RECOGNITION OF SOME SIMPLE GROUPS BY CHARACTER DEGREE GRAPH AND ORDER

BEHROOZ KHOSRAVI, BEHNAM KHOSRAVI, BAHMAN KHOSRAVI
and ZAHRA MOMEN

Communicated by Vasile Brînzănescu

The character degree graph of a finite group G is the graph whose vertices are the prime divisors of the irreducible character degrees of G and two distinct vertices p_1 and p_2 are joined by an edge if $p_1 p_2$ divides some irreducible character degree of G. In this paper, we prove that some finite simple groups are uniquely determined by their character degree graphs and their orders.

AMS 2010 Subject Classification: 20C15, 20D05, 20D60.

Key words: character degree graph, order, simple group, irreducible characters.

1. INTRODUCTION AND PRELIMINARY RESULTS

Let G be a finite group, $\mathrm{Irr}(G)$ be the set of irreducible characters of G, and denote by $\mathrm{cd}(G)$, the set of irreducible character degrees of G.

The *character degree graph* of G, which is denoted by $\Gamma(G)$, is defined as follows: the vertices of this graph are the prime divisors of the irreducible character degrees of the group G and two distinct vertices p_1 and p_2 are joined by an edge if there exists an irreducible character degree of G which is divisible by $p_1 p_2$. This graph was introduced in [13] and studied by many authors (see [11, 14]).

Recently, there has been much interest in the influence of arithmetical conditions on degrees of irreducible characters of a group G on the structure of G. A finite group G is called a K_3-group if $|G|$ has exactly three distinct prime divisors. Recently Chen *et al.* in [15] and [16] proved that all simple K_3-groups and the Mathieu groups are uniquely determined by their orders and one or both of their largest and second largest irreducible character degrees.

Let p be an odd prime number. In [6] the authors proved that the simple group $L_2(p)$ is uniquely determined by its order and its largest and second largest irreducible character degrees. In [10] it is proved that the simple group $L_2(p^2)$ is uniquely determined by its character degree graph and its order.

The first author was in part supported by a grant from IPM (grant no. 93200043).

In [7], the authors proved that if G is a finite group such that $|G| = 2|L_2(p^2)|$, $p^2 \in \mathrm{cd(G)}$ and there does not exist any $\theta \in \mathrm{Irr}(G)$ such that $2p \mid \theta(1)$, then G has a unique nonabelian composition factor isomorphic to $L_2(p^2)$. In [5] it is proved that the projective special linear group $L_2(q)$ is uniquely determined by its group order and its largest irreducible character degree when q is a prime or when $q = 2^a$ for an integer $a \geq 2$ such that $2^a - 1$ or $2^a + 1$ is a prime.

In [8] it is proved that if p is an odd prime number and G is a finite group such that $|G| = |L_2(p^2)|$, $p^2 \in \mathrm{cd}(G)$ and there does not exist any $\theta \in \mathrm{Irr}(G)$ such that $2p \mid \theta(1)$, then $G \cong L_2(p^2)$.

The goal of this paper is to introduce a new characterization for some simple groups. In fact for a simple group S in this list we prove that *if G is a finite group such that $|G| = |S|$ and $\Gamma(G) = \Gamma(S)$, then $G \cong S$.* Also by an example we show that this result is not true for all simple groups.

If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$, then the inertia group of θ in G is $I_G(\theta) = \{g \in G \mid \theta^g = \theta\}$. If the character $\chi = \sum_{i=1}^{k} e_i\chi_i$, where for each $1 \leq i \leq k$, $\chi_i \in \mathrm{Irr}(G)$ and e_i is a natural number, then each χ_i is called an irreducible constituent of χ.

Lemma 1.1 ((Itô's Theorem) [3, Theorem 6.15]). *Let $A \trianglelefteq G$ be abelian. Then $\chi(1)$ divides $|G : A|$, for all $\chi \in \mathrm{Irr}(G)$.*

Lemma 1.2 ([3, Theorems 6.2, 6.8, 11.29]). *Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr(G)}$. Let θ be an irreducible constituent of χ_N and suppose $\theta_1 = \theta, \ldots, \theta_t$ are the distinct conjugates of θ in G. Then $\chi_N = e\sum_{i=1}^{t}\theta_i$, where $e = [\chi_N, \theta]$ and $t = |G : I_G(\theta)|$. Also $\theta(1) \mid \chi(1)$ and $\chi(1)/\theta(1) \mid |G : N|$.*

Lemma 1.3 ((Itô-Michler Theorem) [2]). *Let $\rho(G)$ be the set of all prime divisors of the elements of $\mathrm{cd}(G)$. Then $p \notin \rho(G)$ if and only if G has a normal abelian Sylow p-subgroup.*

Lemma 1.4 ([15, Lemma]). *Let G be a nonsolvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Lemma 1.5 ((Gallagher's Theorem) [3, Corollary 6.17]). *Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}(G)$ be such that $\chi_N = \theta \in \mathrm{Irr}(N)$. Then the characters $\beta\chi$ for $\beta \in \mathrm{Irr}(G/N)$ are irreducible distinct for distinct β and all of the irreducible constituents of θ^G.*

Lemma 1.6 ((Palfy's Theorem) [11, Theorem 4.1]). *Let $\rho(G)$ be the set of all prime divisors of the character degrees of G. Let G be a solvable group and $\pi \subseteq \rho(G)$. If $|\pi| \geq 3$, then there exist primes $p, q \in \pi$ and a degree $a \in \mathrm{cd}(G)$*

so that pq divides a. In other words, any three primes in $\rho(G)$ must have an edge in $\Gamma(G)$ that is incident to two of those primes.

LEMMA 1.7 ([6, Main Theorem 1]). *Let p be an odd prime number. If G is a finite group such that (i) $|G| = |L_2(p)|$, (ii) $p \in \mathrm{cd}(G)$, (iii) $\mathrm{cd}(G)$ has an even integer, (iv) there does not exist any element $a \in \mathrm{cd}(G)$ such that $2p \mid a$, then $G \cong L_2(p)$.*

LEMMA 1.8 ([5, Theorem B]). *Let G be a group. Assume that either $2^a - 1$ or $2^a + 1$ is a prime. Then $G \cong L_2(2^a)$ if and only if $|G| = |L_2(2^a)|$ and $b(G) = b(L_2(2^a))$, where $b(G)$ is the largest irreducible character degree of G.*

LEMMA 1.9 ([16, Lemma 2]). *Let G be a finite solvable group of order $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$, where $p_1, p_2, \dots, p_n$ are distinct primes. If $(kp_n + 1) \nmid p_i^{\alpha_i}$, for each $i \leq n - 1$ and $k > 0$, then the Sylow p_n-subgroup is normal in G.*

If n is an integer and r is a prime number, then we write $r^{\alpha} \| n$, when $r^{\alpha} \mid n$ but $r^{\alpha+1} \nmid n$. Also if r is a prime number we denote by $\mathrm{Syl}_r(G)$, the set of Sylow r-subgroups of G and we denote by $n_r(G)$, the number of elements of $\mathrm{Syl}_r(G)$. If H is a characteristic subgroup of G, we write H ch G. All other notations are standard and we refer to [1].

2. MAIN RESULTS

THEOREM 2.1. *Let p be an odd prime number and $q = p$ or $q = p^2$. If $q \geq 5$ and G is a finite group of order $|L_2(q)|$, such that $\Gamma(G) = \Gamma(L_2(q))$, then $G \cong L_2(q)$.*

Proof. If $q = p^2$, then the result follows by the main theorem of [10]. So let $q = p$. By assumptions p and 2 are some vertices of $\Gamma(G)$ and $2 \nsim p$ in $\Gamma(G)$. Therefore using Lemma 1.7 we get that $G \cong L_2(p)$. □

THEOREM 2.2. *Let α be a positive integer such that $2^{\alpha} - 1$ or $2^{\alpha} + 1$ is a prime. Then $L_2(2^{\alpha})$ is characterizable by order and character degree graph.*

Proof. Let G be a finite group of order $|L_2(2^{\alpha})|$ such that $\Gamma(G) = \Gamma(L_2(2^{\alpha}))$. We consider two cases:

Case (I) Let $p = 2^{\alpha} + 1$ be a prime number. Then 2 and p are two isolated vertices of the character degree graph of G. Also the prime divisors of $2^{\alpha} - 1$ construct a connected component of $\Gamma(G)$ which is a clique. Therefore since the degree of each irreducible character divides the order of G we conclude that p is the largest element of $\mathrm{cd}(G)$. Now we get the result using Lemma 1.8.

Case (II) Let $p = 2^\alpha - 1$ be a prime number. By (I) we can suppose that $p \geq 7$. Then by Palfy's Theorem, G is nonsolvable and so by Lemma 1.4, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. Also there exists an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = p$.

If $p \mid |H|$, then by Lemma 1.2, $\chi_H \in \mathrm{Irr}(H)$, since $p \| |G|$. We know that K/H has an irreducible character of even degree and so using Gallagher's Theorem we get that $2 \sim p$ in $\Gamma(G)$, which is a contradiction.

If $p \mid |G/K|$, then by Lemma 1.4 we get that $p \mid |\mathrm{Out}(K/H)|$. Also K/H is a direct product of k copies of a simple group S and so $\mathrm{Out}(K/H) \cong \mathrm{Out}(S) \wr S_k$. Since $60^p \leq |K/H| \leq (p+1)(p+2)$, we conclude that $p \mid |\mathrm{Out}(S)|$. Since $p \geq 7$ and $p \nmid |S|$ it follows that S is a nonabelian simple group of Lie type over $\mathrm{GF}(q)$, where $q = r^\beta$ and $p \mid \beta$. Now $2^p \leq q \leq |K/H| \leq (p+1)(p+2)$, which is a contradiction.

Therefore $p \mid |K/H|$. By assumptions $p^2 \nmid |G|$, which implies that K/H is a nonabelian simple group. Also $2 \nsim p$ in $\Gamma(G)$ and so $2 \nsim p$ in $\Gamma(K/H)$ and specially $\Gamma(K/H)$ is not complete. In [14] finite simple groups, with noncomplete character degree graphs are determined . By considering them we have:

$$K/H \in \{J_1, M_{11}, M_{23}, A_8, {}^2B_2(q), L_3(q), U_3(q), L_2(q)\}.$$

Since 2 is an isolated vertex in the character degree graph of G we see that the only possibility for K/H is $L_2(p+1)$ where $p = 2^\alpha - 1$. Hence $H = 1$ and $G = K$. Therefore $L_2(2^\alpha)$ is characterizable by its order and its character degree graph. □

Using the above theorem it follows that $L_2(4)$, $L_2(8)$, $L_2(16)$ and $L_2(32)$ are characterizable by order and character degree graph. In the next theorem we prove the same result for $L_2(64)$.

Theorem 2.3. *If G is a finite group of order $|L_2(64)|$, such that $\Gamma(G) = \Gamma(L_2(64))$, then $G \cong L_2(64)$.*

Proof. By [1] we see that $\Gamma(G)$ has three connected components and 2 is an isolated vertex of $\Gamma(G)$. Hence by Palfy's Theorem, G is not a solvable group and so G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [1], it follows that K/H is isomorphic to $Sz(8)$, $L_3(4)$, $L_4(2)$, A_7, $L_2(13)$, $L_2(8)$, $L_2(7)$, A_6 or A_5. Now we consider each possibility separately. Since 2 is an isolated vertex of $\Gamma(G)$ and $\Gamma(K/H)$ is a subgraph of $\Gamma(G)$ we get that $K/H \ncong Sz(8)$, $L_3(4)$, $L_4(2)$, A_7, A_6, $L_2(7)$ and $L_2(13)$. If $K/H \cong L_2(8)$, then $|G/K| = 1$, since $|G/K| \mid 3$. Therefore

$|H| = 2^3 \cdot 5 \cdot 13$. Since $5 \sim 13$ in $\Gamma(G)$, we get that G has an irreducible character χ such that $65 \mid \chi(1)$ and so H has an irreducible character θ such that $65 \mid \theta(1)$, by Lemma 1.2. Hence $65^2 < |H|$, which is a contradiction.

If $K/H \cong A_5$, then $|H| = 2^\alpha \cdot 3 \cdot 7 \cdot 13$, where $\alpha = 3$ or 4. Using Lemma 1.2 we get that H has an irreducible character of degree 13. Therefore $O_{13}(H) = 1$ by Itô-Michler Theorem. If H is a solvable group, then by Lemma 1.9 we get that $O_{13}(H) \neq 1$, which is a contradiction. Therefore H is a nonsolvable group and so H has a nonabelian chief factor M_{i+1}/M_i which is isomorphic to $L_2(7)$ or $L_2(13)$. In each case the vertex 2 is not an isolated vertex of $\Gamma(M_{i+1}/M_i)$ and so of $\Gamma(M_{i+1})$. Therefore since M_{i+1} is a subnormal subgroup of G we get that 2 is not an isolated vertex of $\Gamma(G)$, which is a contradiction.

Hence $K/H \cong L_2(64)$ which implies that $G \cong L_2(64)$ and the result follows. □

LEMMA 2.4. *Let G be a solvable group of order $2^\alpha \cdot 5 \cdot 7$, where $1 \leq \alpha \leq 6$. Then $O_5(G) \neq 1$ or $O_7(G) \neq 1$.*

Proof. If $1 \leq \alpha \leq 3$, then using Lemma 1.9 we get that Sylow 5-subgroup of G is a normal subgroup of G.

Let $\alpha = 4$. If $O_5(G) = 1$ and $O_7(G) = 1$, then M, a normal minimal subgroup of G is an elementary abelian subgroup of order 2^β. If $\beta < 4$, then R/M, the Sylow 5-subgroup of G/M is a normal subgroup and so $O_5(R) \neq 1$, which implies that $O_5(G) \neq 1$. Therefore suppose that M, the Sylow 2-subgroup of G is an elementary abelian subgroup of order 16. Let Q be a Sylow 7-subgroup of G. If Q is not a normal subgroup of G, then G has 8 Sylow 7-subgroups and so $|N_G(Q)| = 70$. Let H be a Hall subgroup of order 35 in $N_G(Q)$. Then $G = HM$. We claim that H is a maximal subgroup of G. Otherwise let L be a subgroup of G such that $H \leq L \leq G$. Then $M \cap L$ is a normal subgroup of M and L. Therefore $M \cap L \trianglelefteq G$. Since M is a normal minimal subgroup of G we get that $M \cap L = 1$ or M. Also $G = LM$ and since $H \leq L$ by Dedekind modular law we get that $(M \cap L)H = MH \cap L = L$. If $M \cap L = 1$, then $L = H$ and if $M \cap L = M$, then $L = G$. Hence H is a maximal subgroup of G, which is a contradiction since $|N_G(Q)| = 70$.

If $\alpha = 5$ or $\alpha = 6$, then exactly similar to the previous case we get the result and we omit the details for abbreviation. □

THEOREM 2.5. *The alternating groups A_n, where $5 \leq n \leq 8$, are characterizable by order and character degree graph.*

Proof. We know that $A_5 \cong L_2(5)$ and $A_6 \cong L_2(9)$. So the result follows by Theorem 2.1. So in the sequel we consider $n = 7$ and $n = 8$.

Case I. Let $n = 7$ and G be a finite group such that $|G| = 2^3 \cdot 3^2 \cdot 5 \cdot 7$ and $\Gamma(G) = \Gamma(A_7)$.

Using [1] we know that the character degree graph of A_7 is a complete graph on four vertices $\{2, 3, 5, 7\}$. Hence $O_7(G) = 1$ and $O_5(G) = 1$, by Itô's theorem and there exists an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $21 \mid \chi(1)$. If G is a solvable group, then by considering a Hall subgroup H of G, where $|G : H| = 5$ we get a normal subgroup $N = H_G$ of G such that $G/N \hookrightarrow S_5$. Also the order of solvable subgroups of S_5 which are divisible by 5 are 5, 10 and 20. Therefore $|G/N|$ is a divisor of 20 and so $7 \mid |N|$. Then using Lemma 1.2 we get that N has an irreducible character θ such that $21 \mid \theta(1)$. Since $\theta(1)^2 < |N|$, it follows that $|N| = 504$ and so $H = N \lhd G$. On the other hand, there exists $\eta \in \mathrm{Irr}(G)$ such that $35 \mid \eta(1)$. Let $\varphi \in \mathrm{Irr}(H)$ such that $e = [\eta_H, \varphi] \neq 0$. Then $\eta(1) = et\varphi(1)$, where $t = |G : I_G(\varphi)|$ and $et \mid 5$. Therefore $7 \mid \varphi(1)$ and since $[\eta_H, \eta_H] = e^2 t \leq 5$, it follows that $e = 1$ and $t = 5$. Hence there exist 5 irreducible characters of degree divisible by 7 in H. Therefore $1 + 5 \cdot 7^2 + 21^2 \leq |H| = 504$, which is a contradiction. So G is a nonsolvable group and by Lemma 1.4, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [1], it follows that K/H is isomorphic to A_5, A_6, A_7, $L_2(7)$ or $L_2(8)$. If $K/H \cong A_5$ or A_6, then $7 \mid |H|$ and the Sylow 7-subgroup of H is a normal subgroup of H which implies that $O_7(G) \neq 1$, and this is a contradiction. If $K/H \cong L_2(7)$ or $L_2(8)$, then $5 \mid |H|$ and similarly to above $O_5(G) \neq 1$, which is a contradiction. Therefore $K/H \cong A_7$ and so $G \cong A_7$.

Case II. Let $n = 8$.

The character degree graph of A_8 is obtained by removing the edge between 2 and 3 from the complete graph on vertex set $\{2, 3, 5, 7\}$. Let G be a finite group of order $|A_8| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ such that $\Gamma(G) = \Gamma(A_8)$. Since 5 and 7 are the vertices of $\Gamma(G)$, by Lemma 1.3 we get that $O_5(G) = 1$ and $O_7(G) = 1$.

First we prove that G is a nonsolvable group. On the contrary let G be a solvable group. By the above discussion we get that $O_2(G) \neq 1$ or $O_3(G) \neq 1$. Now we consider each possibility separately.

Let $O_3(G) \neq 1$. Since 3 is a vertex in $\Gamma(G)$ by Lemma 1.3 we get that $|O_3(G)| = 3$. Now we consider the Hall subgroup $M/O_3(G)$ of $G/O_3(G)$ such that $|G/O_3(G) : M/O_3(G)| = 3$. As we mentioned above we get a normal subgroup $N \lhd G$ such that $|N| = 2^5 \cdot 3 \cdot 5 \cdot 7$ or $|N| = 2^6 \cdot 3 \cdot 5 \cdot 7$. Again by considering T a Hall subgroup of N such that $|N : T| = 3$, we get a normal subgroup F of N such that $O_5(F) \neq 1$ or $O_7(F) \neq 1$, by Lemma 2.4, which is a contradiction. Hence $O_3(G) = 1$.

Therefore $O_2(G) \neq 1$. Let $|O_2(G)| = 2^\alpha$, where $1 \leq \alpha \leq 6$. If $4 \leq \alpha \leq 6$, then both the Sylow 5-subgroup and the Sylow 7-subgroup of $G/O_2(G)$ are normal subgroups of $G/O_2(G)$ by Lemma 1.9. So we get a normal subgroup Q of G such that $|Q| = 2^\alpha \cdot 5 \cdot 7$. Now we get a contradiction by Lemma 2.4.

If $\alpha = 3$, then the Sylow 5-subgroup of $G/O_2(G)$ is a normal subgroup and so we get a normal subgroup Q of G such that $|Q| = 40$ and so $O_5(Q) \neq 1$, which implies that $O_5(G) \neq 1$, a contradiction.

If $\alpha \leq 2$, then $E/O_2(G) = O_3(G/O_2(G)) \neq 1$, since $O_5(G) = 1$ and $O_7(G) = 1$. If $|E/O_2(G)| = 9$, then using Lemma 2.4 we get a contradiction. So $|E/O_2(G)| = 3$. Since $O_3(G) = 1$ we get that $\alpha = 2$. Then by considering a Hall subgroup L such that $|G/E : L/E| = 3$, we get a normal subgroup M of G such that $|M| = 2^\gamma \cdot 3 \cdot 5 \cdot 7$. Again by considering T a Hall subgroup of M such that $|M : T| = 3$, we get a normal subgroup F of M such that $O_5(F) \neq 1$ or $O_7(F) \neq 1$, by Lemma 2.4, which is a contradiction.

Therefore G is not a solvable group and so by Lemma 1.4, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [1], it follows that K/H is isomorphic to A_5, A_6, A_7, A_8, $L_2(7)$, $L_2(8)$ or $L_3(4)$. Since $2 \sim 3$ in $\Gamma(A_7)$ and $\Gamma(L_2(7))$ so these cases are not occurred. If $K/H \cong L_2(8)$, then $|H| \cdot |G/K| = 2^3 5$ and so $O_5(G) \neq 1$, which is a contradiction. If $K/H \cong L_3(4)$, then $H = 1$ and $G/K = 1$. Therefore $G \cong L_3(4)$, which is a contradiction since $2 \nsim 7$ in $\Gamma(L_3(4))$ but $2 \sim 7$ in $\Gamma(A_8)$.

Let $K/H \cong A_6$. Then $|G/K| \mid 4$ and $|H| \cdot |G/K| = 56$. If $G/K \neq 1$, then the Sylow 7-subgroup of H is a normal subgroup of H which implies that $O_7(G) \neq 1$, a contradiction. Therefore $G = K$ and $|H| = 56$. By Lemma 1.2 we see that H has an irreducible character of degree 7 and so H is the Frobenius group $2^3 : 7$. It is a well known result that $|\mathrm{Aut}(H)| = 168$. Also since H is a normal subgroup of G we get that $G/C_G(H) \hookrightarrow \mathrm{Aut}(H)$. Therefore $2^3 \cdot 3 \cdot 5 \mid |C_G(H)|$ and $HC_G(H) \trianglelefteq G$. Then $C_G(H)/(H \cap C_G(H)) \cong HC_G(H)/H \trianglelefteq G/H \cong A_6$ and since H is centerless we get that $C_G(H) \cong A_6$. Therefore $G \cong H \times A_6$ and so $3 \nsim 5$ in $\Gamma(G)$, which is a contradiction.

Let $K/H \cong A_5$. Then $|H| = 2^3 \cdot 3 \cdot 7$ or $|H| = 2^4 \cdot 3 \cdot 7$. If H is a nonsolvable group, then the only nonsolvable composition factor of H is isomorphic to $L_2(7)$. But $2 \sim 3$ in $\Gamma(L_2(7))$, which is a contradiction. Therefore H is a solvable group.

We know that $|G/K| \mid 2$. Therefore we consider two subcases:

(i) Let $|G/K| = 2$. So $|H| = 2^3 \cdot 3 \cdot 7$. We know that there exists $\chi \in \mathrm{Irr}(G)$ such that $15 \mid \chi(1)$. Therefore K has an irreducible character η such that $15 \mid \eta(1)$. Now by considering a Hall subgroup M of index 3 in H we get that $M \lhd H$, since $O_7(G) = 1$. Hence M is a normal subgroup of K and so $C_K(M) \lhd K$. Therefore $T = MC_K(M) \unlhd K$. Also $M \cap C_K(M) = 1$ which implies that $T \cong M \times C_K(M)$. On the other hand, $K/C_K(M) \hookrightarrow \mathrm{Aut}(M)$, which is a group of order 168, since as we mentioned above M is the Frobenius group $2^3 : 7$. This implies that $K = MC_K(M)$ or $|K : MC_K(M)| = 3$.

If $K = MC_K(M)$, then $|K : C_K(M)| = 56$ implies that there exists an irreducible character $\theta \in \mathrm{Irr}(C_K(M))$ such that $15 \mid \theta(1)$, by Lemma 1.2, which is a contradiction since $|C_K(M)| \leq 225$. Therefore $|K : MC_K(M)| = 3$. Let Q be a Sylow 3-subgroup of H. Then $H = QM$. Since K/H is a simple group and $MC_K(M) \lhd K$ we get that $H \not\leq MC_K(M)$. Therefore $K = QMC_K(M)$ and so

$$A_5 \cong \frac{K}{H} \cong \frac{QMC_K(M)}{H} \cong C_K(M).$$

On the other hand if $L = C_K(M)$, then $L \lhd K$ and so $K/C_K(L) \hookrightarrow \mathrm{Aut}(L) \cong S_5$, which implies that $2^2 \cdot 3 \cdot 7 \mid |C_K(L)|$. Since L is centerless we get that $K \cong C_K(L) \times L$, where $C_K(L) \cong M \cdot 3$ and $L \cong A_5$. Now since $\mathrm{cd}(C_K(L)) \subseteq \{1, 3, 7\}$ and $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$, we get that if $3 \sim 5$ in $\Gamma(K)$, then $2 \sim 3$ in $\Gamma(K)$, which is a contradiction.

(ii) So suppose that $|G/K| = 1$ and so $|H| = 2^4 \cdot 3 \cdot 7$. Similarly to the above case by considering a subgroup of index 3 in H, we get a normal subgroup M of H of order $2^3 \cdot 7$ or $2^4 \cdot 7$.

For the first case, as we mentioned above we get that M is a Frobenius group of order 56 and so R, the Sylow 2-subgroup of M is a normal subgroup of M, which is an elementary abelian subgroup. Using this we conclude that G has a normal subgroup of order $2^3 \cdot 7$, which is a Frobenius group. If H has a normal subgroup M of order $2^4 \cdot 7$, then by Itô-Michler theorem we get that a Sylow 7-subgroup of M is not a normal subgroup of M. If Sylow 2-subgroup of M is a normal subgroup of M, then easily we can prove that G has a normal subgroup of order $2^3 \cdot 7$, which is a Frobenius group. Otherwise if Sylow 2-subgroup of M is not a normal subgroup of M, then there exist 7 Sylow 2-subgroups in M. Now using Theorem 1.16 in [3] we get that there exist two Sylow 2-subgroup P_1 and P_2 such that $|P_1 \cap P_2| = 8$. On the other hand $P_1 \cap P_2 \lhd P_1$ and $P_1 \cap P_2 \lhd P_2$, which implies that $N_H(P_1 \cap P_2) \geq 32$ and so $P_1 \cap P_2 \lhd H$. Therefore we get that G has a normal subgroup of order 56.

Now in each case we get that G has a normal subgroup of order 56, which is a Frobenius group and we get a contradiction similarly to (i).

Finally $K/H \cong A_8$ and so $G \cong A_8$ and the result follows. □

THEOREM 2.6. *The finite simple groups $L_3(q)$, where $2 \leq q \leq 4$ are characterizable by order and character degree graph.*

Proof. We know that $L_3(2) \cong L_2(7)$ and the result follows by Theorem 3.1. So we consider the following two cases:

Case I. Let $q = 3$ and G be a finite group such that $|G| = 2^4 \cdot 3^3 \cdot 13$ and $\Gamma(G) = \Gamma(L_3(3))$, a complete graph on the vertex set $\{2, 3, 13\}$. Then obviously $O_{13}(G) = 1$.

Similarly to the previous theorems first we prove that G is a nonsolvable group. On the contrary let G be a solvable group and M be a normal minimal subgroup of G. Hence M is a p-elementary abelian subgroup where $p = 2$ or $p = 3$ and $|M| \neq 27$ and $|M| \neq 16$.

If $|M| = 3^k$, where $1 \leq k \leq 2$, then using Lemma 1.9 it follows that the Sylow 13-subgroup of G/M is a normal subgroup and so $O_{13}(G) \neq 1$, which is a contradiction. Therefore $|M| = 2^k \leq 8$. By considering a Hall subgroup of G/M of order $3^3 13$, we get a normal subgroup N of G such that $13 \mid |N/M|$ and $|N/M| \mid 3^3 \cdot 13$. Since $O_{13}(G) = 1$, we get that $|N/M| = 3^3 \cdot 13$. Hence $|N| = 2^k \cdot 3^3 \cdot 13$. We know that $3 \sim 13$ in $\Gamma(G)$ and so N has an irreducible character θ such that its degree is divisible by 39. Since $\theta(1)^2 = 1521 < |N|$, it follows that $\theta(1) = 39$, $|M| = 8$ and $|N| = 2^3 \cdot 3^3 \cdot 13$. Let $\xi \in \mathrm{Irr}(M)$ such that $e = [\theta_M, \xi] \neq 0$. Therefore $39 = et$, where $t = |N : I_N(\xi)|$. Since $t \leq |M| = 8$, we get that $(e, t) = (13, 3)$ or $(39, 1)$. But in each case we have $13^2 \cdot 3 \leq [\theta_M, \theta_M] \leq |N : M| = 3^3 13$, which is a contradiction. Therefore G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. By the classification of finite simple groups and [1], it follows that K/H is isomorphic to $L_3(3)$. Hence $G \cong L_3(3)$.

Case II. Let $q = 4$ and G be a finite group such that $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ and $\Gamma(G) = \Gamma(L_3(4))$. We know that the character degree graph of $L_3(4)$ consists of a complete graph on the vertex set $\{3, 5, 7\}$ and there exists an edge between 2 and 5. Since the order of $L_3(4)$ is equal to the order of A_8 exactly similar to the proof of the last theorem we get that G is nonsolvable and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is isomorphic to $L_3(4)$. Therefore $G \cong L_3(4)$. □

Remark 2.7. We note that easily we can find some nonabelian simple groups which are not characterizable by order and character degree graph. As an example we construct a solvable group with the same order and character degree graph as $U_3(3)$. We know that this group is of order $2^5 \cdot 3^3 \cdot 7$ and its character degree graph is a complete graph on vertices $\{2, 3, 7\}$. Let H be the Frobenius group of order $56 = 2^3 \cdot 7$ and P be an extraspecial group of order 3^3 and exponent 3. We know that P has an automorphism σ of order 2 such that σ does not centralize $Z(P)$, the center of P and $\mathrm{cd}(H) = \{1, 7\}$. Let

K be the semidirect product of $\langle\sigma\rangle$ acting on P. Therefore $|K| = 2 \cdot 3^3$ and $\mathrm{cd}(K) = \{1, 2, 6\}$. Let $C \cong \mathbb{Z}_2$, and $G = H \times K \times C$. Then $|G| = 2^5 \cdot 3^3 \cdot 7$ and easily we get that $\mathrm{cd}(G) = \{1, 2, 6, 7, 14, 42\}$. Thus, $\Gamma(G) = \Gamma(U_3(3)$ and $|G| = |U_3(3)|$.

Acknowledgments. The authors would like to thank the referee for very valuable suggestions and his/her encouragement which seriously improved the results of the paper.
The first author would like to thank the Institute for Research in Fundamental sciences (IPM) for the financial support.

REFERENCES

[1] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of finite groups.* Oxford University Press, Oxford 1985.

[2] B. Huppert, *Character theory of finite groups.* de Gruyter, Berlin/New York, 1998.

[3] I.M. Isaacs, *Character theory of finite groups.* Academic Press, New York, 1976.

[4] I.M. Isaacs, *Character degree graphs and normal subgroups.* Trans. Amer. Math. Soc. **356 (3)** (2004), 1155–1183.

[5] Q. Jiang and C. Shao, *Recognition of* $L_2(q)$ *by its group order and largest irreducible character degree.* Monatsh. Math. **176 (3)** (2015), 413–422.

[6] B. Khosravi, B. Khosravi and B. Khosravi, *Recognition of* $\mathrm{PSL}(2,p)$ *by order and some information on its character degrees where* p *is a prime.* Monatsh. Math. **175 (2)** (2014), 277–282.

[7] B. Khosravi, B. Khosravi and B. Khosravi, *Some extensions of* $\mathrm{PSL}(2,p^2)$ *are uniquely determined by their complex group algebras.* Comm. Algebra **43 (8)** (2015), 3330–3341.

[8] B. Khosravi, B. Khosravi, B. Khosravi and Z. Momen, *A new characterization for the simple group* $\mathrm{PSL}(2,p^2)$ *by order and some character degrees.* Czech. Math. J. **65 (1)** (2015), 271–280.

[9] B. Khosravi, *Groups with the same order and large character degrees as* $\mathrm{PGL}(2,9)$. Quasigroups Related Systems **21 (2)** (2013), 239–243.

[10] B. Khosravi, B. Khosravi, B. Khosravi and Z. Momen, *Recognition of the Simple group* $\mathrm{PSL}(2,p^2)$ *by character degree graph and order.* Monatsh Math. **178 (2)** (2015), 251–257.

[11] M.L. Lewis, *An overview of graphs associated with character degrees and conjugacy class sizes in finite groups.* Rocky Mountain J. Math. **38 (1)** (2008), 175–211.

[12] M.L. Lewis and D.L. White, *Nonsolvable groups with no prime dividing three character degrees.* J. Algebra. **336** (2011), 158–183.

[13] O. Manz, R. Staszewski and W. Willems, *On the number of componenets of a graph related to character degrees.* Proc. Amer. Math. Soc. **103** (1988), 31–37.

[14] D.L. White, *Degree graphs of simple groups.* Rocky Mountain J. Math. **39 (5)** (2009), 1713–1739.

[15] H. Xu, G.Y. Chen and Y. Yan, *A new characterization of simple* K_3*-groups by their orders and large degrees of their irreducible characters.* Comm. Algebra **42 (12)** (2014), 5374–5380.

[16] H. Xu, Y. Yan and G.Y. Chen, *A new characterization of Mathieu-groups by the order and one irreducible character degree.* J. Ineq. App. **209** (2013), 1–6

Received 29 January 2014

Institute for Research
in Fundamental sciences (IPM),
School of Mathematics,
P.O.Box: 19395–5746,
Tehran, Iran

Amirkabir University of Technology
(Tehran Polytechnic)
Dept. of Pure Math.,
Faculty of Math. and Computer Sci.
424, Hafez Ave., Tehran 15914,
Iran

Institute for Advanced Studies
in Basic Sciences,
Department of Mathematics,
Zanjan 45137-66731,
Iran

Qom University of Technology,
Department of Mathematics,
Faculty of Science,
Qom, Iran
khosravibbb@yahoo.com