SYMMETRIC LIFTING OPERATOR ACTING ON SOME SPACES OF ANALYTIC FUNCTIONS

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Communicated by Lucian Beznea

In this paper, we study the cases where a function in the spaces of analytic functions in the unit disc is symmetrically lifted to the spaces of analytic functions in the bidisc, and for this end we apply a characterization for Bergman and Dirichlet type spaces.

AMS 2010 Subject Classification: 47B38, 32A36.

Key words: Symmetric lifting operator, Dirichlet type spaces, Bergman spaces.

1. INTRODUCTION

Let $\mathbb{D}$ the open unit disc in the complex plane $\mathbb{C}$. For any $\alpha > -1$ we consider the weighted area measure

$$\mathrm{d}A_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \mathrm{d}A(z),$$

where $\mathrm{d}A$ is the normalized area measure on $\mathbb{D}$.

For $p > 0$ and $\alpha > -1$ we denote by $A^p_\alpha(\mathbb{D})$ the space of analytic functions f in $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^p \mathrm{d}A_\alpha(z) < \infty.$$

These are called the weighted Bergman spaces with the standard weight. See [1], [2], and [6] for the theory of Bergman spaces.

We define by $\mathcal{D}^p_\alpha(\mathbb{D})$ the space of analytic functions f in $\mathbb{D}$ such that $f' \in A^p_\alpha(\mathbb{D})$, so

$$\int_{\mathbb{D}} |f'(z)|^p \mathrm{d}A_\alpha(z) < \infty.$$

These are called the Dirichlet type spaces.

The Pseudo-hyperbolic and hyperbolic metrics on $\mathbb{D}$ are given by

$$\rho(z,w) = \left|\frac{z-w}{1-\bar{z}w}\right|,$$

and

$$\beta(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}$$

respectively.

We mention that the hyperbolic metric is also called the Bergman metric and sometimes the Poincaré metric.

Characterizations for the Bergman spaces have appeared in [7] and [8]. A characterization for the Bergman spaces with euclidean, hyperbolic and pseudo-hyperbolic metrics have been obtained in [11] as follows:

THEOREM 1.1. *Suppose $p > 0$, $\alpha > -1$ and f is analytic in $\mathbb{D}$. Then the following conditions are equivalent.*
(a) f belongs to $A^P_\alpha(\mathbb{D})$.
(b) There exists a continuous function g in $L^p(\mathbb{D}, \mathrm{d}A_\alpha)$ such that

$$|f(z) - f(w)| \leq \rho(z, w)(g(z) + g(w))$$

for all z and w in $\mathbb{D}$.
(c) There exists a continuous function g in $L^p(\mathbb{D}, \mathrm{d}A_\alpha)$ such that

$$|f(z) - f(w)| \leq \beta(z, w)(g(z) + g(w))$$

for all z and w in $\mathbb{D}$.
(d) There exists a continuous function g in $L^p(\mathbb{D}, \mathrm{d}A_{p+\alpha})$ such that

$$|f(z) - f(w| \leq |z - w|(g(z) + g(w))$$

for all z and w in $\mathbb{D}$.

Similar characterizations for the Hardy-Sobolev spaces have appeared in [3] and [4]. For the classical Bloch space $\mathcal{B}$, in [6] and [12], it was shown that any analytic function f in $\mathbb{D}$ belongs to $\mathcal{B}$ if and only if there is a positive constant C, such that

$$|f(z) - f(w)| \leq C\beta(z, w).$$

Let $\mathbb{D}^2 = \mathbb{D} \times \mathbb{D}$ be the bidisc in $\mathbb{C}^2$ and let $H(\mathbb{D}^2)$ denote the space of all holomorphic functions in $\mathbb{D}^2$. Similarly $H(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$. For $p > 0$ and $\alpha > -1$ we define $A^p_\alpha(\mathbb{D}^2)$ as the space of functions $f \in H(\mathbb{D}^2)$ such that

$$\int_{\mathbb{D}} \int_{\mathbb{D}} |f(z, w)|^p \mathrm{d}A_\alpha(z) \mathrm{d}A_\alpha(w) < \infty.$$

These are also called weighted Bergman spaces.

We define $\mathcal{D}^p_\alpha(\mathbb{D}^2)$ as the space of functions $f \in H(\mathbb{D}^2)$ such that

$$\int_{\mathbb{D}} \int_{\mathbb{D}} |Rf(z, w)|^p \mathrm{d}A_\alpha(z) \mathrm{d}A_\alpha(w) < \infty,$$

where $Rf(z, w)$ is the directional derivative of f at (z, w) in the radial direction, and is given by

$$Rf(z,w) = z.\frac{\partial f}{\partial z}(z,w) + w.\frac{\partial f}{\partial w}(z,w).$$

We consider the symmetric lifting operator

$$L : H(\mathbb{D}) \to H(\mathbb{D}^2)$$

defined by

$$L(f)(z,w) = \frac{f(z) - f(w)}{z - w}.$$

For any $f \in \mathcal{D}^p_\alpha(\mathbb{D}^2)$ and any arbitrary point $(z,w) \in \mathbb{D}^2$, we write

$$R(L(f))(z,w) = \frac{\partial L(f)}{\partial z}.z + \frac{\partial L(f)}{\partial w}.w,$$

and we call it the radial derivative of L.

With a direct calculation we obtain

$$R(L(f))(z,w) = \frac{f'(z)z^2 + f'(w)w^2 - f'(z)wz - f(z)z}{(z-w)^2}$$
$$+ \frac{f(w)z - f'(w)wz + f(z)w - f(w)w}{(z-w)^2},$$

so we have

$$R(L(f))(z,w) = \frac{(zf'(z) - wf'(w))(z-w) + (f(w) - f(z))(z-w)}{(z-w)^2}$$
$$= \frac{zf'(z) - wf'(w)}{z-w} + \frac{f(w) - f(z)}{z-w}.$$

Some properties of the weighted composition operators are being studied in [9] and [10] by Vaezi. In this article we study the action of symmetric lifting operator on Bergman and Dirichlet type spaces.

2. SYMMETRIC LIFTING OPERATOR ON BERGMAN SPACES

The action of the symmetric lifting operator on Bergman spaces has been studied by Wulan and Zhu in [11].

Some results about the lifting operator on the Bergman spaces is also given in [5] by Hassanlou and Vaezi.

THEOREM 2.1. *Suppose $\alpha > -1$, $p > 0$. Then the symmetric lifting operator L maps $A^p_\alpha(\mathbb{D})$ boundedly into $A^p_{\alpha+p}(\mathbb{D}^2)$.*

Proof. Given $f \in A^p_\alpha(\mathbb{D})$, we apply Theorem 1.1 (d) to find a function $g \in L^p(\mathbb{D}, \mathrm{d}A_{\alpha+p})$ such that

$$|f(z) - f(w)| \leq |z - w|(g(z) + g(w)),$$

for $z, w \in \mathbb{D}$. Then there exists a constant $C = C_p$ such that

$$|L(f)(z,w)|^p \leq C_p(g(z)^p + g(w)^p).$$

It follows that

$$\int_{\mathbb{D}}\int_{\mathbb{D}} |L(f)(z,w)|^p \mathrm{d}A_{\alpha+p}(z)\mathrm{d}A_{\alpha+p}(w) \leq$$

$$C_p \int_{\mathbb{D}}\int_{\mathbb{D}} (|g(z)|^p + |g(w)|^p)(1-|z|^2)^{\alpha+p}(1-|w|^2)^{\alpha+p}\mathrm{d}A(z)\mathrm{d}A(w).$$

But for any $z, w \in \mathbb{D}$, we have $(1-|z|^2)^{\alpha+p} \leq 1$ and also $(1-|w|^2)^{\alpha+p} \leq 1$. So we can write

$$\int_{\mathbb{D}}\int_{\mathbb{D}} |L(f)(z,w)|^p \mathrm{d}A_{\alpha+p}(z)\mathrm{d}A_{\alpha+p}(w) \leq 2C_p \int_{\mathbb{D}} |g(z)|^p (1-|z|^2)^{\alpha+p}\mathrm{d}A(z).$$

This shows that L maps $A^p_\alpha(\mathbb{D})$ into $A^p_{\alpha+p}(\mathbb{D}^2)$.

Since L is linear and two spaces A^p_α and $A^p_{\alpha+p}$ are Banach spaces, we can use the closed graph theorem to show that

$$L : A^p_\alpha(\mathbb{D}) \to A^p_{\alpha+p}(\mathbb{D}^2)$$

is bounded. $\square$

The following theorem is proved in [11] by Wulan and Zhu.

THEOREM 2.2. *Suppose $\alpha > -1$ and $0 < p < \alpha + 2$. Then the symmetric lifting operator L maps $A^p_\alpha(\mathbb{D})$ boundedly into $A^p_\alpha(\mathbb{D}^2)$. Moreover, this is no longer true when $p > \alpha + 2$.*

If $\alpha > -1$, $p > \alpha + 2$ and β is determined by $2(\beta + 1) = p + \alpha$, we have $\beta < p + \alpha$, since $p + \alpha > 0$. So $A^p_\beta(\mathbb{D}^2) \subset A^p_{\alpha+p}(\mathbb{D}^2)$. In this special case Wulan and Zhu in [11] have proved the following theorem.

THEOREM 2.3. *Suppose $\alpha > -1$, $p > \alpha + 2$, and β is determined by*

$$2(\beta + 1) = p + \alpha.$$

Then the symmetric lifting operator L maps $A^p_\alpha(\mathbb{D})$ boundedly into $A^p_\beta(\mathbb{D}^2)$.

We prove the result for the case $p = \alpha + 2$. In the sequel we need the following standard estimate.

LEMMA 2.4. *For any $\alpha > -1$ and any real β, let*

$$I_{\alpha,\beta}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-z\overline{w}|^{2+\alpha+\beta}}\mathrm{d}A(w), \quad z \in \mathbb{D}.$$

Then

$$I_{\alpha,\beta}(z) = \begin{cases} 1 & \beta < 0 \\ \log \frac{1}{1-|z|^2} & \beta = 0 \\ \frac{1}{(1-|z|^2)^\beta} & \beta > 0 \end{cases}$$

Proof. See [6]. □

It is well known that for $\alpha > -1$ and $p > 0$, $A^p_\alpha(\mathbb{D}^2) \subset A^p_{\alpha+1}(\mathbb{D}^2)$. In the case $p = \alpha + 2$ we prove the following theorem.

THEOREM 2.5. *Suppose* $\alpha > -1$ *and* $p = \alpha + 2$. *Then the symmetric lifting operator* L *maps* $A^p_\alpha(\mathbb{D})$ *boundedly into* $A^p_{\alpha+1}(\mathbb{D}^2)$.

Proof. Given $f \in A^p_\alpha(\mathbb{D})$, we apply Theorem 1.1 (b) to find a function $g \in L^p(\mathbb{D}, \mathrm{d}A_\alpha)$ such that

$$|f(z) - f(w)| \le \rho(z,w)(g(z) + g(w))$$

for any $z, w \in \mathbb{D}$.

Then there exists a constant $C = C_p$ such that

$$|L(f)(z,w)|^p \le C\Big[\frac{g(z)^p}{|1-\overline{z}w|^p} + \frac{g(w)^p}{|1-\overline{z}w|^p}\Big].$$

It follows that

$$\int_{\mathbb{D}}\int_{\mathbb{D}} |L(f)(z,w)|^p \mathrm{d}A_{\alpha+1}(z)\mathrm{d}A_{\alpha+1}(w) \le \int_{\mathbb{D}}\int_{\mathbb{D}} |L(f)(z,w)|^p \mathrm{d}A_{\alpha+1}(z)\mathrm{d}A_\alpha(w)$$

$$\le 2C \int_{\mathbb{D}} g(z)^p \mathrm{d}A_{\alpha+1}(z) \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-\overline{z}w|^{2+\alpha}} \mathrm{d}A(w).$$

By using Lemma 2.4 (in the case $\beta = 0$) and the boundedness of $(1-|z|^2)\log\frac{1}{1-|z|^2}$, for $z \in \mathbb{D}$, there exists another constant $C > 0$ such that

$$\int_{\mathbb{D}}\int_{\mathbb{D}} |L(f)(z,w)|^p \mathrm{d}A_{\alpha+1}(z)\mathrm{d}A_{\alpha+1}(w) \le C\int_{\mathbb{D}} g(z)^p \mathrm{d}A_\alpha(z).$$

This proves the theorem. □

3. SYMMETRIC LIFTING OPERATOR ON DIRICHLET TYPE SPACES

In this section, we study the action of the symmetric lifting operator on Dirichlet type spaces $\mathcal{D}^p_\alpha$.

THEOREM 3.1. *Suppose* $\alpha > -1$, $p > 0$. *Then the symmetric lifting operator* L *maps* $\mathcal{D}^p_\alpha(\mathbb{D})$ *boundedly into* $\mathcal{D}^p_{\alpha+p}(\mathbb{D}^2)$.

Proof. Suppose $f \in \mathcal{D}^p_\alpha(\mathbb{D})$. To show that $L(f) \in \mathcal{D}^p_{\alpha+p}(\mathbb{D}^2)$ it is sufficient to show that $R(L(f)) \in A^p_{\alpha+p}(\mathbb{D}^2)$. But as we obtained earlier

$$R(L(f))(z,w) = \frac{zf'(z) - wf'(w)}{z-w} + \frac{f(w)-f(z)}{z-w}$$

for any $z, w \in \mathbb{D}$. Since $\frac{f(w)-f(z)}{z-w} = -L(f)(z,w)$, according to Theorem 2.1, $\frac{f(w)-f(z)}{z-w} \in A^p_{\alpha+p}(\mathbb{D}^2)$. So it remains to show that $\frac{zf'(z)-wf'(w)}{z-w} \in A^p_{\alpha+p}(\mathbb{D}^2)$.

Since $f' \in A^p_\alpha(\mathbb{D})$, if we define $h(z) = zf'(z)$ for any $z \in \mathbb{D}$, then $h(z) \in A^p_\alpha(\mathbb{D})$. According to Theorem 1.1 there exists a function $g \in L^p(\mathbb{D}, \mathrm{d}A_\alpha)$ such that

$$|h(z) - h(w)| \leq \rho(z,w)(g(z) + g(w)).$$

It follows that

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|zf'(z) - wf'(w)|^p}{|z-w|^p} \mathrm{d}A_{\alpha+p}(z)\mathrm{d}A_{\alpha+p}(w) \leq$$

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|z-w|^p|g(z)+g(w)|^p}{|z-w|^p|1-\overline{z}w|^p} \mathrm{d}A_{\alpha+p}(z)\mathrm{d}A_{\alpha+p}(w).$$

According to Lemma 2.4 (for $\beta < -1$), there exists a constant C such that

$$\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|g(z)+g(w)|^p}{|1-\overline{z}w|^p} \mathrm{d}A_{\alpha+p}(z)\mathrm{d}A_{\alpha+p}(w) \leq$$

$$2C \int_{\mathbb{D}} |g(z)|^p \mathrm{d}A_{\alpha+p}(z) \int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha+p}}{|1-\overline{z}w|^p} \mathrm{d}A(w) \leq$$

$$2C \int_{\mathbb{D}} |g(z)|^p \mathrm{d}A_\alpha(z) < \infty.$$

The closed graph theorem gives us the boundedness of L. This completes the proof. □

THEOREM 3.2. *Suppose $\alpha > -1$ and $0 < p < \alpha + 2$. Then the symmetric lifting operator L maps $\mathcal{D}^p_\alpha(\mathbb{D})$ boundedly into $\mathcal{D}^p_\alpha(\mathbb{D}^2)$.*

Proof. Given $f \in \mathcal{D}^p_\alpha(\mathbb{D})$, since $\mathcal{D}^p_\alpha(\mathbb{D}) \subset A^p_\alpha(\mathbb{D})$, according to Theorem 2.2, $L(f) \in A^p_\alpha(\mathbb{D}^2)$. But we have shown that $R(L(f))(z,w) = \frac{zf'(z)-wf'(w)}{z-w} - L(f)(z,w)$. So as in the proof of Theorem 3.1, it is enough to show that

$$\frac{zf'(z) - wf'(w)}{z-w} \in A^p_\alpha(\mathbb{D}^2).$$

Using Theorem 1.1, we find a function $g \in L^p(\mathbb{D}, \mathrm{d}A_\alpha)$ such that

$$|zf'(z) - wf'(w)| \leq \rho(z,w)(g(z) + g(w)).$$

It follows that

$$\int_{\mathbb{D}}\int_{\mathbb{D}}\Big|\frac{zf'(z)-wf'(w)}{z-w}\Big|^p \mathrm{d}A_\alpha(z)\mathrm{d}A_\alpha(w) \le \int_{\mathbb{D}}\int_{\mathbb{D}}\frac{|g(z)+g(w)|^p}{|1-\overline{z}w|^p}\mathrm{d}A_\alpha(z)\mathrm{d}A_\alpha(w)$$

$$\le 2C\int_{\mathbb{D}}|g(z)|^p\mathrm{d}A_\alpha(z)\int_{\mathbb{D}}\frac{(1-|w|^2)^\alpha}{|1-\overline{z}w|^p}\mathrm{d}A(w).$$

Since Lemma 2.4 in the case $\beta < 0$ gives an upper bound for $\int_{\mathbb{D}}\frac{(1-|w|^2)^\alpha}{|1-\overline{z}w|^p}\mathrm{d}A(w)$ and we also supposed that $g \in L^p(\mathbb{D}, \mathrm{d}A_\alpha)$, the above inequality implies that

$$\int_{\mathbb{D}}\int_{\mathbb{D}}\Big|\frac{zf'(z)-wf'(w)}{z-w}\Big|^p \mathrm{d}A_\alpha(z)\mathrm{d}A_\alpha(w) < \infty$$

and then $\frac{zf'(z)-wf'(w)}{z-w} \in A^p_\alpha(\mathbb{D}^2)$. Via the closed graph theorem it follows that the operator

$$L : \mathcal{D}^p_\alpha(\mathbb{D}) \to \mathcal{D}^p_\alpha(\mathbb{D}^2)$$

is bounded. □

THEOREM 3.3. *Suppose that $\alpha > -1$, $p > \alpha + 2$, and that β is determined by*

$$2(\beta + 1) = p + \alpha.$$

Then the symmetric lifting operator L maps $\mathcal{D}^p_\alpha(\mathbb{D})$ boundedly into $\mathcal{D}^p_\beta(\mathbb{D}^2)$.

Proof. Given $f \in \mathcal{D}^p_\alpha(\mathbb{D})$, according to Theorem 2.3, $L(f) \in A^p_\beta(\mathbb{D}^2)$. Similarly to the previous proofs, it is enough to show that

$$\frac{zf'(z)+wf'(w)}{z-w} \in A^p_\beta(\mathbb{D}^2).$$

The theorem can be easily proved by using Theorem 1.1 and Lemma 2.4 in the case $\beta > 0$. □

THEOREM 3.4. *Suppose $\alpha > -1$ and $p = \alpha + 2$. Then the symmetric lifting operator L maps $\mathcal{D}^P_\alpha(\mathbb{D})$ boundedly into $\mathcal{D}^p_{\alpha+1}(\mathbb{D}^2)$.*

Proof. Similarly to Theorem 3.1, the proof of the theorem follows by using Theorem 1.1, Lemma 2.4 in the case $\beta = 0$, the boundedness of $(1-|z|^2)\log\frac{1}{1-|z|^2}$, for $z \in \mathbb{D}$ and Theorem 2.5. □

Acknowledgments. The authors wish to express their gratitude to the referees for giving valuable comments and suggestions improving the final version of the manuscript.

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Received 15 January 2014

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