ON COHOMOLOGICALLY COMPLETE INTERSECTIONS IN COHEN-MACAULAY RINGS

WAQAS MAHMOOD

Communicated by Vasile Brînzănescu

An ideal I of a local Cohen-Macaulay ring $(R, \mathfrak{m})$ is called a cohomologically complete intersection if $H^i_I(R) = 0$ for all $i \neq c :=$ height(I). Here $H^i_I(R)$, $i \in \mathbb{Z}$ denotes the local cohomology of R with respect to I. For instance, a set-theoretic complete intersection is a cohomologically complete intersection. Here we study cohomologically complete intersections from various homological points of view. As a main result it is shown that the vanishing $H^i_I(M) = 0$ for all $i \neq c$ is completely encoded in homological properties of $H^c_I(M)$. These results extend those of Hellus and Schenzel (see [13, Theorem 0.1]) shown in the case of a local Gorenstein ring. In particular we get a characterization of cohomologically complete intersections in a Cohen-Macaulay ring in terms of the canonical module.

AMS 2010 Subject Classification: 13D45.

Key words: local cohomology, cohomologically complete intersection, local duality.

1. INTRODUCTION

Let $(R, \mathfrak{m})$ denote a local Noetherian ring. For an ideal $I \subset R$ it is a rather difficult question to determine the smallest number $n \in \mathbb{N}$ of elements $a_1, ..., a_n \in R$ such that $\operatorname{Rad} I = \operatorname{Rad}(a_1, ..., a_n)R$. This number is called the arithmetic rank, $\operatorname{ara} I$, of I. By Krull's generalized principal ideal theorem it follows that $\operatorname{ara} I \geq \operatorname{height}(I)$. Of a particular interest is the case whenever $\operatorname{ara} I = \operatorname{height}(I)$. If this equality holds then I is called a set-theoretic complete intersection.

For the ideal I let $H^i_I(\cdot), i \in \mathbb{Z}$, denote the local cohomology functor with respect to I, see [6] for its definition and basic results. The cohomological dimension, $\operatorname{cd}(I)$, defined by

$$\operatorname{cd}(I) = \sup\{i \in \mathbb{Z} | H^i_I(R) \neq 0\}$$

is another invariant related to the ideal I. It is well known that

$$\operatorname{grade} I \leq \operatorname{height}(I) \leq \operatorname{cd}(I) \leq \operatorname{ara} I.$$

This research was partially supported by Higher Education Commission, Pakistan.

In particular, if I is a set-theoretically complete intersection it follows that $\operatorname{height}(I) = \operatorname{cd}(I)$. The converse does not hold in general (see [12, Remark 2.1(ii)]). Not so much is known about ideals with the property of $\operatorname{grade} I = \operatorname{cd}(I)$. We call those ideals cohomologically complete intersections. In this paper we continue with the investigations of cohomologically complete intersections, in particular when I is an ideal in a Cohen-Macaulay ring $(R, \mathfrak{m})$. In this case I is a cohomologically complete intersection if $\operatorname{height}(I) = \operatorname{cd}(I)$.

As an application of our main results there is a characterization of cohomologically complete intersections. In fact, this provides a large number of necessary conditions for an ideal to be a set-theoretic complete intersection in a local Cohen-Macaulay ring.

THEOREM 1.1. *Let* $(R, \mathfrak{m})$ *be a ring of dimension* $n, I \subseteq R$ *an ideal of* $\operatorname{grade}(I, M) = c$, *and* $M \neq 0$ *a maximal Cohen-Macaulay* R*-module with* $\operatorname{id}_R(M) < \infty$. *Then the following conditions are equivalent:*

(a) $H^i_I(M) = 0$ *for all* $i \neq c$.

(b) *For all* $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M)$ *the natural homomorphism*

$$H^{h(\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) \to H^{\dim(M_\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$$

is an isomorphism and $H^i_{\mathfrak{p}R_\mathfrak{p}}(H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) = 0$ *for all* $i \neq h(\mathfrak{p}) = \dim(M_\mathfrak{p}) - c$.

(c) *For all* $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M)$ *the natural homomorphism*

$$\operatorname{Ext}^{h(\mathfrak{p})}_{R_\mathfrak{p}}(k(\mathfrak{p}), H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) \to \operatorname{Ext}^{\dim(M_\mathfrak{p})}_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p})$$

is an isomorphism and $\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) = 0$ *for all* $i \neq h(\mathfrak{p}) = \dim(M_\mathfrak{p}) - c$.

(d) *For all* $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M)$ *the natural homomorphism*

$$K(\hat{M}_\mathfrak{p}) \to \operatorname{Ext}^c_{\hat{R}_\mathfrak{p}}(H^c_{I\hat{R}_\mathfrak{p}}(\hat{M}_\mathfrak{p}), K(\hat{R}_\mathfrak{p}))$$

is an isomorphism and $\operatorname{Ext}^i_{\hat{R}_\mathfrak{p}}(H^c_{I\hat{R}_\mathfrak{p}}(\hat{M}_\mathfrak{p}), K(\hat{R}_\mathfrak{p})) = 0$ *for all* $i \neq h(\mathfrak{p}) = \dim(M_\mathfrak{p}) - c$.

(e) $H^i_I(R) = 0$ *for all* $i \neq c$, *that is* I *is a cohomologically complete intersection.*

Note that the natural homomorphisms, in (b), (c), and (d) of the above Theorem, can be constructed by truncation complex (see Definition 4.1 and Theorem 4.3). Moreover $K(\hat{M}_\mathfrak{p})$ denotes the canonical module of $\hat{M}_\mathfrak{p}$, see [18] for its definition and basic properties.

In the case of $M = R$ a local Gorenstein ring the equivalence of the conditions (a), (b), (c), (d) was shown by Hellus and Schenzel as the main result of their paper (see [13, Theorem 0.1]). The new point of view here is the generalization to any maximal Cohen-Macaulay module of finite injective dimension. This implies a large bunch of new necessary conditions for an ideal to become set-theoretically a complete intersection.

As a byproduct of our investigations there is another characterization of a Cohen-Macaulay ring. H. Bass conjectured in his paper (see [1]) that if a Noetherian local ring $(R, \mathfrak{m})$ possesses a non-zero finitely generated R-module M such that its injective dimension $\operatorname{id}_R(M)$ is finite then R is Cohen-Macaulay ring. This was proved by M. Hochster (see [9]) in the equicharacteristic case and finally by P. Roberts (see [16]). We add here the following characterization of a local Cohen-Macaulay ring.

THEOREM 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I an ideal of R, and $M \neq 0$ a finitely generated R-module. Suppose that $H^i_I(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M)$, then the following are equivalent:*

(1) $\operatorname{id}_R(M) < \infty$.

(2) $\operatorname{id}_R(H^c_I(M)) < \infty$.

If one of the equivalent conditions is satisfied then R is a Cohen-Macaulay ring.

The paper is organized as follows. In the Section 2 we recall a few preliminaries used in the sequel of the paper. In Section 3 we make a comment to the Local Duality for a Cohen-Macaulay ring. In Section 4 we describe the truncation complex as it was introduced by Hellus and Schenzel (see [13, Definition 2.1]) and use it for the proofs of our main results. In Section 5 we conclude with a few applications.

2. PRELIMINARIES

In this section, we will fix the notation of the paper and summarize a few preliminaries and auxiliary results. We always assume that $(R, \mathfrak{m})$ is a local commutative Noetherian ring with $\mathfrak{m}$ as a maximal ideal and $k = R/\mathfrak{m}$ denotes the residue field. Furthermore $E = E_R(k)$ denotes the injective hull of k.

Let $I \subseteq R$ be an ideal of R. For an R-module M let $H^i_I(M)$, $i \in \mathbb{Z}$ denote the local cohomology modules of M with respect to I (see [6] for its definition). We define the grade and the cohomological dimension

$$\operatorname{grade}(I, M) = \inf\{i \in \mathbb{Z} : H^i_I(M) \neq 0\} \text{ and } \operatorname{cd}(I, M) = \sup\{i \in \mathbb{Z} : H^i_I(M) \neq 0\}.$$

For a finitely generated R-module M this notion of grade coincides with the usual one on the maximal length of an M-regular sequence contained in I. For any R-module X, $\operatorname{id}_R(X)$ stands for the injective dimension of X.

Moreover, if M is finitely generated, we define $\operatorname{height}_M I = \operatorname{height} IR/\operatorname{Ann}_R M$. Then $\operatorname{grade}(I, M) \leq \operatorname{height}_M I \leq \operatorname{cd}(I, M)$. In the case of $M = R$

in addition it follows that height $I \leq \operatorname{ara} I \leq \operatorname{cd} I$. Furthermore for a Cohen-Macaulay R-module M it turns out that $\operatorname{height}_M I = \dim M - \dim M/IM$ for any ideal $I \subset R$. For a Cohen-Macaulay R-module M it is clear that $\operatorname{height}_M I = \operatorname{grade}(I, M)$.

Remark 2.1. Let $\underline{x} = x_1, \ldots, x_r \in I$ denote a system of elements of R such that $\operatorname{Rad} I = \operatorname{Rad}(\underline{x})R$. We consider the Čech complex $\check{C}_{\underline{x}}$ with respect to $\underline{x} = x_1, ..., x_r$. That is

$$\check{C}_{\underline{x}} = \overset{r}{\underset{i}{\otimes}} \check{C}_{x_i},$$

where $\check{C}_{x_i}$ is the complex $0 \to R \to R_{x_i} \to 0$. Then $\check{C}_{\underline{x}}$ has the following form

$$0 \to R \to \overset{r}{\underset{i=1}{\oplus}} R_{x_i} \to ... \to R_{x_1 x_2 \ldots x_r} \to 0.$$

By $\check{D}_{\underline{x}}$ we denote the truncation of $\check{C}_{\underline{x}}$ by R. That is, we have $\check{D}^i_{\underline{x}} = \check{C}^i_{\underline{x}}$ for all $i \neq 0$ and $\check{D}^0_{\underline{x}} = 0$.

So there is a short exact sequence of complexes of flat R-modules

$$0 \to \check{D}_{\underline{x}} \to \check{C}_{\underline{x}} \to R \to 0,$$

where $\check{C}_{\underline{x}} \to R$ is the identity in homological degree zero.

For an arbitrary complex of R-modules X it follows (see [19, Theorem 1.1]) that

$$H^i(\check{C}_{\underline{x}} \otimes_R X) \cong H^i_I(X) \text{ for all } i \in \mathbb{Z}.$$

Now let us summarize a few well-known facts about grade and local cohomology. For basic notions on grade as well as other notions of commutative algebra we refer to Matsumura's textbook (see [15]). For the facts on homological algebra needed in this paper see [21]. Also we denote $\hat{R}$ for the completion of R with respect to the maximal ideal.

PROPOSITION 2.2. *For finitely generated R-modules M, N and $I \subseteq R$ be an ideal we have*

(a) $\operatorname{grade}(I, M) = \inf\{\operatorname{depth}(M_{\mathfrak{p}}) : \mathfrak{p} \in \operatorname{Supp}_R(M) \cap V(I)\}$.

(b) *Suppose that* $\operatorname{Supp}_R(N) \subseteq \operatorname{Supp}_R(M)$, *then* $\operatorname{cd}(I, N) \leq \operatorname{cd}(I, M)$.

In particular $\operatorname{cd}(I, M) \leq \operatorname{cd}(I)$.

Proof. The statement (a) is shown in [3, Proposition 1.2.10]. For the proof of (b) we refer to [4, Theorem 2.2]. □

In the context of our paper we are interested in Cohen-Macaulay rings and modules. A non-zero finitely generated R-module M is called maximal Cohen-Macaulay module if $\operatorname{depth}(M) = \dim(R)$.

Proposition 2.3. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay ring of dimension* n *and* $I \subseteq R$ *an ideal. For a maximal Cohen-Macaulay module* $M \neq 0$ *we consider the following conditions:*

(a) $\operatorname{grade}(I) = \operatorname{cd}(I)$ *and*

(b) $\operatorname{grade}(I, M) = \operatorname{cd}(I, M)$.

Then (a) implies (b), while the converse is also true provided that $\operatorname{Supp}_R(M) = \operatorname{Spec}(R)$.

Proof. Let $\mathfrak{p} \in \operatorname{Supp}_R(M)$. Because of $\operatorname{depth}(M_\mathfrak{p}) = \operatorname{depth}(R_\mathfrak{p}) = \dim(R_\mathfrak{p})$ (recall that M is maximal Cohen-Macaulay module) it follows that

$$\operatorname{grade}(I) \leq \operatorname{grade}(I, M) \leq \operatorname{cd}(I, M) \leq \operatorname{cd}(I)$$

So (a) implies (b). If $\operatorname{Supp}_R(M) = \operatorname{Spec}(R)$ we have that $\operatorname{grade}(I) = \operatorname{grade}(I, M)$ and $\operatorname{cd}(I, M) = \operatorname{cd}(I)$ (see Proposition 2.2) and the converse holds also. □

Notation 2.4. As usual we use the symbol " $\cong$ " in order to denote an isomorphism of modules. In contrast to that we use the symbol " $\xrightarrow{\sim}$ " in the following context:

Let $X \to Y$ be a morphism of complexes such that it induces an isomorphism in cohomologies, i.e. a quasi-isomorphism. Then we write $X \xrightarrow{\sim} Y$. That is " $\xrightarrow{\sim}$ " indicates that there is a morphism of complexes in the right direction.

Moreover if $X \xrightarrow{\sim} Y$ is a quasi-isomorphism and $F_R^{\cdot}$ is a complex of flat R-modules bounded above. Then it induces a quasi-isomorphism $F_R^{\cdot} \otimes_R X \xrightarrow{\sim} F_R^{\cdot} \otimes_R Y$. Similar results are true for $\operatorname{Hom}_R(., E_R^{\cdot})$ respectively $\operatorname{Hom}_R(P_R^{\cdot}, .)$ for $E_R^{\cdot}$ a bounded below complex of injective R-modules respectively $P_R^{\cdot}$ a bounded above complex of projective R-modules. For details we refer to [7]. This is the only fact we need of the theory of derived functors.

Note that the following Lemma remains true for an arbitrary Noetherian ring and an arbitrary ideal.

Lemma 2.5. *Let* $(R, \mathfrak{m})$ *be a ring of dimension* n. *Let* X *be a complex of* R-*modules such that* $\operatorname{Supp}_R(H^i(X)) \subseteq V(\mathfrak{m})$ *for all* $i \in \mathbb{Z}$. *Then* $H^i_\mathfrak{m}(X) \cong H^i(X)$ *for all* $i \in \mathbb{Z}$.

Proof. Let $\underline{x} = x_1, \ldots, x_n \in \mathfrak{m}$ denote a system of parameters of R. Then we have the following short exact sequence of complexes of flat R-modules

$$0 \to \check{D}_{\underline{x}} \to \check{C}_{\underline{x}} \to R \to 0.$$

Apply the functor $\cdot \otimes_R X$ to this sequence and we get the following long exact sequence of cohomology modules

$$\cdots \to H^i(\check{D}_{\underline{x}} \otimes_R X) \to H^i_\mathfrak{m}(X) \to H^i(X) \to \cdots$$

Now we claim that $\check{D}_{\underline{x}} \otimes_R X$ is an exact complex. This follows because

$$\bigoplus_j H^i(R_{x_j} \otimes_R X) \cong \bigoplus_j R_{x_j} \otimes_R H^i(X) = 0$$

This is true since $\operatorname{Supp}_R(H^i(X)) \subseteq V(\mathfrak{m})$ and cohomology commutes with exact functors. So it proves that $H^i(X) \cong H^i_{\mathfrak{m}}(X)$ for all $i \in \mathbb{Z}$, as required. $\square$

In the following, we need a result that was originally proved by Hellus and Schenzel (see [13, Proposition 1.4]) by a spectral sequence. Here we will give an elementary proof without using spectral sequences.

Proposition 2.6. *Let $(R, \mathfrak{m})$ be a local ring. Let X be an arbitrary R-module. Then for any integer $s \in \mathbb{N}$ the following conditions are equivalent:*
(1) $H^i_{\mathfrak{m}}(X) = 0$ *for all* $i < s$.
(2) $\operatorname{Ext}^i_R(k, X) = 0$ *for all* $i < s$.
If one of the above conditions holds, then there is an isomorphism

$$\operatorname{Hom}_R(k, H^s_{\mathfrak{m}}(X)) \cong \operatorname{Ext}^s_R(k, X).$$

Proof. We prove the statement by an induction on s. First let us consider that $s = 0$. Because of $\operatorname{Supp}(k) = \{\mathfrak{m}\} = V(\mathfrak{m})$ the injection $\Gamma_{\mathfrak{m}}(X) \subseteq X$ induces an isomorphism

$$\operatorname{Hom}_R(k, \Gamma_{\mathfrak{m}}(X)) \cong \operatorname{Hom}_R(k, X).$$

Recall that $\operatorname{Hom}_R(k, X/\Gamma_{\mathfrak{m}}(X)) = 0$ since $X/\Gamma_{\mathfrak{m}}(X)$ is not $\mathfrak{m}$-torsion. Because of $\operatorname{Supp}(\Gamma_{\mathfrak{m}}(X)) \subseteq V(\mathfrak{m})$ it follows that $\Gamma_{\mathfrak{m}}(X) = 0$ if and only if $\operatorname{Hom}_R(k, \Gamma_{\mathfrak{m}}(X)) = 0$ which proves the claim for $s = 0$.

Now consider $s + 1$ and assume that the statement is true for all $i \leq s$. Since $\operatorname{Supp}(H^s_{\mathfrak{m}}(X)) \subseteq V(\mathfrak{m})$, by using the last isomorphism applied for i = s, it follows the equivalence of the vanishing of $H^i_{\mathfrak{m}}(X)$ and $\operatorname{Ext}^i_R(k, X)$ for all $i \leq s$.

So it remains to prove that

$$\operatorname{Hom}_R(k, H^{s+1}_{\mathfrak{m}}(X)) \cong \operatorname{Ext}^{s+1}_R(k, X)$$

To this end let $E^{\cdot}_R(X)$ be a minimal injective resolution of X, then

$$\operatorname{Hom}_R(k, (E^{\cdot}_R(X))^i) = 0$$

for all $i \leq s$. That is we have the following exact sequence

$$0 \to H^{s+1}_{\mathfrak{m}}(X) \to \Gamma_{\mathfrak{m}}(E^{\cdot}_R(X))^{s+1} \to \Gamma_{\mathfrak{m}}(E^{\cdot}_R(X))^{s+2}$$

Because of $\operatorname{Hom}_R(k, E^{\cdot}_R(X)) \cong \operatorname{Hom}_R(k, \Gamma_{\mathfrak{m}}(E^{\cdot}_R(X)))$ it follows that $\operatorname{Ext}^{s+1}_R(k, X) \cong \operatorname{Hom}_R(k, H^{s+1}_{\mathfrak{m}}(X))$. $\square$

Proposition 2.7. *Let* $(R, \mathfrak{m})$ *be a local ring and* $M \neq 0$ *a finitely generated* R*-module. Let* $I \subseteq R$ *be an ideal of* R *with* $H_I^i(M) = 0$ *for all* $i \neq c$ *and* $H_I^c(M) \neq 0$. *Then* $c = \operatorname{grade}(IR_\mathfrak{p}, M_\mathfrak{p})$ *for all* $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R M$. *If* M *is in addition a Cohen-Macaulay module then* $c = \operatorname{height}_{M_\mathfrak{p}} IR_\mathfrak{p}$ *for all* $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R M$.

Proof. Since $c = \operatorname{grade}(I, M) \leq \operatorname{grade}(IR_\mathfrak{p}, M_\mathfrak{p})$ for all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R M$. Suppose that there is a prime ideal $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R M$ such that $\operatorname{grade}(IR_\mathfrak{p}, M_\mathfrak{p}) = h > c$. Then

$$0 \neq H_{IR_\mathfrak{p}}^h(M_\mathfrak{p}) \cong H_{IR_\mathfrak{p}}^h(M \otimes_R R_\mathfrak{p}) \cong H_I^h(M) \otimes_R R_\mathfrak{p}.$$

But it implies that $H_I^h(M) \neq 0$, $h > c$, which is a contradiction to our assumption. □

In case of a not necessarily finitely generated R-module X we put $\dim X = \dim \operatorname{Supp}_R X$. Note that this agrees with the usual notion $\dim X = \dim R/\operatorname{Ann}_R X$ if X is finitely generated.

3. LOCAL DUALITY THEOREM FOR A COHEN-MACAULAY RING

We want to prove a variation of the Local Duality Theorem for a Cohen-Macaulay ring. Let $(R, \mathfrak{m})$ denote a local ring which is the factor ring of a local Gorenstein ring $(S, \mathfrak{n})$ with $\dim(S) = t$. Let N be a finitely generated R-module. Then by the Local Duality Theorem there is an isomorphism

$$H_\mathfrak{m}^i(N) \cong \operatorname{Hom}_R(\operatorname{Ext}_S^{t-i}(N, S), E)$$

for all $i \in \mathbb{N}$ (see [6]). Under these circumstances we define

$$K(N) := \operatorname{Ext}_S^{t-r}(N, S), \dim(N) = r$$

as the canonical module of N. It was introduced by Schenzel (see [18]) as the generalization of the canonical module of a ring (see *e.g.* [3]).

For our purposes here we need an extension of the Local Duality of a local Gorenstein ring to a local Cohen-Macaulay ring which is valid also for modules that are not necessarily finitely generated. A more general result of Lemma 3.1 was proved by Hellus (see [11, Theorem 6.4.1]). We include here a proof of the particular case for sake of completeness.

Lemma 3.1 (Local Duality). *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay ring of dimension* n. *Let* M *be an arbitrary* R*-module. Then there are functorial isomorphisms*

$$\operatorname{Hom}_R(H_\mathfrak{m}^i(M), E) \cong \operatorname{Ext}_R^{n-i}(M, K(\hat{R}))$$

for all $i \in \mathbb{N}$.

Proof. By Cohen's Structure Theorem any complete local ring $(R, \mathfrak{m})$ is a homomorphic image of a regular local ring and – in particular – of local Gorenstein ring. Let $\hat{R}$ be the m-adic completion of R which is homomorphic image of a local Gorenstein (regular) ring $(S, \mathfrak{n})$ with $\dim(S) = t$. Then $H^n_{\mathfrak{m}}(R) \cong H^n_{\mathfrak{m}\hat{R}}(\hat{R})$ because any R-module X with support contained in $V(\mathfrak{m})$ admits the structure of an $\hat{R}$-module such that $X \cong X \otimes_R \hat{R}$. Then by the Local Duality Theorem for Gorenstein rings (see [6]) there is an isomorphism $\operatorname{Hom}_R(H^n_{\mathfrak{m}}(R), E) \cong \operatorname{Ext}^{t-n}_S(\hat{R}, S)$.

For a system of parameters $\underline{x} = x_1, \ldots, x_n$ let $\check{C}_{\underline{x}}$ denote the Čech complex with respect to $\underline{x}$. Because R is a Cohen-Macaulay ring $H^i_{\mathfrak{m}}(R) \cong H^i(\check{C}_{\underline{x}}) = 0$ for all $i \neq n$. That is, $\check{C}_{\underline{x}}$ is a flat resolution of $H^n_{\mathfrak{m}}(R)[-n]$. Therefore

$$\operatorname{Tor}^R_{n-i}(M, H^n_{\mathfrak{m}}(R)) \cong H^i(\check{C}_{\underline{x}} \otimes_R M) \cong H^i_{\mathfrak{m}}(M)$$

for all $i \in \mathbb{Z}$ and an arbitrary R-module M. Now we take the Matlis dual $\operatorname{Hom}_R(\cdot, E)$ of this isomorphism and get

$$\operatorname{Hom}_R(H^i_{\mathfrak{m}}(M), E) \cong \operatorname{Ext}^{n-i}_R(M, K(\hat{R}))$$

as it is easily seen by the adjunction isomorphism. $\square$

Note that the above version of the Local Duality holds for an arbitrary R-module M. That is what we need in the sequel.

COROLLARY 3.2. *Let $(R, \mathfrak{m})$ denote a Cohen-Macaulay ring of dimension n. Let $I \subseteq R$ be an ideal with* $\operatorname{grade}(I, M) = c$ *for an R-module M. Then there are isomorphisms*

$$\operatorname{Ext}^{n-i}_R(H^c_I(M), K(\hat{R})) \cong \varprojlim \operatorname{Ext}^{n-i}_R(\operatorname{Ext}^c_R(R/I^r, M), K(\hat{R}))$$

for all $i \in \mathbb{N}$.

Proof. By Local Duality (Lemma 3.1), we have the following isomorphisms

$$\operatorname{Ext}^{n-i}_R(H^c_I(M), K(\hat{R})) \cong \operatorname{Hom}_R(H^i_{\mathfrak{m}}(H^c_I(M)), E)$$

for all $i \in \mathbb{Z}$. The module on the right side is isomorphic to

$$\operatorname{Hom}_R(\varinjlim H^i_{\mathfrak{m}}(\operatorname{Ext}^c_R(R/I^r, M)), E)$$

since $H^c_I(M) \cong \varinjlim \operatorname{Ext}^c_R(R/I^r, M)$ and because the local cohomology commutes with direct limits. Finally

$$\operatorname{Ext}^{n-i}_R(H^c_I(M), K(\hat{R})) \cong \varprojlim \operatorname{Hom}_R(H^i_{\mathfrak{m}}(\operatorname{Ext}^c_R(R/I^r, M)), E)$$

since the Hom-functor in the first variable transforms a direct system into an inverse system, see [20, Proposition 6.4.6]. Then the claim follows since

$$\operatorname{Hom}_R(H^i_{\mathfrak{m}}(\operatorname{Ext}^c_R(R/I^r, M)), E) \cong \operatorname{Ext}^{n-i}_R(\operatorname{Ext}^c_R(R/I^r, M), K(\hat{R}))$$

as it is true again by Local Duality (see Lemma 3.1). $\square$

4. THE TRUNCATION COMPLEX

Let $(R, \mathfrak{m})$ be a local ring of dimension n. Let $M \neq 0$ denote a finitely generated R-module and $\dim M = n$. Let $I \subseteq R$ be an ideal of R with $\operatorname{grade}(I, M) = c$. Suppose that $E_R^{\cdot}(M)$ is a minimal injective resolution of M. Then it follows from (Matlis [14] or Gabriel [5]) that

$$E_R^{\cdot}(M)^i \cong \bigoplus_{\mathfrak{p} \in \operatorname{Supp} M} E_R(R/\mathfrak{p})^{\mu_i(\mathfrak{p}, M)},$$

where $\mu_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})}(\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}))$. We get $\Gamma_I(E_R(R/\mathfrak{p})) = 0$ for all $\mathfrak{p} \notin V(I)$ and $\Gamma_I(E_R(R/\mathfrak{p})) = E_R(R/\mathfrak{p})$ for all $\mathfrak{p} \in V(I)$. Moreover $\mu_i(\mathfrak{p}, M) = 0$ for all $i < c$ since $\operatorname{grade}(I, M) = c$. Whence for all $i < c$ it follows that

$$\Gamma_I(E_R^{\cdot}(M))^i \cong \bigoplus_{\mathfrak{p} \in V(I) \cap \operatorname{Supp} M} \Gamma_I(E_R(R/\mathfrak{p}))^{\mu_i(\mathfrak{p}, M)} = 0.$$

Therefore $H^c_I(M)$ is isomorphic to the kernel of $\Gamma_I(E_R^{\cdot}(M))^c \to \Gamma_I(E_R^{\cdot}(M))^{c+1}$. Whence there is an embedding of complexes of R-modules $H^c_I(M)[-c] \to \Gamma_I(E_R^{\cdot}(M))$. Here, we assume the following convention: if $C^{\cdot} := (C^i)_{i \in \mathbb{Z}}$ is a complex and $k \in \mathbb{Z}$, we shall denote by $C[k]^{\cdot}$ the complex defined by the rule $C[k]^i := C^{i+k}$.

Definition 4.1. Let $C_M^{\cdot}(I)$ be the cokernel of the above embedding. It is called the truncation complex. So there is a short exact sequence of complexes of R-modules

$$0 \to H^c_I(M)[-c] \to \Gamma_I(E_R^{\cdot}(M)) \to C_M^{\cdot}(I) \to 0.$$

In particular it follows that $H^i(C_M^{\cdot}(I)) = 0$ for all $i \leq c$ or $i > n$ and $H^i(C_M^{\cdot}(I)) \cong H^i_I(M)$ for all $c < i \leq n$.

The advantage of the truncation complex is that it separates the information of cohomology modules of $H^i_I(M)$ for $i = c$ from those with $i \neq c$.

In the following we need some preparations of completion. For an R-module N we denote by $\Lambda^I(N) = \varprojlim N/I^\alpha N$ the I-adic completion of N. Its left derived functors are denoted by $\mathrm{L}_i\Lambda^I(\cdot), i \in \mathbb{Z}$ (see [21] for more details). Note that $\mathrm{L}_i\Lambda^I(N) = 0$ for all $i > 0$ and a finitely generated R-module N as follows since $\Lambda^I(\cdot)$ is exact on the category of finitely generated R-modules.

Let us consider the Čech complex $\check{C}_{\underline{x}}$ with respect to $\underline{x} = x_1, \ldots, x_r \in I$ such that $\operatorname{Rad}(\underline{x})R = \operatorname{Rad} I$. Then by [19, Section 4] there is a quasi-isomorphism $L_{\underline{x}} \xrightarrow{\sim} \check{C}_{\underline{x}}$ of complexes of R-modules, where $L_{\underline{x}}$ is a bounded complex of free R-modules (depending on $\underline{x} = x_1, \ldots, x_r$).

THEOREM 4.2. *Let I denote an ideal of a ring $(R,\mathfrak{m})$. Let $X^{\cdot}$ denote a complex of flat R-modules. Then there are natural isomorphisms*

$$\mathrm{L}_i\Lambda^I(X^{\cdot}) \cong H_i(\mathrm{Hom}_R(L_{\underline{x}}, X^{\cdot})) \cong H_i(\mathrm{Hom}_R(\check{C}_{\underline{x}}, E^{\cdot}))$$

for all $i \in \mathbb{Z}$, where $E^{\cdot}$ denotes an injective resolution of $X^{\cdot}$.

Proof. See [19, Theorem 1.1]. □

THEOREM 4.3. *Fix the previous notation. Let $M \neq 0$ be a maximal Cohen-Macaulay module with $\dim M/IM = d$. We put $c = \dim M - \dim M/IM = \mathrm{grade}(I,M)$. Then we have the following results:*

(a) *There are an exact sequence*

$$0 \to H^{n-1}_{\mathfrak{m}}(C^{\cdot}_M(I)) \to H^d_{\mathfrak{m}}(H^c_I(M)) \to H^n_{\mathfrak{m}}(M) \to H^n_{\mathfrak{m}}(C^{\cdot}_M(I)) \to 0$$

and isomorphisms $H^{i-1}_{\mathfrak{m}}(C^{\cdot}_M(I)) \cong H^{i-c}_{\mathfrak{m}}(H^c_I(M))$ for all $i \neq n, n+1$.

(b) *There are an exact sequence*

$$0 \to \mathrm{Ext}^{n-1}_R(k, C^{\cdot}_M(I)) \to \mathrm{Ext}^d_R(k, H^c_I(M)) \to \mathrm{Ext}^n_R(k, M) \to \mathrm{Ext}^n_R(k, C^{\cdot}_M(I))$$

and isomorphisms $\mathrm{Ext}^{i-c}_R(k, H^c_I(M)) \cong \mathrm{Ext}^{i-1}_R(k, C^{\cdot}_M(I))$ for all $i < n$.

(c) *Assume in addition that R is a Cohen-Macaulay ring. There are an exact sequence*

$$0 \to \mathrm{Ext}^0_R(C^{\cdot}_M(I), K(\hat{R})) \to K(\hat{M}) \to \mathrm{Ext}^c_R(H^c_I(M), K(\hat{R})) \to \mathrm{Ext}^1_R(C^{\cdot}_M(I), K(\hat{R})) \to 0$$

and isomorphisms $\mathrm{Ext}^{i+c}_R(H^c_I(M), K(\hat{R})) \cong \mathrm{Ext}^{i+1}_R(C^{\cdot}_M(I), K(\hat{R}))$ for all $i > 0$.

Proof. (*a*) Let $\underline{x} = x_1, \ldots, x_n \in \mathfrak{m}$ denote a system of parameters of R. We tensor the short exact sequence of the truncation complex by the Čech complex $\check{C}_{\underline{x}}$. Then the resulting sequence of complexes remains exact because $\check{C}_{\underline{x}}$ is a complex of flat R-modules. That is, there is the following short exact sequence of complexes of R-modules

$$0 \to \check{C}_{\underline{x}} \otimes_R H^c_I(M)[-c] \to \check{C}_{\underline{x}} \otimes_R \Gamma_I(E^{\cdot}_R(M)) \to \check{C}_{\underline{x}} \otimes_R C^{\cdot}_M(I) \to 0.$$

Now we look at the cohomology of the complex in the middle. Since $\Gamma_I(E^{\cdot}_R(M))$ is a complex of injective R-modules the natural morphism

$$\Gamma_{\mathfrak{m}}(E^{\cdot}_R(M)) \cong \Gamma_{\mathfrak{m}}(\Gamma_I(E^{\cdot}_R(M))) \to \check{C}_{\underline{x}} \otimes_R \Gamma_I(E^{\cdot}_R(M))$$

induces an isomorphism in cohomology (see [19, Theorem 1.1]). Because M is a maximal Cohen-Macaulay R-module the only non-vanishing local cohomology module is $H^n_{\mathfrak{m}}(M)$. So the result follows from the long exact cohomology sequence.

(*b*) Let $F^{\cdot}_R(k)$ be a free resolution of k. Apply the functor $\operatorname{Hom}_R(F^{\cdot}_R(k), .)$ to the short exact sequence of the truncation complex. Then it induces the following short exact sequences of complexes of R-modules

$$0 \to \operatorname{Hom}_R(F^{\cdot}_R(k), H^c_I(M))[-c] \to \operatorname{Hom}_R(F^{\cdot}_R(k), \Gamma_I(E^{\cdot}_R(M))) \\ \to \operatorname{Hom}_R(F^{\cdot}_R(k), C^{\cdot}_M(I)) \to 0.$$

Now $\operatorname{Hom}_R(F^{\cdot}_R(k), \Gamma_I(E^{\cdot}_R(M))) \xrightarrow{\sim} \Gamma_I(\operatorname{Hom}_R(F^{\cdot}_R(k), E^{\cdot}_R(M)))$ since $F^{\cdot}_R(k)$ is a right bounded complex of finitely generated free R-modules. Since any R-module of the complex $\operatorname{Hom}_R(F^{\cdot}_R(k), E^{\cdot}_R(M))$ is injective. By [19, Theorem 1.1] it implies that the complex $\Gamma_I(\operatorname{Hom}_R(F^{\cdot}_R(k), E^{\cdot}_R(M)))$ is quasi-isomorphic to $\check{C}_{\underline{y}} \otimes_R \operatorname{Hom}_R(F^{\cdot}_R(k), E^{\cdot}_R(M))$ where $\underline{y} = y_1, \ldots, y_r \in I$ such that $\operatorname{Rad} I = \operatorname{Rad}(\underline{y})R$.

Now tensoring with a right bounded complex of flat R-modules preserves the quasi-isomorphisms and any R-module of $E^{\cdot}_R(M)$ is injective. It induces the following quasi-isomorphism

$$\check{C}_{\underline{y}} \otimes_R \operatorname{Hom}_R(k, E^{\cdot}_R(M)) \xrightarrow{\sim} \check{C}_{\underline{y}} \otimes_R \operatorname{Hom}_R(F^{\cdot}_R(k), E^{\cdot}_R(M)).$$

But the complex on the left side is isomorphic to $\operatorname{Hom}_R(k, E^{\cdot}_R(M))$. This is true because of each R-module of the complex $\operatorname{Hom}_R(k, E^{\cdot}_R(M))$ has support in $V(\mathfrak{m})$. Therefore the complex $\operatorname{Hom}_R(F^{\cdot}_R(k), \Gamma_I(E^{\cdot}_R(M)))$ is quasi-isomorphic to $\operatorname{Hom}_R(k, E^{\cdot}_R(M))$. Then the result follows by the long exact cohomology sequence. To this end note that the assumption on the maximal Cohen-Macaulayness of M implies that $\operatorname{Ext}^i_R(k, M) = 0$ for all $i < n$.

(c) Let $E^{\cdot}_R(K(\hat{R}))$ be a minimal injective resolution of $K(\hat{R})$. We apply the functor $\operatorname{Hom}_R(., E^{\cdot}_R(K(\hat{R})))$ to the short exact sequence of the truncation complex. Then we have a short exact sequence of complexes of R-modules

$$0 \to \operatorname{Hom}_R(C^{\cdot}_M(I), E^{\cdot}_R(K(\hat{R}))) \to \operatorname{Hom}_R(\Gamma_I(E^{\cdot}_R(M)), E^{\cdot}_R(K(\hat{R}))) \\ \to \operatorname{Hom}_R(H^c_I(M), E^{\cdot}_R(K(\hat{R})))[c] \to 0.$$

First we investigate the complex in the middle. There is a quasi-isomorphism of complexes $\Gamma_I(E^{\cdot}_R(M)) \xrightarrow{\sim} L_{\underline{x}} \otimes E^{\cdot}_R(M)$ (see [19, Theorem 1.1]). That is the complex in the middle is quasi-isomorphic to $X = \operatorname{Hom}_R(L_{\underline{x}} \otimes E^{\cdot}_R(M), E^{\cdot}_R(K(\hat{R})))$. By Hom-Tensor adjunction X is isomorphic to the complex $\operatorname{Hom}_R (L_{\underline{x}}, \operatorname{Hom}_R(E^{\cdot}_R(M), E^{\cdot}_R(K(\hat{R}))))$. Since $E^{\cdot}_R(M)$ is an injective resolution of M and $L_{\underline{x}}$ is a bounded complex of free R-modules. So there are the following quasi-isomorphisms of complexes

$$\operatorname{Hom}_R(E^{\cdot}_R(M), E^{\cdot}_R(K(\hat{R}))) \xrightarrow{\sim} \operatorname{Hom}_R(M, E^{\cdot}_R(K(\hat{R}))), \text{ and}$$

$$X \xrightarrow{\sim} \operatorname{Hom}_R(L_{\underline{x}}, \operatorname{Hom}_R(M, E^{\cdot}_R(K(\hat{R})))).$$

That is, in order to compute the homology of X it will be enough to compute the homology of $\operatorname{Hom}_R(L_{\underline{x}}, \operatorname{Hom}_R(M, E^{\cdot}_R(K(\hat{R}))))$. By virtue of Theorem 4.2 there is the following spectral sequence

$$E_2^{i,j} = \mathrm{L}_i\Lambda^I(\operatorname{Ext}^j_R(M, K(\hat{R}))) \Rightarrow E_\infty^{i+j} = H^{i+j}(\operatorname{Hom}_R(L_{\underline{x}}, \operatorname{Hom}_R(M, E^{\cdot}_R(K(\hat{R})))))$$

Since $\operatorname{Ext}^j(M, K(\hat{R}))$ is finitely generated for all $j \in \mathbb{Z}$. So it degenerates to isomorphisms $\mathrm{L}_0\Lambda^I(\operatorname{Ext}^j_R(M, K(\hat{R}))) \cong H^j(\operatorname{Hom}_R(L_{\underline{x}}, \operatorname{Hom}_R(M, E^{\cdot}_R(K(\hat{R})))))$ for all $j \in \mathbb{Z}$. This finally proves that $H^j(X) \cong \mathrm{L}_0\Lambda^I(\operatorname{Ext}^j_R(M, K(\hat{R}))))$ for all $j \in \mathbb{Z}$.

Since M is a maximal Cohen-Macaulay R-module and R is a Cohen-Macaulay ring it follows by the Local Duality Theorem (see Lemma 3.1) that $H^j(X) = 0$ for all $j \neq 0$ and $H^j(X) \cong L_0\Lambda^I(\operatorname{Hom}_R(M, K(\hat{R})))$ for $j = 0$. But $\operatorname{Hom}_R(M, K(\hat{R})) \cong K(\hat{M})$ and $L_0\Lambda^I(K(\hat{M})) = K(\hat{M})$ since $\hat{R}$ is homomorphic image of a local Gorenstein ring. Then the long exact sequence of local cohomology provides the statements in (c). □

4.1. NECESSARY CONDITION OF $H^i_I(M) = 0$, FOR ALL $i \neq c$

In the following let $(R, \mathfrak{m})$ denote a local ring of dimension n. Let $M \neq 0$ be a finitely generated R-module with $\dim M = n$. Let $I \subset R$ be an ideal such that $c = \operatorname{grade}(I, M)$.

Corollary 4.4. *Let $M \neq 0$ be a maximal Cohen-Macaulay R-module such that $H^i_I(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M)$. If $\dim_R(M/IM) = n - c$, then:*

(a) *The natural homomorphism*

$$H^d_{\mathfrak{m}}(H^c_I(M)) \to H^n_{\mathfrak{m}}(M)$$

is an isomorphism and $H^i_{\mathfrak{m}}(H^c_I(M)) = 0$ for all $i \neq d$.

(b) *The natural homomorphism*

$$\operatorname{Ext}^d_R(k, H^c_I(M)) \to \operatorname{Ext}^n_R(k, M)$$

is an isomorphism and $\operatorname{Ext}^i_R(k, H^c_I(M)) = 0$ for all $i < d$.

(c) *Suppose that R is in addition a Cohen-Macaulay ring. The natural homomorphism*

$$K(\hat{M}) \to \operatorname{Ext}^c_R(H^c_I(M), K(\hat{R}))$$

is an isomorphism and $\operatorname{Ext}^{i+c}_R(H^c_I(M), K(\hat{R})) = 0$ for all $i > 0$.

Proof. Because of assumption $C^{\cdot}_M(I)$ is an exact complex.

First we prove part (a). Apply the Čech complex $\check{C}_{\underline{x}} \otimes_R \cdot$ to the short exact sequence of the truncation complex since $\check{C}_{\underline{x}}$ is a complex of flat R-modules so $\check{C}_{\underline{x}} \otimes_R C^{\cdot}_M(I)$ is exact. Whence result follows from Theorem 4.3.

Now we prove (b). Let $F^{\cdot}_R(k)$ be a free resolution of k. Then apply $\operatorname{Hom}_R(F^{\cdot}_R(k), .)$ to the short exact sequence of the truncation complex. Since $\operatorname{Hom}_R(F^{\cdot}_R(k), C^{\cdot}_M(I))$ is an exact complex the result follows from Theorem 4.3.

Finally, we prove (c). Let $E^{\cdot}_R(K(\hat{R}))$ be a minimal injective resolution of $K(\hat{R})$. Then apply $\operatorname{Hom}_R(., E^{\cdot}_R(K(\hat{R})))$ to the short exact sequence of the truncation complex. Since $\operatorname{Hom}_R(C^{\cdot}_M(I)), E^{\cdot}_R(K(\hat{R})))$ is an exact complex the result follows from Theorem 4.3. □

LEMMA 4.5. *Let $(R, \mathfrak{m})$ be ring, $I \subseteq R$ an ideal, and $M \neq 0$ a maximal Cohen-Macaulay R-module. Suppose that $H^i_I(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M)$ and $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap V(I)$. Then*

(1) *$H^i_{IR_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $i \neq c$.*
(2) *The natural homomorphism*

$$H^i_{\mathfrak{p}R_\mathfrak{p}}(H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) \to H^{\dim(M_\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$$

is an isomorphism for $i = h(\mathfrak{p}) = \dim(M_\mathfrak{p}) - c$ and $H^i_{\mathfrak{p}R_\mathfrak{p}}(H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) = 0$ for all $i \neq h(\mathfrak{p}) = \dim(M_\mathfrak{p}) - c$.

Proof. It is follow from Proposition 2.7 and Corollary 4.4 (a). □

4.2. CONVERSE (SUFFICIENT CONDITION OF $H^i_I(M) = 0$, FOR ALL $i \neq c$)

Before we can prove Theorem 4.10 which is one of the main result of this section we need the following Lemma. For the following technical result we need a few details on derived categories and derived functors. For all these facts we refer to the Lecture Note by R. Hartshorne (see [7]). We are grateful to P. Schenzel for suggesting this argument to us.

LEMMA 4.6. *Let $M \neq 0$ be a finitely generated R-module such that $\dim_R(M) = n = \dim(R)$. Let $c = \operatorname{grade}(I, M)$ where $I \subseteq R$ is an ideal. Then the following are equivalent:*

(1) *The natural homomorphism*

$$\operatorname{Ext}^{i-c}_R(k, H^c_I(M)) \to \operatorname{Ext}^i_R(k, M)$$

is an isomorphism for all $i \in \mathbb{Z}$.
(2) *The natural homomorphism*

$$H^{i-c}_\mathfrak{m}(H^c_I(M)) \to H^i_\mathfrak{m}(M)$$

is an isomorphism for all $i \leq n$.

Proof. First of all note that the statement in (2) is equivalent to the isomorphism for all $i \in \mathbb{Z}$. This follows because of $\dim_R(H^c_I(M)) \leq \dim_R(M/IM) \leq \dim_R(M) - \operatorname{grade}(I, M) = n - c = d$ since $\operatorname{Supp}_R(H^c_I(M)) \subseteq V(I) \cap \operatorname{Supp}_R(M)$. By applying $\mathrm{R}\Gamma_{\mathfrak{m}}(\cdot)$ to the short exact sequence of the truncation complex it follows that the assumption in (2) is equivalent to the fact that $\mathrm{R}\Gamma_{\mathfrak{m}}(C^{\cdot}_M(I))$ is an exact complex.

Moreover by applying the derived functor $\mathrm{R}\operatorname{Hom}(k, \cdot)$ to the short exact sequence of the truncation complex it follows that the statement in (1) is equivalent to the fact that $\mathrm{R}\operatorname{Hom}(k, C^{\cdot}_M(I))$ is an exact complex.

Now we prove that (2) implies (1). If $\mathrm{R}\Gamma_{\mathfrak{m}}(C^{\cdot}_M(I))$ is an exact complex the same is true for

$$\mathrm{R}\operatorname{Hom}(k, C^{\cdot}_M(I)) \cong \mathrm{R}\operatorname{Hom}(k, \mathrm{R}\Gamma_{\mathfrak{m}}(C^{\cdot}_M(I)))$$

since $C^{\cdot}_M(I)$ is a left bounded complex.

In order to prove that (1) implies (2) consider the short exact sequence (in fact an exact triangle in the derived category)

$$0 \to \mathfrak{m}^{\alpha}/\mathfrak{m}^{\alpha+1} \to R/\mathfrak{m}^{\alpha+1} \to R/\mathfrak{m}^{\alpha} \to 0.$$

for $\alpha \in \mathbb{N}$. By applying the derived functor $\mathrm{R}\operatorname{Hom}(\cdot, C^{\cdot}_M(I))$ it provides a short exact sequence of complexes

$$0 \to \mathrm{R}\operatorname{Hom}(R/\mathfrak{m}^{\alpha}, C^{\cdot}_M(I)) \to \mathrm{R}\operatorname{Hom}(R/\mathfrak{m}^{\alpha+1}, C^{\cdot}_M(I)) \\ \to \mathrm{R}\operatorname{Hom}(\mathfrak{m}^{\alpha}/\mathfrak{m}^{\alpha+1}, C^{\cdot}_M(I)) \to 0.$$

By induction on α it follows that $\mathrm{R}\operatorname{Hom}(R/\mathfrak{m}^{\alpha}, C^{\cdot}_M(I))$ is an exact complex for all $\alpha \in \mathbb{N}$. Since $\varinjlim \mathrm{R}\operatorname{Hom}(R/\mathfrak{m}^{\alpha}, C^{\cdot}_M(I)) \cong \mathrm{R}\Gamma_{\mathfrak{m}}(C^{\cdot}_M(I))$ it follows that $\mathrm{R}\Gamma_{\mathfrak{m}}(C^{\cdot}_M(I))$ is exact. This finishes the proof of the Lemma. □

Corollary 4.7. *Let $M \neq 0$ be a maximal Cohen-Macaulay module over a Cohen-Macaulay ring R of dimension n and $I \subseteq R$ an ideal of* $\operatorname{grade}(I, M) = c$. *Then the following are equivalent:*

(1) *The natural homomorphism*

$$\operatorname{Ext}^{i-c}_R(k, H^c_I(M)) \to \operatorname{Ext}^i_R(k, M)$$

is an isomorphism for $i \geq n$ and $\operatorname{Ext}^i_R(k, H^c_I(M)) = 0$ for all $i < d = n - c$.

(2) *The natural homomorphism*

$$H^d_{\mathfrak{m}}(H^c_I(M)) \to H^n_{\mathfrak{m}}(M)$$

is an isomorphism and $H^i_{\mathfrak{m}}(H^c_I(M)) = 0$ for all $i \neq d = n - c$.

Proof. This is an immediate consequence of Lemma 4.6. □

In case we have $\operatorname{id}_R(M) < \infty$ for the R-module M in Corollary 4.7 we get the following Corollary. Note that this Corollary provides us a new relation between the canonical module and the Ext module.

COROLLARY 4.8. *Let $M \neq 0$ be a maximal Cohen-Macaulay module of finite injective dimension over a local ring R of dimension n and $I \subseteq R$ an ideal of* $\operatorname{grade}(I, M) = c$. *Then the following are equivalent:*

(1) *The natural homomorphism*

$$\operatorname{Ext}^d_R(k, H^c_I(M)) \to \operatorname{Ext}^n_R(k, M)$$

is an isomorphism and $\operatorname{Ext}^i_R(k, H^c_I(M)) = 0$ *for all $i \neq d = n - c$.*

(2) *The natural homomorphism*

$$H^d_{\mathfrak{m}}(H^c_I(M)) \to H^n_{\mathfrak{m}}(M)$$

is an isomorphism and $H^i_{\mathfrak{m}}(H^c_I(M)) = 0$ for all $i \neq d = n - c$.

(3) *The natural homomorphism*

$$K(\hat{M}) \to \operatorname{Ext}^c_R(H^c_I(M), K(\hat{R}))$$

is an isomorphism and $\operatorname{Ext}^{n-i}_R(H^c_I(M), K(\hat{R})) = 0$ *for all $i \neq d = n - c$.*

Proof. Since M is of finite injective dimension so by [16] it follows that R is Cohen-Macaulay ring. Note that the equivalence of (1) ⇔ (2) follows from Corollary 4.7. Recall that $\operatorname{Ext}^i_R(k, M) = 0$ for all $i \neq n$ since $\operatorname{id}_R(M) = n$ under the assumption.

Next we proof the equivalence of (2) ⇔ (3).

In fact this is a consequence of the generalized Local Duality (see Lemma 3.1) and Matlis Duality. □

Remark 4.9. (1) Let $M \neq 0$ be a maximal Cohen-Macaulay module of finite injective dimension over a local ring R. Suppose that $I \subseteq R$ is an ideal with $c = \operatorname{grade}(I, M)$. If any of the equivalent conditions of Corollary 4.8 holds then it follows that all the Bass numbers of $H^c_I(M)$ are zero except for $i = d = \dim_R(M) - c$ see Proposition 2.7.

(2) Let R be a complete local ring of dimension n and $M \neq 0$ be a maximal Cohen-Macaulay module of finite injective dimension. Let $I \subseteq R$ be an ideal of $c = \operatorname{grade}(I, M)$. Then $K(R)$ exists and by [17, Theorem 4.1] it follows that $M \cong \oplus K(R)$. Therefore if R is complete so it is enough to prove Corollary 4.8 for $K(R)$ instead of M.

(3) In case R possesses a maximal Cohen-Macaulay module of finite injective dimension $M \neq 0$ the ring R is Cohen-Macaulay and it follows that $\operatorname{Supp}_R M = \operatorname{Spec} R$. To this end first note that $\operatorname{Supp}_{\hat{R}} \hat{M} = \operatorname{Spec} \hat{R}$ since $\hat{M}$ is isomorphic

to a direct sum of $K(\hat{R})$ and $\operatorname{Ann} K(\hat{R}) = (0)$. This implies $\operatorname{Ann}_R M = 0$ which implies the claim.

Now we can prove one of our main result as follows:

THEOREM 4.10. *Let $M \neq 0$ be a maximal Cohen-Macaulay module of $\operatorname{id}_R(M) < \infty$ over a local ring R of dimension n and $I \subseteq R$ an ideal of $\operatorname{grade}(I, M) = c$. The following conditions are equivalent:*

(a) *$H^i_I(M) = 0$ for all $i \neq c$.*

(b) *For all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M)$ the natural homomorphism*

$$H^{h(\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) \to H^{\dim(M_\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})$$

is an isomorphism and $H^i_{\mathfrak{p}R_\mathfrak{p}}(H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) = 0$ for all $i \neq h(\mathfrak{p}) = \dim(M_\mathfrak{p}) - c$.

(c) *For all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M)$ the natural homomorphism*

$$\operatorname{Ext}^{h(\mathfrak{p})}_{R_\mathfrak{p}}(k(\mathfrak{p}), H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) \to \operatorname{Ext}^{\dim(M_\mathfrak{p})}_{R_\mathfrak{p}}(k(\mathfrak{p}), (M_\mathfrak{p}))$$

is an isomorphism and $\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), H^c_{IR_\mathfrak{p}}(M_\mathfrak{p})) = 0$ for all $i \neq h(\mathfrak{p}) = \dim(M_\mathfrak{p}) - c$.

(d) *For all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M)$ the natural homomorphism*

$$K(\hat{M}_\mathfrak{p}) \to \operatorname{Ext}^c_{\hat{R}_\mathfrak{p}}(H^c_{I\hat{R}_\mathfrak{p}}(\hat{M}_\mathfrak{p}), K(\hat{R}_\mathfrak{p}))$$

is an isomorphism and $\operatorname{Ext}^i_{\hat{R}_\mathfrak{p}}(H^c_{I\hat{R}_\mathfrak{p}}(\hat{M}_\mathfrak{p}), K(\hat{R}_\mathfrak{p})) = 0$ for all $i \neq h(\mathfrak{p}) = \dim(M_\mathfrak{p}) - c$.

(e) *$H^i_I(R) = 0$ for all $i \neq c$, that is I is a cohomologically complete intersection.*

Proof. Firstly we prove the equivalence of (a) and (e). By Remark 4.9 the assumptions imply that $\operatorname{Supp}_R M = \operatorname{Spec} R$ and the equivalence follows by Proposition 2.3.

For the proof that (a) implies (b) see Lemma 4.5. By Corollary 4.8 the equivalence of (b), (c), (d) is easily seen by passing to the localization.

Now we prove that (b) implies (a)

We proceed by induction on $\dim(M/IM)$. If $\dim(M/IM) = 0$ then $V(I) \cap \operatorname{Supp}_R(M) \subseteq V(\mathfrak{m})$ so statement (b) holds for $\mathfrak{p} = \mathfrak{m}$. By the definition of the truncation complex it implies that $H^i_\mathfrak{m}(C^\cdot_M(I)) = 0$ for all $i \in \mathbb{Z}$.

Now we apply Lemma 2.5 for $X = C^\cdot_M(I)$. Because of $\operatorname{Supp}_R(H^i(C^\cdot_M(I))) \subseteq V(\mathfrak{m})$ (recall that $H^i(C^\cdot_M(I)) \cong H^i_I(M)$ for all $i \neq c$ and $H^c(C^\cdot_M(I)) = 0$ (see the Definition 4.1)). This proves the vanishing of $H^i_I(M)$ for all $i \neq c$.

Now let $\dim(M/IM) > 0$ then

$$\dim(M_\mathfrak{p}/IM_\mathfrak{p}) < \dim(M/IM)$$

for all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}_R(M) \setminus \{\mathfrak{m}\}$. By the induction hypothesis it implies that

$$H^i_{IR_\mathfrak{p}}(M_\mathfrak{p}) = 0$$

for all $i \neq c$ and all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}(M) \setminus \{\mathfrak{m}\}$. That is $\operatorname{Supp}(H^i_I(M)) \subseteq V(\mathfrak{m})$ for all $i \neq c$. It implies that $H^i_\mathfrak{m}(C^{\cdot}_M(I)) = 0$ for $i \leq c$ and $i > n$. Also for $c < i \leq n$ there is an isomorphism

$$H^i_\mathfrak{m}(C^{\cdot}_M(I)) \cong H^i(C^{\cdot}_M(I)) \cong H^i_I(M).$$

On the other side note that by the assumption for $\mathfrak{p} = \mathfrak{m}$ we have $H^i_\mathfrak{m}(C^{\cdot}_M(I)) = 0$ for all $i \in \mathbb{Z}$ (by Theorem 4.3) and hence by virtue of the last isomorphism it gives the result. This completes the proof of the Theorem. □

Remark 4.11. Let us discuss the necessity of the local conditions in Theorem 4.10. It is not enough to assume the statements (b), (c), (d) in the Theorem 4.10 for $\mathfrak{p} = \mathfrak{m}$. That is, we do need these statements for all $\mathfrak{p} \in V(I) \cap \operatorname{Supp}(M)$. This is shown by Hellus and Schenzel (see [13, Example 4.1]) where (b) is true for $\mathfrak{p} = \mathfrak{m}$ but not for a localization at $\mathfrak{p}$. That is, these three properties (b), (c), (d) do not localize.

In the next we are interested in the injective dimension of the local cohomology module. As a consequence it provides a new characterization of a local ring to be Cohen-Macaulay.

THEOREM 4.12. *Let $(R, \mathfrak{m})$ be a ring and I an ideal of R. Suppose that $M \neq 0$ is a finitely generated R-module such that $H^i_I(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M)$ then the following are equivalent:*

(1) $\operatorname{id}_R(M) < \infty$.
(2) $\operatorname{id}_R(H^c_I(M)) < \infty$.

Each of the equivalent conditions implies that R is a Cohen-Macaulay ring.

Proof. Let $E^{\cdot}_R(M)$ be a minimal injective resolution of M. Since $H^i_I(M) = 0$ for all $i \neq c$ so $H^c_I(M)[-c] \to \Gamma_I(E^{\cdot}_R(M))$ is an isomorphism of complexes in cohomology. If $\operatorname{id}_R(M) < \infty$, then $\Gamma_I(E^{\cdot}_R(M))$ is a finite resolution of $H^c_I(M)[-c]$ by injective R-modules. This proves (1) ⇒ (2).

For the converse statement note that the above quasi-isomorphism of complexes induces the following isomorphism

$$\operatorname{Ext}^{i-c}_R(k, H^c_I(M)) \cong \operatorname{Ext}^i_R(k, M)$$

so it follows that $\operatorname{id}_R(M) < \infty$ since $\operatorname{Ext}^{i-c}_R(k, H^c_I(M)) = 0$ for $i \gg 0$.

Note that if one of the equivalent conditions in Theorem 4.12 hold then R will be a Cohen-Macaulay ring (see [16]). □

Note that Zargar and Zakeri define a relative Cohen-Macaulay R-module $M \neq 0$ with respect to the ideal $I \subseteq R$ which is actually equivalent to the fact that $H_I^i(M) = 0$ for all $i \neq c = \operatorname{grade}(I, M)$. They have shown that if M is a relative Cohen-Macaulay R-module with respect to I, then $\operatorname{id}_R(H_I^c(M)) = \operatorname{id}_R(M) - c$ (see [22, Theorem 2.6]).

5. APPLICATIONS

In this section, we will give some applications of our main results one of which is an extension of Hellus and Schenzel's result (see [13, Lemma 4.3]).

PROPOSITION 5.1. *With the notation in Theorem* 4.10 *suppose in addition that $K(R)$ exists and consider the following condition:*

(f) $H_I^i(K(R)) = 0$ *for all* $i \neq c$.

then (f) is equivalent to all the conditions of Theorem 4.10.

Proof. If R is a Cohen-Macaulay ring then $K(R)$ is a maximal Cohen-Macaulay module of finite injective dimension. Then the equivalence follows from Theorem 4.10 by view of Proposition 2.3. □

Now the following Proposition is a generalization of an application of Hellus and Schenzel (see [13, Lemma 4.3]).

PROPOSITION 5.2. *Let $(R, \mathfrak{m})$ be a ring, $I \subseteq R$ an ideal of* $\operatorname{grade}(I, M) = c$ *for an R-module $M \neq 0$. Suppose that $\underline{x} = (x_1, ..., x_j) \in I$ is an M-regular sequence for $1 \leq j \leq c$ then the following are equivalent:*

(a) $H_{IR/\underline{x}}^i(M/\underline{x}M) = 0$ *for all* $i \neq c - j$.

(b) $H_I^i(M) = 0$ *for all* $i \neq c$ *and* $\underline{x}$ *is* $\operatorname{Hom}_R(H_I^c(M), E)$*- regular.*

Proof. By induction on the length of the M-regular sequence $\underline{x}$ it will be enough to prove the equivalence of the statements in (a) and (b) for $j = 1$. Let us assume that $j = 1$ and $x_1 = x$.

Firstly, we prove that (a) implies (b). Since x is M-regular so we have the following short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

Applying Γ_I to this sequence yields the following long exact sequence of local cohomology

$$... \to H_I^{i-1}(M/xM) \to H_I^i(M) \xrightarrow{x} H_I^i(M) \to H_I^i(M/xM) \to ... \tag{5.1}$$

If $i < c$ then there is the following exact sequence

$$0 \to H_I^i(M) \xrightarrow{x} H_I^i(M) \tag{5.2}$$

Let $r \in H^i_I(M)$ then there exists $n \in \mathbb{N}$ such that $rI^n = 0$ it implies that $r = 0$ because of sequence 5.2. It follows that $H^i_I(M) = 0$ for all $i < c$. Similarly we can prove that $H^i_I(M) = 0$ for all $i > c$ so $H^i_I(M) = 0$ for all $i \neq c$. Now applying $\mathrm{Hom}_R(-, E)$ to sequence 5.1 and substituting $i = c$ then x is $\mathrm{Hom}_R(H^c_I(M), E)$- regular follows from the following short exact sequence

$$0 \to \mathrm{Hom}_R(H^c_I(M), E) \xrightarrow{x} \mathrm{Hom}_R(H^c_I(M), E) \to \mathrm{Hom}_R(H^{c-1}_{IR/xR}(M/xM), E) \to 0$$

Now we will prove that the assertion (b) implies (a).

If $i \leq c-2$ or $i > c$ then it follows from sequence 5.1 and Independence of the Base Ring Theorem (see [2, Theorem 4.2.1]) that $H^i_{IR/xR}(M/xM) = 0$ for all $i \leq c-2$ or $i > c$. If $i = c$ then there is the following exact sequence

$$H^c_I(M) \xrightarrow{x} H^c_I(M) \to H^c_{IR/xR}(M/xM) \to 0$$

it induces the following exact sequence

$$0 \to \mathrm{Hom}_R(H^c_{IR/xR}(M/xM), E) \to \mathrm{Hom}_R(H^c_I(M), E) \xrightarrow{x} \mathrm{Hom}_R(H^c_I(M), E).$$

Since x is $\mathrm{Hom}_R(H^c_I(M), E)$- regular so $\mathrm{Hom}_R(H^c_{IR/xR}(M/xM), E)$ being the kernel of the morphism $\mathrm{Hom}_R(H^c_I(M), E) \xrightarrow{x} \mathrm{Hom}_R(H^c_I(M), E)$ is zero. Therefore result follows from [10, Remark 3.11]. □

Acknowledgments. The author is grateful to the reviewer for suggestions to improve the manuscript. Moreover the author is grateful to P. Schenzel for suggesting the argument of Lemma 4.6. Actually this argument is very much helpful to get more vanishing of the Bass numbers of the local cohomological module.

REFERENCES

[1] H. Bass, *On the ubiquity of Gorenstein rings.* Math. Zeitschr. **82** (1963), 8–28.

[2] M. Brodmann and Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications.* Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1998.

[3] W. Bruns and J. Herzog, *Cohen-Macaulay rings.* Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1998.

[4] K. Divaani-Aazar, R. Naghipour and M. Tousi, *Cohomological dimension of certain algebraic varieties.* Proc. Amer. Math. Soc. **130** (2002), 3537–3544.

[5] P. Gabriel, *Objects Injectifs Dans Les Catégories abéliennes.* Sém. Dubreil **12** (1959).

[6] A. Grothendieck, *Local Cohomology. Notes by R. Hartshorne.* Lecture Notes in Mathematics **41**, Springer, 1967.

[7] R. Hartshorne, Residues and duality. Lecture Note in Mathematics **20**, Springer, 1966.

[8] R. Hartshorne, *Cohomological Dimension of Algebraic Varieties.* Annals of Math. **88** (1968), 403–450.

[9] M. Hochster, *Topics in the homological theory of modules over commutative rings.* CBMS. Regional Conference Series in Mathematics **24**, Providence, R.I.: American Mathematical Society (AMS), 1975.

[10] C. Huneke, *Lectures on Local Cohomology (with an Appendix by Amelia Taylor).* Contemporary Mathematics **436** (2007), 51–100.

[11] M. Hellus, *Local Cohomology and Matils Duality.* arXiv:math/0703124v1 [math.AC] 2007.

[12] M. Hellus and J. Stückrad, *Local Cohomology and Matils Duality.* Universitatis Iagellonicae Acta Mathematica **45** (2007), 63–70.

[13] M. Hellus and P. Schenzel, *On Cohomologically Complete Intersections.* J. Algebra **320** (2008), 3733–3748.

[14] E. Matlis, *Injective Modules over Noetherian rings.* Pacific J. Math. **8** (1958), 511–528.

[15] H. Matsumura, *Commutative Ring Theory.* Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1986.

[16] P. Roberts, *Le théoréme d'intersection,* (The intersection theorem) (French) C. R. Acad. Sci., Paris, Sér. I **304** (1987), 177–180.

[17] R. Sharp, *On Gorenstein Modules over a Complete Cohen-Macaulay Local Ring.* Q. J. Math. **22** (1971), 425–434.

[18] P. Schenzel, *On The Use of Local Cohomology in Algebra and Geometry.* In: Six Lectures in Commutative Algebra, Proceed. Summer School on Commutative Algebra at Centre de Recerca Matemàtica, (Ed.: J. Elias, J.M. Giral, R.M. Miró-Roig, S. Zarzuela), Progr. Math. **166** (1998), 241–292.

[19] P. Schenzel, *Proregular Sequences, Local Cohomology, and Completion.* Math. Scand, **92** (2003), 161–180.

[20] L. Vermani, *An Elementary Approach to Homological Algebra.* Chapman and Hall/CRC (Monographs and Surveys in Pure and Applied Mathematics), 2003.

[21] C. Weibel, *An introduction to Homological Algebra.* Cambridge University Press, 1994.

[22] M.R. Zargar and H. Zakeri, *On Injective and Gorenstein Dimensions of Local Cohomology Modules.* arXiv:1204.2394v4 [math.AC] 2012.

Received 22 January 2014

Quaid-i-Azam University,
Islamabad, Pakistan
waqassms@gmail.com