LINEAR LEGENDRE MULTI-WAVELETS METHODS FOR SOLVING SYSTEMS OF FREDHOLM INTEGRAL EQUATIONS

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In this paper, continuous Legendre multi-wavelets are utilized as a basis in a practical direct method to approximate the solutions of the Fredholm integral equations system. To begin with we describe the characteristic of Legendre multi-wavelets and will go on to indicate that through this method a system of Fredholm integral equations can be reduced to an algebraic equation. Finally, numerical results of some examples show that the method is practical and has high accuracy.

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Key words: Fredholm integral equation, system of integral equations, Legendre multi-wavelets, direct method, Multi-resolution of analysis (MRA), algebraic equations.

1. INTRODUCTION

Many physical problems are modeled in the form of Fredholm integral equations, such problems as potential theory and Dirichlet problems which discussed in [1] and [2], electrostatics [3], mathematical problems of radiative equilibrium [4], the particle transport problems of astrophysics and reactor theory [5], and radiative heat transfer problems which discussed in [6–7]. For such equations as well as a system of such equations, various techniques such as iterative, extrapolation, Galerkin, collocation, quadrature, projection, spline, orthogonal polynomial, and multiple grid methods have been presented to determine desired solutions (see *e.g.* [1, 8, 9] and the references therein). These methods include analytical and numerical approaches. In principle, analytical solution is the most desired result in theory and it is almost unobtainable for most practical problems. Although classical numerical methods can cope with a majority of complicated problems related to a system of integral equations, the obtained results cannot be expressed in simple form. Therefore some more sophisticated numerical methods called approximate methods were proposed. In comparison with classical numerical methods, one of the advantages of approximate methods lies in that it can give a solution in an analytic form with an acceptable error. As a result, up-to-date approximate methods remain of much interest in spite of advanced numerical methods accompanied with the help of modern computers. Usual approximate methods include iterative methods, series expansion in terms of certain orthogonal functions, perturbation technique, and so on. Furthermore, the system of integral equations plays a basic role to many physical, biological and engineering models. For instance, in several heat transfer problems in physics, the equations are usually replaced by system of integral equations [8]. Also, many well-known models for neural networks in biomathematics, nuclear reactor dynamics problems and thermo-elasticity problems are also based on these systems [10–11]. Moreover, integro-differential equations, which are important in many practical problems, can be transformed to integral equations systems [12].

Let us consider the system of linear Fredholm integral equations of the form:

$$F(x) = G(x) + \int_{\Gamma} K(x,t)F(t)\mathrm{d}t, \qquad x \in \Gamma = [0,1] \ , \tag{1}$$

where,

$$\begin{aligned} &F(x) = [f_1(x), f_2(x), ..., f_n(x)]^T, \\ &G(x) = [g_1(x), g_2(x), ..., g_n(x)]^T, \\ &K(x,t) = [k_{i,j}(x,t)]^T, \qquad i,j = 1,2,...,n. \end{aligned}$$

In system (1) the known kernel $K(x,t)$ is continuous, the function $G(x)$ is given, and $F(x)$ is the solution to be determined [12].

There have been considerable interests in solving integral equation (1). In addition to the well-known techniques, there are several new techniques for solving integral equation systems, such as Haar functions method [13], Adomian decomposition method [14], Block-Pulse functions [15], Rung-kutta method [16], Tau method [17], Newton-Taw method [18],Taylor collocation method [19], Sinc function basis [20], homotopy perturbation method [21–22], Biorthogonal systems method [23], triangular functions method [24], Fast multiscale Galerkin methods [25], reproducing kernel method [26]. As we know, it is important to select a suitable basis function in numerical methods for system of integral equations. One of the most attractive proposals made in the recent years was an idea connected to the application of wavelets as basis functions in the method of moments [27]. The wavelet technique allows the creation of very fast algorithms when compared to the algorithms ordinarily used and the main advantage of the wavelet technique is its ability to transform complex problems into a system of algebraic equations. Various wavelet basis are applied. In addition to the conventional Duabechies wavelets, Haar wavelets [28], linear B-splines [29], Walsh functions [30] have been used.

In this paper, we present the application of the linear Legendre multi-wavelets as basis functions in collocations method for numerical solution of the system of Fredholm integral equations (1). The method is tested with the numerical examples.

2. PROPERTIES OF LEGENDRE MULTI-WAVELETS

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as [30].

$$\varphi_{a,b}(t) = |a|^{-1/2}\phi(\frac{t-b}{a}), \qquad a, b \in \mathbb{R}, a \neq 0.$$

If we restrict the parameters a and b to discrete values as $a = 2^{-k}, b = n2^{-k}$, then

$$\varphi_{k,n}(t) = 2^{-k/2}\phi(2^k t - n),$$

form an orthogonal basis [30].

The linear Legendre multi-wavelets are described in [31] and applied in [32–34]. For constructing the linear Legendre multi-wavelets, at first we describe the following scaling functions:

$$\varphi_0(t) = 1, \qquad \varphi_1(t) = \sqrt{3}(2t-1), \qquad 0 \leq t \leq 1.$$

Now let $\psi^0(t)$ and $\psi^1(t)$ be the corresponding mother wavelets, then by Multiresolution of analysis (MRA) and applying suitable conditions [31] on $\psi^0(t)$ and $\psi^1(t)$ the explicit formula for linear Legendre mother wavelets will obtained as:

$$\psi^0(t) = \begin{cases} -\sqrt{3}(4t-1), & 0 \leq t \leq \frac{1}{2}, \\ \sqrt{3}(4t-3), & \frac{1}{2} \leq t \leq 1, \end{cases} \tag{2}$$

$$\psi^1(t) = \begin{cases} 6t-1, & 0 \leq t \leq \frac{1}{2}, \\ 6t-5, & \frac{1}{2} \leq t \leq 1, \end{cases} \tag{3}$$

and the family $\{\psi^j_{k,n}\} = \{2^{k/2}\psi^j(2^k t - n)\}$, k is any nonnegative integer $n = 0, 1, ..., 2^k - 1$ and $j = 0, 1$, forms an orthonormal basis for $L^2(\mathbb{R})$.

3. FUNCTION APPROXIMATION

A function $f(t)$ defined over $[0, 1)$ may be expanded as:

$$f(t) = f_0\varphi_0(t) + f_1\varphi_1(t) + \sum_{k=0}^{\infty}\sum_{j=0}^{1}\sum_{n=0}^{\infty} f^j_{k,n}\psi^j_{k,n}(t), \tag{4}$$

where,

$$f_0 = < f(t), \varphi_0(t) >, \qquad f_1 = < f(t), \varphi_1(t) >, \qquad f^j_{k,n} = < f(t), \psi^j_{k,n}(t) >. \tag{5}$$

In Eq. (5), $< .,. >$ denoting the inner product. We noted that the inner product is in $L^2(\mathbb{R})$. Now, we can consider truncated series of Eq. (4) as follows:

$$f(t) \approx f_0\varphi_0(t) + f_1\varphi_1(t) + \sum_{k=0}^{M}\sum_{j=0}^{1}\sum_{n=0}^{2^k-1} f^j_{k,n}\psi^j_{k,n}(t) = C^T\phi = \sum_{i=1}^{2^{M+2}} c_i\phi_i(t) \tag{6}$$

where,

$$\begin{aligned} C &= [f_0, f_1, f^0_{0,0}, f^1_{0,0}, ..., f^0_{M,0}, f^0_{M,1}, ..., f^0_{M,(2^M-1)}, ..., f^1_{M,0}, f^1_{M,1} \\ &\quad , ..., f^1_{M,(2^M-1)}]^T, \\ \phi &= [\varphi_0(t), \varphi_1(t), \psi^0_{0,0}(t), \psi^1_{0,0}(t), ..., \psi^0_{M,0}(t), \psi^0_{M,1}(t), ..., \psi^0_{M,(2^M-1)}(t) \\ &\quad , ..., \psi^1_{M,0}(t), \psi^1_{M,1}(t), ..., \psi^1_{M,(2^M-1)}(t)]^T, \end{aligned}$$

and M is a nonnegative integer. Furthermore, a function $k(x,t) \in L^2([0,1] \times [0,1])$ may be approximated as:

$$k(x,t) \approx \varphi^T(x)K\varphi(t), \tag{7}$$

where, K is $(2^{M+2}) \times (2^{M+2})$ matrix with:

$$K_{i,j} = < \varphi_i(x), < k(x,t), \varphi_j(t) >> . \tag{8}$$

The integration of the product of two Legendre multi-wavelets vector function is obtained as,

$$I = \int_0^1 \varphi(t)\varphi^T(t), \tag{9}$$

where, I is an identity matrix.

4. SOLVING THE SYSTEM OF FREDHOLM INTEGRAL EQUATIONS

In this section, we use Legendre multi-wavelets direct method to convert equation (1) to algebraic system of linear equations $AX = b$ and then solve this system by the robust and iterative solver such as Krylov subspace iteration methods, SOR method, preconditioning models and analytical solver (see *e.g.* [35–40] and the references quoted there). We assume that Eq. (1) has a unique solution. However, the necessary and sufficient conditions for existence and uniqueness of the solution of system (1) could be found in [12].

Consider the ith equation of (1),

$$f_i(x) = g_i(x) + \int_0^1 \sum_{j=1}^{n} k_{i,j}(x,t) f_i(t)\mathrm{d}t, \qquad i = 1, 2, ..., n. \tag{10}$$

We approximate f_i, g_i and $k_{i,j}$ by (4)-(7) as follows:

$$f_i(x) \approx C_i^T \phi(x), g_i(x) \approx G_i^T \phi(x), k_{i,j}(x,t) \approx \phi^T(x) K_{i,j} \phi(t). \tag{11}$$

By substituting relation (11) in (10) we have,

$$\begin{aligned} \varphi^T(x)C_i &= \varphi^T(x)G_i + \textstyle\int_0^1 \sum\limits_{j=1}^{n} \varphi^T(x)K_{i,j}\varphi(t)\varphi^T(x)C_j dt \\ &= \varphi^T(x)G_i + \varphi^T(x) \sum\limits_{j=1}^{n} K_{i,j}(\int\limits_0^1 \varphi(t)\varphi^T(t)dt)C_j \\ &= \varphi^T(x)G_i + \varphi^T(x) \sum\limits_{j=1}^{n} K_{i,j}C_j. \end{aligned} \tag{12}$$

Therefore, we have the following algebraic system of linear equations:

$$C_i = G_i + \sum_{j=1}^{n} K_{i,j} C_j. \tag{13}$$

By solving this linear system, we can find the vector C_i, so,

$$f_i(x) \approx C_i^T \phi(x), \qquad i = 1, 2, ...n. \tag{14}$$

5. NUMERICAL EXPRIMENTS

In this section, we give some numerical experiments to illustrate the results obtained in previous sections. All the numerical experiments presented in this section were computed by a Maple 16 on a PC with a 1.86 GHz 32-bit processor and 1 GB memory.

Example 5.1. Consider the following linear system of Fredholm integral equations:

$$\begin{cases} u(x) = \frac{x}{18} + \frac{17}{36} + \int\limits_0^1 (\frac{x+t}{3})(u(t)+v(t))\mathrm{d}t, \\ v(x) = x^2 - \frac{19x}{12} + 1 + \int\limits_0^1 (x.t)(u(t)+v(t))\mathrm{d}t. \end{cases}$$

With the exact solutions $u(x) = x + 1$ and $v(x) = 1 + x^2$.

TABLE 1

Shows the results of Example 5.1

x	Error for $u(x)$	Error for $v(x)$
0.0	0	8.437524563517452e-001
0.1	0	4.583333669999856e-003
0.2	0	4.166662600000315e-004
0.3	0	4.166661900000257e-004
0.4	0	4.583333879999874e-003
0.5	0	1.041666597500002e-002
0.6	0	4.583334199999900e-003
0.7	0	4.166656999999852e-004
0.8	0	4.166658000002155e-004
0.9	0	4.583334299999686e-003

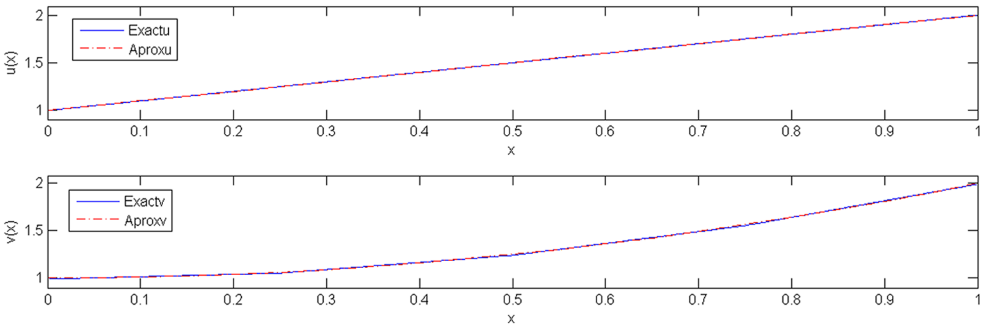

Fig. 1 – Comparison plot of exact and approximation solution of Example 5.1, for $M = 1$.

TABLE 2

Shows the results of Example 5.2

x	Error for $u(x)$	Error for $v(x)$
0.0	1.176114627000000e-026	1.562502645000000e-001
0.1	3.327658000000511e-005	4.582723310000000e-003
0.2	3.108605999999070e-005	4.178689700000000e-004
0.3	2.889554000001793e-005	4.184190000000000e-004
0.4	2.670502000001740e-005	4.581073300000000e-003
0.5	2.451449999996136e-005	1.041943410000000e-002
0.6	2.232398000001634e-005	4.580142900000000e-003
0.7	2.013345999996030e-005	4.202801000000000e-004
0.8	1.794294000001528e-005	4.206607000000000e-004
0.9	1.575241999995924e-005	4.579000300000000e-003

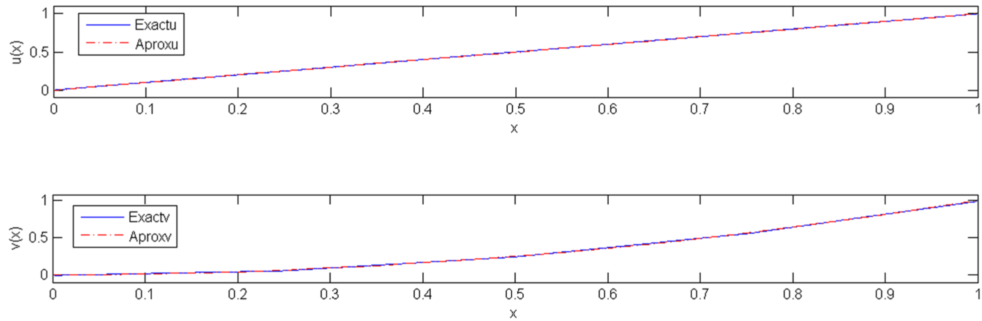

Fig. 2 – Comparison plot of exact and approximation solution of Example 5.2, for $M = 1$.

Example 5.2. Consider the following system of Fredholm integral equations:

$$\begin{cases} u(x) = \frac{11x}{6} + \frac{11}{15} - \int\limits_0^1 (x+t)u(t)\mathrm{d}t - \int\limits_0^1 (x+2t^2)v(t)\mathrm{d}t, \\ v(x) = \frac{5}{4}x^2 + \frac{x}{4} - \int\limits_0^1 (xt^2)u(t)\mathrm{d}t - \int\limits_0^1 (x^2t)v(t)\mathrm{d}t. \end{cases}$$

With the exact solutions $u(x) = x$ and $v(x) = x^2$. For $M = 1$, Table 2 and Fig. 2 are the numerical results for Example 5.2.

TABLE 3

Shows the results of Example 5.3

x	Error for $u(x)$	Error for $v(x)$
0.0	8.788906725000000e-001	9.732308201000000e-007
0.1	2.556600000000000e-003	4.750000000000000e-008
0.2	1.832660000000000e-004	3.030000000000000e-008
0.3	3.715530000000000e-004	1.428000000000000e-007
0.4	3.389135000000000e-003	4.715000000000000e-007
0.5	9.503396000000001e-003	8.002000000000001e-007
0.6	4.215705000000000e-003	1.440400000000000e-006
0.7	3.015720000000000e-004	2.080500000000000e-006
0.8	6.120040000000000e-004	2.876500000000000e-006
0.9	5.588323000000000e-003	3.828000000000000e-006

Example 5.3. Consider the following system of Fredholm integral equations:

$$\begin{cases} u(x) = \frac{2e^x}{3} - \frac{1}{4} + \int\limits_0^1 (\frac{1}{3}e^x t)u(t)\mathrm{d}t + \int\limits_0^1 t^2 v(t)\mathrm{d}t, \\ v(x) = \frac{3}{2}x - x^2 + \int\limits_0^1 (x^2 e^{-t})u(t)\mathrm{d}t - \int\limits_0^1 xv(t)\mathrm{d}t. \end{cases}$$

With the exact solutions $u(x) = e^x$ and $v(x) = x$. For $M = 1$, Table 3 and Fig. 3 are the numerical results for Example 5.3.

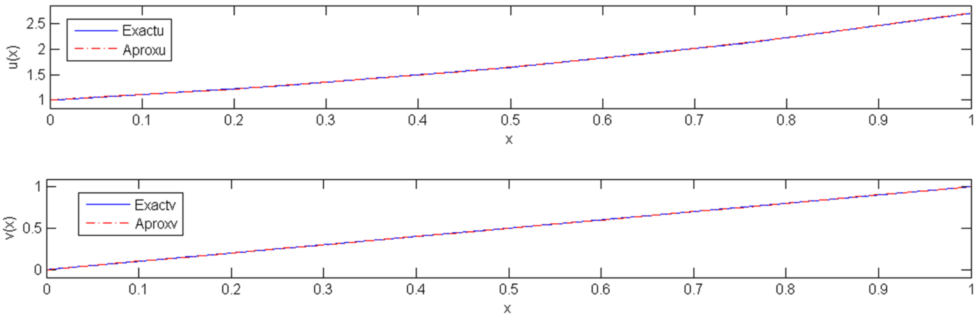

Fig. 3 – Comparison plot of exact and approximation solution of Example 5.3, for $M = 1$.

6. CONCLUSIONS

In this paper, the systems of Fredholm integral equations are investigated and a practical direct method based on Legendre multi-wavelets is proposed. The proposed method is easy to understand and this approximation reduces the system of integral equations to an explicit system of algebraic equations. Finally, illustrative examples are included to demonstrate the validity and applicability of the technique.

REFERENCES

[1] K.E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind.* Cambridge University Press, Cambridge, 1997.

[2] S.G. Mikhlin, *Integral Equations.* 2nd Edition, Pergamon Press, London, 1964.

[3] E.R. Love, *The electrostatic field of two equal circular co-axial conducting discs.* Quart J. Mech. Appl. Math. **2** (1949), 428–451.

[4] E. Hopf, *Mathematical Problems of Radiative Equilibrium.* Cambridge University Press, Cambridge, 1934.

[5] H.G. Kaper and R.B. Kellogg, *Asymptotic behavior of the solution of the integral transport equation in slab geometry.* SIAM J. Appl. Math. **32** (1971), 191–200.

[6] N. Qatanani and I. Barghouthi, *Numerical treatment of the two Dimensional heat radiation integral equation.* J. Comput. Anal. Appl. **7** (3), 319–349.

[7] N. Qatanani and M. Schulz, *Analytical and numerical investigation of the Fredholm integral equation for the heat radiation problem.* Appl. Math. Comput. **175** (2006), 149–170.

[8] R. Kress, *Linear Integral Equations.* Springer-Verlag, NewYork, 1999.

[9] P.K. Kythe and P. Puri, *Computational methods for linear integral equations.* Birkhauser Verlag, Springer, Boston, 2002.

[10] U. Heiden, *Analysis of Neural Networks.* Springer-Verlag, NewYork, 1980.

[11] B.L. Moiseiwitsch, *Integral Equations.* Dover Publications Inc., NewYork, 2005.

[12] L.M. Delves and J.L. Mohamed, *Computational methods for integral equations.* Cambridge University Press, London, 1985.

[13] K. Maleknejad and F. Mirzaee, *Numerical solution of linear Fredholm integral equations system by rationalized Haar functions method.* Int. J. Comput. Math. **80** (2003), 1397–405.

[14] E. Babolian, J. Biazar, and A.R. Vahidi, *The decomposition method applied to system of Fredholm integral equations of the second kind.* Appl. Math. Comput. **148** (2004), 443–452.

[15] K. Maleknejad, M. Sharezaee, and H. Khatami, *Numerical solution of integral equations system of the second kind by Block-Pulse functions.* Appl. Math. Comput. **166** (2005), 15–24.

[16] K. Maleknejad and M. Shahrezaee, *Rung-Kutta method for numerical solution of the system of Volterra integral equation.* Appl. Math. Comput. **149** (2004), 399–410.

[17] J. Pour-Mahamoud, M.Y. Rahimi-Ardabili, and S. Shahmorad, *Numerical solution of system of Fredholm integro-differential equations by the Tau method.* Appl. Math. Comput. **168** (2005), 465–478.

[18] K. Ivaz and B.S. Mostahkam, *Newton-Tau numerical solution of a system of nonlinear Fredholm integral equations of second kind.* Appl. Comput. Math. **5** (2006), 201–208.

[19] M. Gulsu and M. Sezer, *Taylor collocation method for solution of systems of high-order linear FredholmVolterra integro-differential equations.* Int. J. Comput. Math. **83** (2006), 429–448.

[20] J. Rashidinia and M. Zarebnia, *Convergence of approximate solution of system of Fredholm integral equations.* J. Math. Anal. Appl. **333** (2007), 1216–27.

[21] R. K. Saeed, *Homotopy perturbation method for solving system of nonlinear Fredholm integral equations of the second kind.* J. App. Sci. Research **4** (2008), 1166–1173.

[22] H. Aminikhah and M. Salahi, *A new HPM for integral equations.* Appl. Appl. Math. **4** (2009), 122–133.

[23] M.I. Berenguer, D. Gamez, A.I. Garralda-Guillem, M. Ruiz Galan and M.C. Serrano Perez, *Biorthogonal systems for solving Volterra integral equation systems of the second kind.* J. Comput. Appl. Math. **235** (2011), 1873–1875.

[24] H. Almasieh and M. Roodaki, *Triangular functions method for the solution of Fredholm integral equations system.* Ain Shams Engineering J. **3** (2012), 411–416.

[25] S. Ding and H. Yang, *Fast multiscale Galerkin methods for solving ill-posed integral equations via a coupled system under general source conditions.* J. Math. Anal. Appl. **408** (2013), 213–224.

[26] W. Jiang and Z. Chen, *Solving a system of linear Volterra integral equations using the new reproducing kernel method.* Appl. Math. Comput. **219** (2013), 10225–10230.

[27] A.P. Anioutine and A.G. Kyurkchan, *Application of wavelets technique to the integral equations of the method of auxiliary currents.* J. Quant. Spectrosc. Radiat. Transf. **79–80** (2003), 495–508.

[28] B.A. Lewis, *On the numerical solution of Fredholm integral equations of the first kind.* J. Inst. Math. Appl. **16** (1973), 207–220.

[29] C. Sanchez-Avila, *Wavelet domain signal deconvolution with singularity-preserving regularization.* Math. Comput. Simul. **61** (2003)165–176.

[30] A. Boggess and F.J. Narcowich, *A First Course in Wavelets with Fourier Analysis.* Prentice-Hall, 2001.

[31] F. Khellat and S.A. Youse, *The linear mother wavelets operational matrix of integration and its application.* J. Franklin Inst. **343** (2006), 181–190.

[32] Han Danfu and Shang Xufeng, *Numerical solution of Fredholm integral equations of the first kind by using linear Legendre multi-wavelets.* Appl. Math. Comput. **191** (2007), 440–444.

[33] Z. Abbas, S. Vahdati, M. Atan, K. Ariffin, N. Long and N.M. Asri, *Legendre multi-wavelets direct method for linear integro-differential equations.* Applied Mathematical Sciences. **3** (2009), 693–700.

[34] F. Khellat, *Optimal control of linear time-delayed systems by linear Legendre multi-wavelets.* J. Optim. Theory Appl. **143** (2009), 107–121.

[35] D.M. Young, *Iterative solution of large linear systems.* Academic Press, New York, 1971.

[36] H. Saberi Najafi and S.A. Edalatpanah, *Fast iterative method-FIM: Application to the convection-diffusion equation.* J. Inform. Comp. Sci. **6** (2011), 303–313.

[37] H. Saberi Najafi and S.A. Edalatpanah, *On application of Liaos method for system of linear equations.* Ain Shams Engineering J. **4** (2013), 501–505.

[38] H. Saberi Najafi and S.A. Edalatpanah, *A new modified SSOR iteration method for solving augmented linear systems.* Int. J. Comput. Math. **91** (2014), 539–552.

[39] H. Saberi Najafi and S.A. Edalatpanah, *A new family of (I+S)-type preconditioner with some applications.* Comp. Appl. Math. **34** (2015), 917–931.

[40] Y. Saad, *Iterative methods for sparse linear system.* SIAM, 2003.

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