(ii) It is immediately follows from (i). □

One should remember that the amenability case of Theorem 2.1 has been proved by Ghahramani and Laali in [9, Proposition 1.1], but our proof is different.

Recall that a topological algebra $\mathcal{A}$ is said to be essential if $\mathcal{A}^2$ is dense in $\mathcal{A}$. In [8, Proposition 2.1], Esslamzadeh and Shojaee proved that if the Banach algebra $\mathcal{A}$ is approximately weakly amenable, then $\mathcal{A}$ is essential. The same conclusion holds if $\mathcal{A}^{**}$ is weakly amenable [9, Proposition 2.1]. We show this result for the approximate case as follows.

THEOREM 2.2. *Let $\mathcal{A}$ be a Banach algebra. If $(\mathcal{A}^{**}, \Box)$ is approximately weakly amenable, then $\mathcal{A}$ is essential.*

Proof. Assume towards a contradiction that $\mathcal{A}^2$ is not dense in $\mathcal{A}$. Take $a_0 \in \mathcal{A} \backslash \mathcal{A}^2$ and $\lambda \in \mathcal{A}^*$ such that $\lambda|_{\mathcal{A}^2} = 0$ and $\langle \lambda, a_0 \rangle = 1$. Consider the map $D : \mathcal{A}^{**} \longrightarrow \mathcal{A}^{***}; a^{**} \mapsto \langle \lambda, a^{**} \rangle \lambda$. Obviously, D is continuous and linear. For each $a^{**}, b^{**} \in \mathcal{A}^{**}$, there are nets $(a_\alpha), (b_\beta) \subseteq \mathcal{A}$ such that $a^{**} \Box b^{**} = \omega^* - \lim_\alpha \lim_\beta a_\alpha b_\beta$. We have $\langle a^{**} \Box b^{**}, \lambda \rangle = \lim_\alpha \lim_\beta \langle \lambda, a_\alpha b_\beta \rangle = 0$, and so $D(a^{**} \Box b^{**}) = 0$. On the other hand,

$$\langle a^{**} \cdot D(b^{**}), c^{**} \rangle + \langle D(a^{**}) \cdot b^{**}, c^{**} \rangle = \langle D(b^{**}), c^{**} \Box a^{**} \rangle + \langle D(a^{**}), b^{**} \Box c^{**} \rangle$$

$$= \langle b^{**}, \lambda \rangle \langle c^{**} \Box a^{**}, \lambda \rangle$$

$$+ \langle a^{**}, \lambda \rangle \langle b^{**} \Box c^{**}, \lambda \rangle = 0.$$

Thus $D : \mathcal{A}^{**} \longrightarrow \mathcal{A}^{***}$ is a derivation, but it is not approximately inner. In fact $\langle D(a_0), a_0 \rangle = 1$, whereas

$$\lim_\alpha \langle ad_{\lambda_\alpha}(a_0), a_0 \rangle = \lim_\alpha \langle a_0 \cdot \lambda_\alpha - \lambda_\alpha \cdot a_0, a_0 \rangle = \lim_\alpha \langle \lambda_\alpha, a_0^2 - a_0^2 \rangle = 0,$$

for any net $(\lambda_\alpha) \subseteq \mathcal{A}^{***}$. This being a contradiction of $\mathcal{A}^{**}$ is approximately weakly amenable. $\square$

Recall that a character on the Banach algebra $\mathcal{A}$ is a non-zero homomorphism from $\mathcal{A}$ into $\mathbb{C}$. The set of characters on $\mathcal{A}$ is called the character space of $\mathcal{A}$ and denoted by $\Phi_\mathcal{A}$. Also, $\mathcal{A}$ is said to be dual if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $\mathcal{A} = \mathcal{A}_*^*$.

It is shown in part (ii) of [8, Propositions 2.1] that the homomorphic image of an approximately weakly amenable commutative Banach algebra is again approximately weakly amenable. In the next theorem, we generalize this result.

THEOREM 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras.*
  (i) *Suppose that $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ and $\psi : \mathcal{B} \longrightarrow \mathcal{A}$ are continuous homomorphisms such that $\varphi \circ \psi = I_\mathcal{B}$. If $\mathcal{A}$ is approximately weakly amenable, then $\mathcal{B}$ is approximately weakly amenable. Moreover, if $(\mathcal{A}^{**}, \square)$ is approximately weakly amenable, then $(\mathcal{B}^{**}, \square)$ is approximately weakly amenable;*
  (ii) *Suppose that $\mathcal{A}$ is a dual Banach algebra. If $(\mathcal{A}^{**}, \square)$ is approximately weakly amenable then $\mathcal{A}$ is approximately weakly amenable;*
  (iii) *Suppose that $\mathcal{A}$ is commutative. Then $\mathcal{A}$ and $\mathcal{B}$ are approximately weakly amenable if and only if the ℓ^1-direct sum $\mathcal{A} \oplus_1 \mathcal{B}$ is approximately weakly amenable;*
  (iv) *Suppose that $\mathcal{A}$ is weakly amenable. Then $\mathcal{B}$ is approximately weakly amenable if and only if the ℓ^1-direct sum $\mathcal{A} \oplus_1 \mathcal{B}$ is approximately weakly amenable;*
  (v) *Suppose that $\mathcal{A}$ is commutative. Then $\mathcal{A}$ is approximately weakly amenable if and only if $\mathcal{A} \widehat{\otimes} \mathcal{A}$ is approximately weakly amenable;*
  (vi) *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital, and $\phi_1 \in \Phi_\mathcal{A}$, $\phi_2 \in \Phi_\mathcal{B}$. If $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is approximately weakly amenable, then $\mathcal{A}$ and $\mathcal{B}$ are approximately weakly amenable.*

Proof. (i) Let $D : \mathcal{B} \longrightarrow \mathcal{B}^*$ be a derivation. We can consider $\mathcal{B}$ as an $\mathcal{A}$-bimodule with actions $a \cdot x = \varphi(a)x$ and $x \cdot a = x\varphi(a)$ for every $a \in \mathcal{A}, x \in \mathcal{B}$. Hence the map φ^* is an $\mathcal{A}$-module homomorphism, and thus

$$\varphi^* \circ D \circ \varphi(ab) = \varphi^*(D(\varphi(a)) \cdot \varphi(b) + \varphi(a) \cdot D(\varphi(b)))$$
$$= \varphi^* \circ D \circ \varphi(a) \cdot b + a \cdot \varphi^* \circ D \circ \varphi(b),$$

for all $a, b \in \mathcal{A}$. Hence, $\varphi^* \circ D \circ \varphi : \mathcal{A} \longrightarrow \mathcal{A}^*$ is a continuous derivation. Therefore there exists a net $(a_\alpha^*) \subseteq \mathcal{A}^*$ such that $\varphi^* \circ D \circ \varphi(a) = \lim_\alpha (a \cdot a_\alpha^* - a_\alpha^* \cdot a)$ $(a \in \mathcal{A})$. The equality $\varphi \circ \psi = I_\mathcal{B}$ implies $\psi^* \circ \varphi^* = I_{\mathcal{B}^*}$, and thus for every $c \in \mathcal{B}$, we get

$$\begin{aligned} D(c) &= \psi^* \circ \varphi^* \circ D \circ \varphi \circ \psi(c) \\ &= \psi^*(\varphi^* \circ D \circ \varphi(\psi(c)) \\ &= \psi^*(\lim_\alpha(\psi(c) \cdot a_\alpha^* - a_\alpha^* \cdot \psi(c))) \\ &= \lim_\alpha \psi^*(\psi(c) \cdot a_\alpha^* - a_\alpha^* \cdot \psi(c)) \\ &= \lim_\alpha(c \cdot \psi^*(a_\alpha^*) - \psi^*(a_\alpha^*) \cdot c). \end{aligned}$$

The above equalities show that $\mathcal{B}$ is approximately weakly amenable. Since $\varphi^{**} \circ \psi^{**} = I_{\mathcal{B}^{**}}$, $(\mathcal{B}^{**}, \square)$ is approximately weakly amenable.

(ii) According to [3, Theorem 2.15], $(\mathcal{A}_*)^\perp$ is a ω^*-closed ideal in $\mathcal{A}^{**}$ and $\mathcal{A}^{**} = \mathcal{A} \oplus (\mathcal{A}_*)^\perp$. Now, if $\varphi : \mathcal{A}^{**} \longrightarrow \mathcal{A}$ is the projection map and $\psi : \mathcal{A} \longrightarrow \mathcal{A}^{**}$ is the inclusion map, then $\varphi \circ \psi = I_\mathcal{A}$, hence by part (i) we get the desired result.

(iii) It is known that weak amenability and approximate weak amenability coincide for a commutative Banach algebra, and so we deduce the sufficiency part by [8, Proposition 2.2(iii)].

Conversely, the maps $\varphi : \mathcal{A} \oplus \mathcal{B} \longrightarrow \mathcal{A}$; $a \oplus b \mapsto a$ and $\psi : \mathcal{A} \longrightarrow \mathcal{A} \oplus \mathcal{B}$; $a \mapsto a \oplus 0$ are continuous homomorphisms and $\varphi \circ \psi = I_\mathcal{A}$. By (i), $\mathcal{A}$ is approximately weakly amenable. Similarly for $\mathcal{B}$.

(iv) The proof is immediately by [8, Proposition 2.2(iii)] and part (i).

(v) The sufficiency part follows immediately from [15, Proposition 2.6]. For the converse, in light of [2, Corollary 2.8.70] we can suppose that $\mathcal{A}$ has an identity. Consider the homomorphisms $\varphi : \mathcal{A} \widehat{\otimes} \mathcal{A} \longrightarrow \mathcal{A}$ defined by $\varphi(a \otimes b) = ab$ and $\psi : \mathcal{A} \longrightarrow \mathcal{A} \widehat{\otimes} \mathcal{A}$ by $\psi(a) = a \otimes e_\mathcal{A}$, where $e_\mathcal{A}$ is the identity of $\mathcal{A}$. Easily, $\varphi \circ \psi = I_\mathcal{A}$. Now, part (i) shows that $\mathcal{A}$ is approximately weakly amenable.

(vi) One can check that the maps $\varphi : \mathcal{A} \widehat{\otimes} \mathcal{B} \longrightarrow \mathcal{A}$, $\varphi(a \otimes b) = \phi_2(b)a$ and $\psi : \mathcal{A} \longrightarrow \mathcal{A} \widehat{\otimes} \mathcal{B}$, $\psi(a) = a \otimes e_\mathcal{B}$ are homomorphisms so that $\varphi \circ \varphi = I_\mathcal{A}$. It is a consequence of part (i) that $\mathcal{A}$ is approximately weakly amenable. Similarly for $\mathcal{B}$. $\square$

Recall that a linear functional d on $\mathcal{A}$ is a point derivation at $\varphi \in \Phi_\mathcal{A}$ if

$$d(ab) = \varphi(a)d(b) + \varphi(b)d(a) \quad (a, b \in \mathcal{A}).$$

THEOREM 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is approximately weakly amenable.*

(i) *Then both $\mathcal{A}$ and $\mathcal{B}$ are essential;*

(ii) *If $\varphi \in \Phi_{\mathcal{A}}$ and $\psi \in \Phi_{\mathcal{B}}$, then there are no non-zero point derivations on both $\mathcal{A}$ and $\mathcal{B}$;*

(iii) *If $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is approximately weakly amenable, then $\mathcal{A}$ is essential and there is no non-zero point derivation on $\mathcal{A}$.*

Proof. (i) It suffices to consider $\mathcal{A}$. For $\mathcal{B}$ is similar. Suppose that $\overline{\mathcal{A}^2} \neq \mathcal{A}$. Take $a_0 \in \mathcal{A}\backslash\mathcal{A}^2$ and $\lambda \in \mathcal{A}^*$ such that $\lambda|_{\mathcal{A}^2} = 0$ and $\langle\lambda, a_0\rangle = 1$. Also, choose $\mu \in \mathcal{B}^*$ and $b_0 \in \mathcal{B}$ such that $\langle\mu, b_0\rangle = 1$. Define $D : \mathcal{A}\widehat{\otimes}\mathcal{B} \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{B})^*$ by $D(a\otimes b) = \langle\lambda, a\rangle\langle\mu, b\rangle(\lambda\otimes\mu)$ where $a \in \mathcal{A}$ and $b \in \mathcal{B}$. It is easy to see that D is a derivation. Due to approximate weak amenability of $\mathcal{A}\widehat{\otimes}\mathcal{B}$, there exists a net $(x_\alpha) \subseteq (\mathcal{A}\widehat{\otimes}\mathcal{B})^*$ such that $D(a\otimes b) = \lim_\alpha(a\otimes b)\cdot x_\alpha - x_\alpha\cdot(a\otimes b)$ $(a \in \mathcal{A}, b \in \mathcal{B})$. Therefore $\langle D(a_0 \otimes b_0), a_0 \otimes b_0\rangle = 1$. On the other hand, $\lim_\alpha(\langle(a_0 \otimes b_0) \cdot x_\alpha - x_\alpha \cdot (a_0 \otimes b_0), a_0 \otimes b_0\rangle) = 0$. This contradicts our assumption.

(ii) Suppose that d is a non-zero continuous point derivation at $\varphi_1 \in \Phi_{\mathcal{A}}$. We can show that the map $D : \mathcal{A}\widehat{\otimes}\mathcal{B} \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{B})^*$, $D(a \otimes b) = d(a)\psi(b)\varphi_1 \otimes \psi$ is a derivation. Since $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is approximately weakly amenable, there exists a net $(x_\alpha) \subseteq (\mathcal{A}\widehat{\otimes}\mathcal{B})^*$ such that

$$D(a \otimes b) = \lim_\alpha(a \otimes b) \cdot x_\alpha - x_\alpha \cdot (a \otimes b) \qquad (a \in \mathcal{A}, b \in \mathcal{B}).$$

Take $b \in \mathcal{B}$ such that $\psi(b) = 1$. Then $d(a)\varphi_1(a) = 0$, and so $d|_{(\mathcal{A}-ker\varphi_1)} = 0$. Thus, since d is a point derivation at φ_1, and also using (i) we obtain $d = 0$ which is a contradiction.

(iii) The result is a direct consequence of parts (i) and (ii). $\square$

THEOREM 2.5. *Let G be a locally compact group. Then the following are equivalent:*

(i) *$M(G)$ is weakly amenable;*

(ii) *$M(G)$ is approximately weakly amenable;*

(iii) *There is no non-zero, continuous point derivation at a character of $M(G)$;*

(iv) *G is discrete.*

Proof. (i)$\Rightarrow$(ii) is obvious.
(ii)$\Rightarrow$(iii) By [8, Proposition 2.1] and [4, Proposition 1.3].
(iii)$\Rightarrow$(iv) By [6, Theorem 3.2].
(iv)$\Rightarrow$(i) By [6, Theorem 1.2]. $\square$

Let G be a locally compact group. Recall that $LUC(G)$ is the space of bounded left uniformly continuous functions on G under the supremum norm and $C_0(G)$ is the space of continuous functions on G vanishing at infinity.

THEOREM 2.6. *Let G be a locally compact group, and let $(L^1(G)^{**}, \square)$ be approximately weakly amenable. Then $M(G)$ is approximately weakly amenable.*

Proof. By [11, Lemma 1.1], $LUC(G)^* = M(G) \oplus C_0(G)^\perp$ where $C_0(G)^\perp$ is a closed ideal in $LUC(G)^*$. Assume that E is a right identity for $L^1(G)^{**}$ in which $\|E\| = 1$. Then $L^1(G)^{**} = EL^1(G)^{**} \oplus (1 - E)L^1(G)^{**}$ for which $(1 - E)L^1(G)^{**}$ is a closed ideal in $L^1(G)^{**}$. In addition, by [10], $EL^1(G)^{**} = LUC(G)^*$. Therefore the projection maps $P_1 : L^1(G)^{**} \longrightarrow LUC(G)^*$, $P_2 : LUC(G)^* \longrightarrow M(G)$ and the inclusion maps $\iota_1 : LUC(G)^* \longrightarrow L^1(G)^{**}$, $\iota_2 : M(G) \longrightarrow LUC(G)^*$ are homomorphisms such that $P_1 \circ \iota_1 = I_{LUC(G)^*}$ and $P_2 \circ \iota_2 = I_{M(G)}$. By assumption that $L^1(G)^{**}$ is approximately weakly amenable, the above relations and part (i) of Theorem 2.3, we deduce that $LUC(G)^*$ approximately weakly amenable. Consequently, $M(G)$ is approximately weakly amenable. $\square$

COROLLARY 2.7. *For a non-discrete locally compact group G, the Banach algebra $(L^1(G)^{**}, \square)$ is not approximately weakly amenable.*

Proof. The result follows from Theorem 2.5 and Theorem 2.6. $\square$

3. BOUNDED ω^*-APPROXIMATELY WEAK [CYCLIC] AMENABILITY

We first introduce two new notions of amenability; bounded ω^*-approximate weak amenability and bounded ω^*-approximate cyclic amenability as follows:

Definition 3.1. A Banach algebra $\mathcal{A}$ is bounded ω^*-approximately weakly amenable if for every continuous derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$, there is a net $(x_\alpha) \subseteq A^*$, such that the net (ad_{x_α}) is norm bounded in $B(\mathcal{A}, \mathcal{A}^*)$ and

$$D(a) = \omega^* - \lim_\alpha ad_{x_\alpha}(a) \quad (a \in \mathcal{A}).$$

Definition 3.2. A Banach algebra $\mathcal{A}$ is bounded ω^*-approximately cyclic amenable if for every cyclic continuous derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$, there is a net $(x_\alpha) \subseteq \mathcal{A}^*$, such that the net (ad_{x_α}) is norm bounded in $B(\mathcal{A}, \mathcal{A}^*)$ and

$$D(a) = \omega^* - \lim_\alpha ad_{x_\alpha}(a) \quad (a \in \mathcal{A}).$$

Obviously that all notions of weak amenability, approximate weak amenability and bounded ω^*-approximate weak amenability coincide for a commutative Banach algebra. Moreover, if $\mathcal{A}$ is a commutative Banach algebra without identity, then it is bounded ω^*-approximately weakly amenable if and only if $\mathcal{A}^{\#}$ is bounded ω^*-approximately weakly amenable. These facts fail to be true in general. In the following example we give a Banach algebra that is bounded ω^*-approximately weakly [cyclic] amenable but not weakly [cyclic] amenable.

Example 3.3. We are following Example 6.2 of [10]. So we have a Banach algebra $\mathcal{A}$ that is not weakly amenable, but is approximately amenable. In other words, as is showed in the mentioned example for every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ there exists a sequence $(x_n) \subseteq \mathcal{A}^*$ such that $Da = \lim_n ad_{x_n}(a)$ for each $a \in \mathcal{A}$. Hence the sequence (ad_{x_n}) is bounded and thus $\mathcal{A}$ is bounded ω^*-approximately weakly amenable.

It should be mentioned that some properties such as, being essential, not having non-zero point derivation, hold for Banach algebras that are bounded ω^*-approximately weakly amenable hold. The proofs of them are similar to the [approximate] weak amenability case. The following theorem is analogous to Theorem 2.3 and we prove only part (i).

THEOREM 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras.*
 (i) *Suppose that $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ and $\psi : \mathcal{B} \longrightarrow \mathcal{A}$ are continuous homomorphisms such that $\varphi \circ \psi = I_\mathcal{B}$. If $\mathcal{A}$ is bounded ω^*-approximately weakly amenable are, then $\mathcal{B}$ so is. Moreover, if $(\mathcal{A}^{**}, \square)$ is bounded ω^*-approximately weakly amenable, then $(\mathcal{B}^{**}, \square)$ so is;*
 (ii) *Suppose that $\mathcal{A}$ is a dual Banach algebra. If $(\mathcal{A}^{**}, \square)$ is bounded ω^*-approximately weakly amenable, then $\mathcal{A}$ so is;*
 (iii) *Suppose that $\mathcal{A}$ is commutative. Then $\mathcal{A}$ and $\mathcal{B}$ are bounded ω^*-approximately weakly amenable if and only if the ℓ^1-direct sum $\mathcal{A} \oplus_1 \mathcal{B}$ is bounded ω^*-approximately weakly amenable;*
 (iv) *Suppose that $\mathcal{A}$ is weakly amenable. Then $\mathcal{B}$ is bounded ω^*-approximately weakly amenable if and only if the ℓ^1-direct sum $\mathcal{A} \oplus_1 \mathcal{B}$ is bounded ω^*-approximately weakly amenable;*
 (v) *Suppose that $\mathcal{A}$ is commutative. Then $\mathcal{A}$ is bounded ω^*-approximately weakly amenable if and only if $\mathcal{A} \widehat{\otimes} \mathcal{A}$ is bounded ω^*-approximately weakly amenable;*
 (vi) *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital, and $\phi_1 \in \Phi_\mathcal{A}$, $\phi_2 \in \Phi_\mathcal{B}$. If $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is bounded ω^*-approximately weakly amenable, then $\mathcal{A}$ and $\mathcal{B}$ are bounded ω^*-approximately weakly amenable .*

Proof. We follow the argument of the proof of Theorem 2.3. Let $D : \mathcal{B} \longrightarrow \mathcal{B}^*$ be a derivation. Then $\varphi^* \circ D \circ \varphi : \mathcal{A} \longrightarrow \mathcal{A}^*$ is a continuous derivation, and so there exists a net $(a_\alpha^*) \subseteq \mathcal{A}^*$ such that the net $(ad_{a_\alpha^*})$ is bounded and

$$\varphi^* \circ D \circ \varphi(a) = \omega^* - \lim_\alpha ad_{a_\alpha^*}(a) = \omega^* - \lim_\alpha (a \cdot a_\alpha^* - a_\alpha^* \cdot a) \quad (a \in \mathcal{A}).$$

We have $\psi^* \circ \varphi^* = I_{\mathcal{B}^*}$ and also ψ^* is ω^*-continuous. Thus for every $c \in \mathcal{B}$, we get

$$D(c) = \psi^* \circ \varphi^* \circ D \circ \varphi \circ \psi(c) = \psi^*(\varphi^* \circ D \circ \varphi(\psi(c))$$

$$= \omega^* - \lim_\alpha \psi^*(\psi(c) \cdot a_\alpha^* - a_\alpha^* \cdot \psi(c))$$
$$= \omega^* - \lim_\alpha (c \cdot \psi^*(a_\alpha^*) - \psi^*(a_\alpha^*) \cdot c)$$
$$= \omega^* - \lim_\alpha ad_{\psi^*(a_\alpha^*)}(c).$$

Since $ad_{\psi^*(a_\alpha^*)}(c) = \psi^*(ad_{a_\alpha^*}(\psi(c)))$ and the net $(ad_{a_\alpha^*})$ is bounded, the net $(ad_{\psi^*(a_\alpha^*)})$ is bounded. The arguments of other parts in Theorem 2.3 work to finish the proof. $\square$

We should recall that a Banach algebra $\mathcal{A}$ is approximately amenable if and only if $\mathcal{A}$ is ω^*-approximately amenable [14, Theorem 2.1]. Now, in view of [12, Theorem 3.2], we can show that a locally compact group G is amenable if and only if $L^1(G)$ is bounded $[\omega^*]$-approximately amenable.

Using Example 3.3, we present a Banach algebra which is [bounded ω^*-] approximately weakly amenable but it is neither weakly amenable nor approximately amenable.

Example 3.5. Let G be a non-amenable discrete group. Then $\ell^1(G)$ is approximately weakly amenable, but not approximately amenable [12, Theorem 3.2]. Now, consider the Banach algebra $\mathcal{A}$ as in Example 3.3. Then $\mathcal{B} = \mathcal{A} \oplus_1 \ell^1(G)$ equipped with ℓ^1-norm is a Banach algebra. The maps $\varphi_1 : \mathcal{B} \longrightarrow \mathcal{A}$ and $\varphi_2 : \mathcal{A} \longrightarrow \mathcal{B}$ are homomorphisms in which $\varphi_1 \circ \varphi_2 = I_\mathcal{A}$. Since $\ell^1(G)$ is weakly amenable, $\mathcal{B}$ is [bounded ω^*-] approximately weakly amenable by [8, Proposition 2.2(iii)] and the part (iv) of Theorem 3.4. By Theorem 2.3 (i) and [12, Proposition 2.2], $\mathcal{B}$ cannot be weakly amenable nor approximately amenable.

In light of Theorem 3.4, we can prove Theorems 2.4, 2.5 and 2.6 for the bounded ω^*-approximate weak amenability case.

Let $\mathcal{A}$ be a unital Banach algebra, I and J be arbitrary index sets and P be a $J \times I$ matrix over $\mathcal{A}$ such that all of its non-zero entries are invertible and $\|P\|_\infty = \sup\{\|P_{ji}\| : j \in J, i \in I\} \leq 1$. The set $\ell^1(I \times J, \mathcal{A})$, the vector space of all $I \times J$ matrics X over $\mathcal{A}$ with product $X \circ Y = XPY$ is a Banach algebra that we call the ℓ^1-*Munn-algebra* over $\mathcal{A}$ with sandwich matrix P and denote it by $LM(\mathcal{A}, P)$ (for more information see [7]).

THEOREM 3.6. *If $\mathcal{A}$ is bounded ω^*-approximately cyclic amenable Banach algebra, then so is $LM(\mathcal{A}, P)$.*

Proof. Suppose that $\beta \in J, \alpha \in I$ such that $P_{\alpha\beta} \neq 0$ and $q = P_{\alpha\beta}^{-1}$. Let $D : LM(\mathcal{A}, P) \longrightarrow LM(\mathcal{A}, P)^*$ be a bounded cyclic derivation. Define $\widehat{D}$ via

$$\widehat{D} : \mathcal{A} \longrightarrow \mathcal{A}^*, \quad \langle \widehat{D}a, b \rangle = \langle D(qa\varepsilon_{\beta\alpha}), qb\varepsilon_{\beta\alpha} \rangle,$$

for all $a, b \in \mathcal{A}$. Clearly, $\widehat{D}$ is a bounded linear map. By [19, Theorem 2.1], $\widehat{D}$ is a bounded cyclic derivation, hence there exists a net $(\widehat{\psi_\gamma}) \subseteq \mathcal{A}$ such that the net $(ad_{\widehat{\psi_\gamma}})$ is bounded and

$$\widehat{D}(a) = \omega^* - \lim_\gamma ad_{\widehat{\psi_\gamma}}(a) = \omega^* - \lim_\gamma a \cdot \widehat{\psi_\gamma} - \widehat{\psi_\gamma} \cdot a.$$

Put $\psi_\gamma(a\varepsilon_{ij}) = \widehat{\psi_\gamma}(p_{ji}a) + \langle D(q\varepsilon_{\beta j}), a\varepsilon_{i\alpha} \rangle$ for all $i \in I, j \in J, a \in \mathcal{A}$. It is easy to see that $\psi_\gamma \in LM(\mathcal{A}, P)^*$. We wish to show that the net (ad_{ψ_γ}) is bounded. For every $a, b \in \mathcal{A}$, $j, l \in J$ and $i, k \in I$, we have

$$\begin{aligned}
\langle ad_{\psi_\gamma}(a\varepsilon_{ij}), b\varepsilon_{kl} \rangle &= \langle a\varepsilon_{ij} \cdot \psi_\gamma - \psi_\gamma \cdot a\varepsilon_{ij}, b\varepsilon_{kl} \rangle \\
&= \langle \psi_\gamma, b\varepsilon_{kl} \circ a\varepsilon_{ij} \rangle - \langle \psi_\gamma, a\varepsilon_{ij} \circ b\varepsilon_{kl} \rangle \\
&= \langle \psi_\gamma, bP_{li}a\varepsilon_{kj} \rangle - \langle \psi_\gamma, aP_{jk}b\varepsilon_{il} \rangle \\
&= \langle \widehat{\psi_\gamma}, p_{jk}bP_{li}a \rangle + \langle D(q\varepsilon_{\beta j}), bP_{li}a\varepsilon_{k\alpha} \rangle \\
&\quad - \langle \widehat{\psi_\gamma}, P_{li}aP_{jk}b \rangle - \langle D(q\varepsilon_{\beta j}), aP_{jk}b\varepsilon_{i\alpha} \rangle \\
&= \langle ad_{\widehat{\psi_\gamma}}(P_{li}a), P_{jk}b \rangle + \langle D(q\varepsilon_{\beta j}), bP_{li}a\varepsilon_{k\alpha} \rangle \\
&\quad - \langle D(q\varepsilon_{\beta j}), aP_{jk}b\varepsilon_{i\alpha} \rangle.
\end{aligned}$$

Thus $|\langle ad_{\psi_\gamma}(a\varepsilon_{ij}), b\varepsilon_{kl} \rangle| \leq \|ad_{\widehat{\psi_\gamma}}\|\|a\|\|b\| + 2\|D\|\|a\|\|b\|$. On the other hand, the net $(ad_{\widehat{\psi_\gamma}})$ is bounded, so

$$|\langle ad_{\psi_\gamma}(B_1), B_2 \rangle| \leq (\|ad_{\widehat{\psi_\gamma}}\| + 2\|D\|)\|B_1\|_1\|B_2\|_1 \quad (B_1, B_2 \in LM(\mathcal{A}, P)).$$

Now, let $S = a\varepsilon_{ij}$ and $T = b\varepsilon_{kl}$ be non-zero elements in $LM(\mathcal{A}, P)$ and $U = q\varepsilon_{\beta j}$, $V = q\varepsilon_{\beta l}$, $X = a\varepsilon_{i\alpha}$ and $Y = qp_{jk}b\varepsilon_{\beta\alpha}$. Then, $S = X \circ U$, $U \circ T = Y \circ V$ and $\langle D(V \circ X), Y \rangle + \langle DY, V \circ X \rangle = 0$. By [18, Theorem 2.1] we get

$$(3.1) \qquad \langle D(X), U \circ T \rangle = \langle D(V \circ X), Y \rangle - \langle D(V), X \circ Y \rangle.$$

Also

$$\begin{aligned}
\langle D(V \circ X), Y \rangle &= \langle D(qP_{li}a\varepsilon_{\beta\alpha}), qP_{jk}b\varepsilon_{\beta\alpha} \rangle \\
&= \langle \widehat{D}(P_{li}a), P_{jk}b \rangle \\
&= \lim_\gamma \langle (P_{li}.a).\widehat{\psi_\gamma} - \widehat{\psi_\gamma}.(P_{li}a), P_{jk}b \rangle \\
(3.2) \qquad &= \lim_\gamma (\langle \widehat{\psi_\gamma}, P_{jk}bP_{li}a \rangle - \langle \widehat{\psi_\gamma}, P_{li}aP_{jk}b \rangle).
\end{aligned}$$

Applying (3.1) and (3.2), we have

$$\begin{aligned}
\langle D(S), T \rangle &= \langle D(U), T \circ X \rangle + \langle D(X), U \circ T \rangle \\
&= \langle D(U), T \circ X \rangle + \langle D(V \circ X), Y \rangle - \langle D(V), X \circ Y \rangle \\
&= \langle D(q\varepsilon_{\beta j}), bp_{li}a\varepsilon_{k\alpha} \rangle + \lim_\gamma (\langle \widehat{\psi_\gamma}, p_{jk}bp_{li}a \rangle
\end{aligned}$$

$$- \langle \widehat{\psi}_\gamma, p_{li}ap_{jk}b) \rangle - \langle D(q\varepsilon_{\beta l}), ap_{jk}b\varepsilon_{i\alpha} \rangle$$
$$= \lim_\gamma (\langle \psi_\gamma, bp_{li}ap_{kj} \rangle - \langle \psi_\gamma, ap_{jk}b\varepsilon_{il} \rangle)$$
$$= \lim_\gamma (\langle \psi_\gamma, T \circ S \rangle - \langle \psi_\gamma, S \circ T \rangle)$$
$$= \lim_\gamma \langle ad_{\psi_\gamma}(S), T \rangle.$$

The net (ad_{ψ_γ}) is bounded, and thus

$$\langle D(B_1), B_2 \rangle = \lim_\gamma \langle ad_{\psi_\gamma} B_1, B_2 \rangle \quad (B_1, B_2 \in LM(A, P)).$$

The above equality shows that

$$D(B) = \omega^* - \lim_\gamma ad_{\psi_\gamma} B \quad (B \in LM(A, P)).$$

Therefore $LM(A, P)$ is bounded ω^*-approximately cyclic amenable. $\quad\square$

The proof of the following lemma is similar to the proof of [19, Lemma 2.2], so is omitted.

LEMMA 3.7. *If $\mathcal{A}$ is bounded ω^*-approximately weakly amenable Banach algebra, then every continuous derivation $D : LM(\mathcal{A}, P) \longrightarrow LM(\mathcal{A}, P)^*$ is cyclic.*

The next theorem is an immediate consequence of Theorem 3.6 and Lemma 3.7.

THEOREM 3.8. *If $\mathcal{A}$ is bounded ω^*-approximately weakly amenable, then so is $LM(\mathcal{A}, P)$.*

In the upcoming theorem, we show that the converse of Theorems 3.6 and 3.8 are true as long as the sandwich matrix P is square; *i.e.*, the index sets I and J are equal [19, Remark 2.4].

THEOREM 3.9. *Suppose P is a regular square matrix and $LM(\mathcal{A}, P)$ has a bounded approximate identity. Then $\mathcal{A}$ is bounded ω^*-approximately weakly [respectively cyclic] amenable if and if $LM(\mathcal{A}, P)$ is bounded ω^*-approximately weakly [respectively cyclic] amenable.*

Proof. By Theorem 3.8 we need only to prove the converse statement. According to [7], the index set I is finite and $LM(\mathcal{A}, P)$ is topologically iso-morphic to $\mathcal{A}\widehat{\otimes}M_n$ for some $n \in \mathbb{N}$. If $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a bounded derivation, then $D \otimes 1$ is a bounded derivation from $\mathcal{A}\widehat{\otimes}M_n$ to $(\mathcal{A}\widehat{\otimes}M_n)^*$. Moreover, if D is cyclic, then so is $D \otimes 1$. Thus there exists a net $(X_\alpha) \in (\mathcal{A}\widehat{\otimes}M_n)^*$ such that the net (ad_{X_α}) is bounded and $D(\mathcal{B}) = \omega^*-\lim_\alpha ad_{X_\alpha}(\mathcal{B})$ for every $\mathcal{B} \in \mathcal{A}\widehat{\otimes}M_n$. For each α, we put $X_\alpha = \Sigma_{i,j=1}^n x_{ij}^\alpha \otimes \varepsilon_{ij} \quad (x_{ij}^\alpha \in A^*)$. Now, for $a, b \in A$ we get

$$D(a) \otimes \varepsilon_{11} = (D \otimes 1)(a \otimes \varepsilon_{11})$$

$$= \omega^* - \lim_\alpha((a \otimes \varepsilon_{11})(\Sigma_{i,j=1}^n x_{ij}^\alpha \otimes \varepsilon_{ij}) - (\Sigma_{i,j=1}^n x_{ij}^\alpha \otimes \varepsilon_{ij})(a \otimes \varepsilon_{11}))$$
$$= \omega^* - \lim_\alpha(\Sigma_{i=1}^n a. \cdot x_{i1}^\alpha \otimes \varepsilon_{i1} - \Sigma_{j=1}^n x_{1j}^\alpha a \otimes \varepsilon_{1j}).$$

Thus

$$\langle D(a) \otimes \varepsilon_{11}, b \otimes \varepsilon_{11} \rangle = \lim_\alpha \langle \Sigma_{i=1}^n a \cdot x_{i1}^\alpha \otimes \varepsilon_{i1} - \Sigma_{j=1}^n x_{1j}^\alpha \cdot a \otimes \varepsilon_{1j}, b \otimes \varepsilon_{11} \rangle.$$

We have $\langle D(a), b \rangle = \lim_\alpha \langle a \cdot x_{11}^\alpha - x_{11}^\alpha \cdot a, b \rangle$, and so $D(a) = \omega^* - \lim_\alpha(a \cdot x_{11}^\alpha - x_{11}^\alpha \cdot a)$. To complete of the proof it is enough to show that the net $(ad_{x_{11}^\alpha})$ is bounded. For this, we have

$$\langle ad_{x_{11}^\alpha}(a), b \rangle = \langle a \cdot x_{11}^\alpha - x_{11}^\alpha \cdot a, b \rangle$$
$$= \langle \Sigma_{i=1}^n a \cdot x_{i1}^\alpha \otimes \varepsilon_{i1} - \Sigma_{j=1}^n x_{1j}^\alpha \cdot a \otimes \varepsilon_{1j}, b \otimes \varepsilon_{11} \rangle$$
$$= \langle ad_{X_\alpha}(a \otimes \varepsilon_{11}), b \otimes \varepsilon_{11} \rangle.$$

Hence

$$|\langle ad_{x_{11}^\alpha}(a), b \rangle| \le \|ad_{X_\alpha}\| \|a\| \|b\|.$$

Therefore $\mathcal{A}$ is bounded ω^*-approximately weakly [resp. cyclic] amenable. $\quad\square$

REFERENCES

[1] W.G. Bade, P.C. Curtis and H.G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras.* Proc. London Math. Soc. **55** (3) (1987), 359–377.

[2] H.G. Dales, *Banach algebras and automatic continuity.* Clarendon Press, Oxford, 2000.

[3] H.G. Dales and A.T.-M. Lau, *The second duals of Beurling algebras.* Mem. Amer. Math. Soc. **177** (2005).

[4] H.G. Dales, F. Ghahramani and N. Grønbæk, *Derivations into iterated duals of Banach algebras.* Studia Math. **128** (1) (1998), 19–54.

[5] H.G. Dales, F. Ghahramani and Ya. Helemskii, *The amenability of measure algebras.* J. London Math. Soc. **66** (2) (2002) 213–226.

[6] M. Despić and F. Ghahramani, *Weak amenability of group algebras of locally compact groups.* Canad. Math. Bull. **37** (2) (1994) 165–167.

[7] G.H. Esslamzadeh, *Banach algebra structure and amenability of a class of matrix algebras with applications.* J. Funct. Anal. **161** (1999), 364–383.

[8] G.H. Esslamzadeh and B. Shojaee, *Approximate weak amenability of Banach algebras.* Bull. Belg. Math. Soc. Simon Stevin **18** (2011), 415–429.

[9] F. Ghahramani and J. Laali, *Amenability and toplogical centres of the second duals of Banach algebras.* Bull. Aust. Math. Soc. **65** (2002), 191–197.

[10] F. Ghahramani and A.L.M. Lau, *Isometric isomorphisms between the conjugate algebras of group algebras.* Bull. London Math. Soc. **20** (1988), 342–344.

[11] F. Ghahramani, A.L.M. Lau and V. Losert, *Isometric isomorphisms between Banach algebras related to locally compact groups.* Trans. Amer. Math. Soc. **321** (1990), 273–283.

[12] F. Ghahramani and R.J. Loy, *Generalized notions of amenability.* J. Funct. Anal. **208** (2004), 229–260.

[13] F. Ghahramani and R.J. Loy and G.A. Willis, *Amenability and wek amenability of second cojugate Banach algebras.* Proc. Amer. Math. Soc. **124** (1996), 1489–1497.

[14] F. Ghahramani, R.J. Loy and Y. Zhang, *Generalized notions of amenability II.* J. Funct. Anal. **254** (2008), 1776–1810.

[15] N. Grønbæk, *A characterization of weakly amenable Banach algebras.* Studia Math. **94** (1989), 149-162.

[16] N. Grønbæk, *Weak and cyclic amenability for non-commutative Banach algebras.* Proc. Edinburgh. Math. Soc. **35** (1992), 315–328.

[17] B.E. Johnson, *Weak amenability of group algebras.* Bull. London Math. Soc. **23** (3) (1991), 281–284.

[18] B. Shojaee and A. Bodaghi, *Approximate cyclic amenability of Banach algebras.* Math. Sci. **5** (2011), *1*, 25–32.

[19] B. Shojaee, G.H. Esslamzadeh and A. Pourabbas, *First order cohomology of ℓ^1-Munn algebras.* Bull. Iranian Math. Soc. **35** (2009), 211–219.

Department of Mathematics,
Karaj Branch
Islamic Azad University,
Karaj, Iran
shoujaei@kiau.ac.ir

Young Researchers and Elite Club
Islamshahr Branch
Islamic Azad University
Islamshahr, Iran
abasalt.bodaghi@gmail.com