COORBIT SPACES OF SYMBOLS FOR SQUARE INTEGRABLE FAMILIES OF OPERATORS

M. MĂNTOIU

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The aim of this article is to provide a unified construction of coorbit spaces of symbols for the calculus associated to a square-integrable family of bounded Hilbert space operators, going far beyond the group theoretic point of view.

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1. INTRODUCTION

This article, dedicated to coorbit theory, can be seen a continuation of [6], in which the construction of the symbolic calculus associated to the data $(\Sigma, \mu, \pi, \mathcal{H})$ is undertaken. We denote by Σ a Hausdorff locally compact space endowed with a Radon measure μ. It serves as a family of indices for a set of bounded operators $\{\pi(s)\,|\,s \in \Sigma\,\}$ in the Hilbert space $\mathcal{H}$. We do not assume that $\pi(s)$ is unitary and we do not ask anything about the product $\pi(s)\pi(t)$ for $s, t \in \Sigma$. The map $\pi(\cdot)$ is assumed bounded and weakly continuous. The main requirement is relation (2.2), a condition of square integrability extending a wide-spread related concept from group representation theory [7]. As explained in [6], the general setting is meant to cover, unify and extend some new developments in pseudodifferential theory [1–5, 18, 20, 23–27, 29].

Part of the formalism developed in [6] is briefly summarized in Section 2. It is shown how to raise the family π, essentially by integration, to a correspondence $f \mapsto \Pi(f)$ sending a closed subspace of $\mathscr{L}^2(\Sigma; \mu)$ to the ideal of all Hilbert-Schmidt operators in $\mathcal{H}$. Actually the fact that Π is “an integrated form” of π (in the spirit of group representation theory) is only seen a posteriori; the initial construction is just based on the the “representation coefficient” map Φ.

To make further progress, we introduce an extra Fréchet space $\mathcal{G}$, continuously and densely embedded in $\mathcal{H}$. Then duality and topological tensor product techniques generate new spaces, organized in Gelfand triples, as well as isomorphisms extending or restricting the mappings Π and Φ. These Gelfand triples, also useful for coorbit theory, have extra algebraic properties, compatible with the topologies. We also include an extension of the involution and of the symbol composition law to a large subspace, called the Moyal algebra, cf. [19, 28].

In Sections 3,4 and 5, that are the original parts of the present article, we show that coorbit spaces of vectors and symbols can be developed in such a general setting.

Very roughly, in a *coorbit theory* useful Banach spaces are defined by imposing conditions on suitable transformations applied to its elements. We do not feel competent to give a comprehensive overview, a historical presentation or exhaustive references for this topic, also related to (or even including) wavelet theory, so the discussion will be restricted to a framework as close as possible to our setting.

The basic pattern is the fundamental article [10] of Feichtinger and Gröchenig, which constructed coorbit spaces starting with an integrable, irreducible strongly continuous unitary representation $\pi : G \to \mathbb{B}(\mathcal{H})$ of a locally compact group G in a Hilbert space $\mathcal{H}$. This article unified a lot of previous work on function spaces and had a great influence on the subsequent development of the field. Among many others, it contains as a particular case the theory of modulation spaces previously developed by Feichtinger [8, 9], in a classical function space context and also in connection to the Schrödinger representation of the Heisenberg group. Among the many developments we cite [12]. Recent contributions as [11, 30] (see also references therein) extend the theory of coorbit spaces much beyond group theory; they rely on frames $\{w(s) \mid s \in \Sigma\} \subset \mathcal{H}$ indexed by a locally compact space Σ endowed with a Radon measure with respect to which a square integrability condition is required. Our "coorbit spaces of vectors" are essentially covered as a rather particular case, setting $w(s) := \pi(s)^* w$ for some fixed element $w \in \mathcal{H}$. So our main contribution lies in another realm, involving "coorbit spaces of symbols".

Recent work ([12–18, 21, 22, 32–34, 36] and many others), triggered by contributions of Sjöstrand [31], showed the great importance of modulation spaces in the theory of pseudodifferential operators on $\mathbb{R}^n$ or on Abelian locally compact groups. On one hand, "modulation spaces of vectors" constitute convenient spaces on which pseudodifferential operators naturally apply, providing a rich setting going beyond the usual formalism involving Hilbert and Schwartz spaces. But, more importantly, a similar strategy leads to introducing "modulation spaces of symbols", spaces of functions or distributions defined in phase-space, transformed by the pseudodifferential prescription into operators. These spaces of symbols are often better-suited and simpler to use than the traditional spaces used in pseudodifferential theory.

We are going to introduce *coorbit spaces of vectors* and *coorbit spaces of symbols* associated to the above-mentioned square-integrable family π. For the first, in accord with the existing theory [10, 11], it is clear that we must use a defining procedure in terms of ϕ_w, where $\phi_w(u) := \phi_{u,w} = \Phi(u \otimes w)$ for a fixed "window" w. So, for a Banach space $\mathcal{M}$ of functions on Σ, one gets a Banach space of vectors $\mathsf{co}_w(\mathcal{M}) := \phi_w^{-1}(\mathcal{M})$. This goes along the lines of [10] and the setting is more particular than that of [11, 30], so we will be brief about the spaces $\mathsf{co}_w(\mathcal{M})$.

For symbols, one must first construct the analogous mappings Υ_h indexed by "symbol windows" h; it transforms functions on Σ into functions on $\Sigma \times \Sigma$ and constitutes the main tool. But actually, analogous to Φ, there is a naturally defined isomorphism Υ generating the family $\{\Upsilon_h\}_h$ by *localization*, using the tensor product structure: one sets $\Upsilon_h(f) := \Upsilon(f \otimes h)$ for all $f \in \mathscr{G}'(\Sigma \times \Sigma)$. (See [27] for a related construction in a more restricted framework.)

In fact, as shown in Corollary 3.5, Υ can be seen as a Hilbert algebra Gelfand triple isomorphism. We arrange things so that the algebraic structure on the second Hilbert algebra Gelfand triple be independent on our data π; it just involves composition of integral kernels. For suitable windows h, this facilitates studying the algebraic properties of various coorbit spaces obtained by inducing through Υ_h. We think that preservation of the algebraic properties is one of the main merits of the present coorbit construction.

Once the basic properties of the mapping Υ are obtained, a reconstruction formula and the techniques of a coorbit theory are available. The procedure generates Banach spaces $\mathsf{CO}_h(\mathscr{M})$ of "functions" on Σ from Banach spaces $\mathscr{M}$ of "kernels" on $\Sigma \times \Sigma$. Under admissibility conditions one gets information about denseness, duality, interpolation, independence of the window.

The two types of coorbit spaces are related because the two mappings Φ and Υ are intimately connected, as indicated in Propositions 3.4 and 5.3. The main consequence is Corollary 5.4: For suitable Banach spaces $\mathscr{M}, \mathcal{M}_1, \mathcal{M}_2$ (and for suitably tuned windows), the symbols $f \in \mathsf{CO}(\mathscr{M})$ give raise to bounded operators $\Pi(f) : \mathsf{co}_w(\mathcal{M}_1) \to \mathsf{co}_w(\mathcal{M}_2)$ if the kernels $F \in \mathscr{M}$ give raise to bounded integral operators $\mathfrak{Int}(F) : \mathcal{M}_1 \to \mathcal{M}_2$. This second assertion is universal, it has nothing to do with the data π and the symbolic calculus Π and can be checked independently. Thus one gets efficient boundedness criteria for "the pseudodifferential operators" $\Pi(f)$ in terms of coorbit spaces of vectors and symbols.

2. A GENERALIZED SYMBOLIC CALCULUS

Let us fix a Hausdorff locally compact space Σ and a Radon measure μ on Σ with full support. We set $BC(\Sigma)$ for the C^*-algebra of all bounded continuous complex-valued functions on Σ and $\mathscr{L}^q(\Sigma)$ for the usual Lebesgue space of order $q \in [1, \infty]$ on (Σ, μ). The complex Hilbert space $\mathcal{H}$ will always be separable and infinite-dimensional. Its conjugate will be denoted by $\overline{\mathcal{H}}$ and $\mathbb{B}(\mathcal{H})$ stands for the C^*-algebra of bounded linear operators on $\mathcal{H}$. *The starting point of our constructions is a bounded and weakly continuous map* $\pi : \Sigma \to \mathbb{B}(\mathcal{H})$.

HYPOTHESIS 2.1. *The sesquilinear mapping*

$$\phi^\pi \equiv \phi : \mathcal{H} \times \overline{\mathcal{H}} \to BC(\Sigma), \qquad \phi_{u,v}(s) := \langle \pi(s)u, v\rangle . \tag{2.1}$$

is $\mathscr{L}^2(\Sigma; \mu)$*-valued and extends to an isometry* $\Phi^\pi \equiv \Phi : \mathcal{H} \widehat{\otimes} \overline{\mathcal{H}} \to \mathscr{L}^2(\Sigma; \mu) \equiv \mathscr{L}^2(\Sigma)$.

This overstanding hypothesis is equivalent to requiring

$$\int_\Sigma d\mu(s) |\, \langle \pi(s)u, v\rangle \,|^2 = \|\, u\,\|^2 \,\|\, v\,\|^2, \qquad \forall\, u, v \in \mathcal{H}. \tag{2.2}$$

Let us denote by Λ the canonical isomorphism between the Hilbert space tensor product $\mathcal{H} \widehat{\otimes} \overline{\mathcal{H}}$ (the Hilbert completion of the algebraic tensor product $\mathcal{H} \otimes \overline{\mathcal{H}}$) and the ideal $\mathbb{B}_2(\mathcal{H})$ of all the Hilbert-Schmidt operators in $\mathcal{H}$, sending $u \otimes v$ into the rank one operator $\Lambda(u \otimes v) \equiv \lambda_{u,v} := \langle \cdot, v\rangle u$. Then, by Hypothesis 2.1, there is a closed subspace $\mathscr{B}_2(\Sigma) := \Phi\left(\mathcal{H} \widehat{\otimes} \overline{\mathcal{H}}\right)$ of $\mathscr{L}^2(\Sigma)$ which is unitarily equivalent to $\mathcal{H} \widehat{\otimes} \overline{\mathcal{H}}$ and (hence) with $\mathbb{B}_2(\mathcal{H})$. This space $\mathscr{B}_2(\Sigma)$ is the closure in $\mathscr{L}^2(\Sigma)$ of the subspace $\Phi(\mathcal{H} \otimes \overline{\mathcal{H}})$. Clearly, there is a Hilbert space isomorphism $\Pi := \Lambda \circ \Phi^{-1} : \mathscr{B}_2(\Sigma) \to \mathbb{B}_2(\mathcal{H})$ uniquely defined by

$$\Pi\left(\phi_{u,v}\right) = \lambda_{u,v} = \langle \cdot, v\rangle\, u\,, \qquad \forall\, u, v \in \mathcal{H} \tag{2.3}$$

and satisfying for every $f, g \in \mathscr{B}_2(\Sigma)$

$$\mathrm{Tr}\left[\Pi(f)\Pi(g)^*\right] = \langle f, g\rangle_{(\Sigma)} := \int_\Sigma d\mu(s) f(s)\, \overline{g(s)}\,.$$

For any $f \in \mathscr{B}_2(\Sigma)$ one has in weak sense

$$\Pi(f) = \int_\Sigma d\mu(s) f(s)\, \pi(s)^*\,. \tag{2.4}$$

By transport of structure one defines a product and an involution:

$$\star : \mathscr{B}_2(\Sigma) \times \mathscr{B}_2(\Sigma) \to \mathscr{B}_2(\Sigma)\,, \qquad f \star g := \Pi^{-1}[\Pi(f)\Pi(g)]\,,$$

$$^\star : \mathscr{B}_2(\Sigma) \to \mathscr{B}_2(\Sigma)\,, \qquad f^\star := \Pi^{-1}[\Pi(f)^*]\,.$$

One has for all $f, g, h \in \mathscr{B}_2(\Sigma)$

$$\langle f^\star, g^\star\rangle_{(\Sigma)} = \langle g, f\rangle_{(\Sigma)}\,, \tag{2.5}$$

$$\langle f \star g, h\rangle_{(\Sigma)} = \langle f, h \star g^\star\rangle_{(\Sigma)} = \langle g, f^\star \star h\rangle_{(\Sigma)}\,. \tag{2.6}$$

The *-subalgebra $\mathscr{B}_1(\Sigma) := \mathscr{B}_2(\Sigma) \star \mathscr{B}_2(\Sigma)$ is dense in $\mathscr{B}_2(\Sigma)$ and it is isomorphic through Π with the *-algebra $\mathbb{B}_1(\mathcal{H})$ of trace-class operators. We can "compute" the symbol of a trace-class operator T by

$$\left[\Pi^{-1}(T)\right](s) = \mathrm{Tr}\left[T\pi(s)\right], \qquad \mu - a.e.\ s \in \Sigma\,.$$

In [6] rather explicit formulae are given for the algebraic structure $(\star,^\star)$, that will not be used here. But we notice for further use the relations

$$\left\langle \phi_{u_1,v_1}, \phi_{u_2,v_2}\right\rangle_{(\Sigma)} = \langle u_1, u_2\rangle \langle v_2, v_1\rangle \tag{2.7}$$

and

$$\phi_{u_1,v_1} \star \phi_{u_2,v_2} = \langle u_2, v_1\rangle\, \phi_{u_1,v_2}, \qquad \phi^\star_{u,v} = \phi_{v,u}\,, \tag{2.8}$$

valid for every $u, u_1, u_2, v, v_1, v_2 \in \mathcal{H}$, as well as

$$f \star \phi_{u,v} \star g = \phi_{\Pi(f)u, \Pi(g)^* v}\,, \qquad \forall\, f, g \in \mathscr{B}_2(\Sigma),\ u, v \in \mathcal{H}\,. \tag{2.9}$$

In most of the applications some supplementary structure is available. Let $\mathcal{G}$ be a Fréchet space continuously and densely embedded in $\mathcal{H}$. Then the projective tensor product $\mathcal{G}\widehat{\otimes}_p\overline{\mathcal{G}}$ is a Fréchet space continuously and densely embedded in the Hilbert space $\mathcal{H}\widehat{\otimes}\,\overline{\mathcal{H}}$.

Let us set $\mathscr{G}(\Sigma) := \Phi\left[\mathcal{G}\widehat{\otimes}_p\overline{\mathcal{G}}\right] \subset \mathscr{B}_2(\Sigma) \subset \mathscr{L}^2(\Sigma)$. It can be easily seen that $\mathscr{G}(\Sigma) \subset \mathscr{B}_1(\Sigma)$, since $\mathbb{B}_2(\mathcal{H})$ can be identified with the projective tensor product $\mathcal{H}\widehat{\otimes}_p\overline{\mathcal{H}}$.

We endow $\mathscr{G}(\Sigma)$ with the linear topology transported from $\mathcal{G}\widehat{\otimes}_p\overline{\mathcal{G}}$ through Φ, denote by $\mathscr{G}'(\Sigma)$ the topological dual and keep the same notation $\langle\cdot,\cdot\rangle_{(\Sigma)}$ for the duality between $\mathscr{G}(\Sigma)$ and $\mathscr{G}'(\Sigma)$. On the topological duals we are usually going to consider the *-weak topology. When we want to empasize this we will use notations like $\mathcal{G}'_\sigma$ or $\mathscr{G}'(\Sigma)_\sigma$. The use of the strong topology (the topology of uniform convergence on bounded sets) will be indicated by an index β.

As explained in [6], besides $(\mathcal{G}, \mathcal{H}, \mathcal{G}'_\sigma)$, one gets new Gelfand triples $\left(\mathcal{G}\widehat{\otimes}_p\,\overline{\mathcal{G}}, \mathcal{H}\widehat{\otimes}\,\overline{\mathcal{H}}, (\mathcal{G}\widehat{\otimes}_p\,\overline{\mathcal{G}})'_\sigma\right)$ and $(\mathscr{G}(\Sigma), \mathscr{B}_2(\Sigma), \mathscr{G}'(\Sigma)_\sigma)$. They are isomorphic by (various restrictions or extensions of) the mapping Φ.

Note that for any $u, v \in \mathcal{G}'$ one has a well-defined element $\phi_{u,v} := \Phi(u \otimes v) \in \mathscr{G}'(\Sigma)$. It is an easy task to state and prove suitable extensions of the relations (2.7) and (2.8).

Remark 2.2. One has a canonical isomorphism $\left(\mathcal{G}\widehat{\otimes}_p\,\overline{\mathcal{G}}\right)' \sim \mathbb{B}(\mathcal{G}, \mathcal{G}'_\sigma)$ that involves the space of all the linear operators $A : \mathcal{G} \to \mathcal{G}'$ which are continuous, when $\mathcal{G}'$ is endowed with the weak*-topology. Using this, it is easy to deduce that $\Pi = \Lambda \circ \Phi^{-1} : \mathscr{B}_2(\Sigma) \to \mathbb{B}_2(\mathcal{H})$ extends to a linear isomorphism $\Pi : \mathscr{G}'(\Sigma) \to \mathbb{B}(\mathcal{G}, \mathcal{G}'_\sigma)$. Thus *the elements of* $\mathscr{G}'(\Sigma)$ *can be seen as symbols of* *linear operators $T : \mathcal{G} \to \mathcal{G}'$ that are continuous with respect to the weak*-topology on the dual.* The relation

$$\langle \Pi(g)u, v\rangle = \int_\Sigma \mathrm{d}\mu(t) g(t) \langle \pi(t)^* u, v\rangle = \langle g, \phi_{v,u}\rangle_{(\Sigma)}\,, \tag{2.10}$$

valid a priori for $g \in \mathscr{B}_2(\Sigma)$ and $u, v \in \mathcal{H}$, stands also true for $g \in \mathscr{G}'(\Sigma)$ and $u, v \in \mathcal{G}$ with the obvious reinterpretation of the duality $\langle\cdot,\cdot\rangle_{(\Sigma)}$.

We describe now on the extension of the algebraic structure, relying on [28]. The composition law $\star$ extends to bilinear separately continuous mappings

$$\star : \mathscr{G}'(\Sigma) \times \mathscr{G}(\Sigma) \to \mathscr{G}'(\Sigma) \quad \text{and} \quad \star : \mathscr{G}(\Sigma) \times \mathscr{G}'(\Sigma) \to \mathscr{G}'(\Sigma)$$

and the involution * extends to an anti-linear isomorphism $^\star : \mathscr{G}'(\Sigma) \to \mathscr{G}'(\Sigma)$, such that for every $f \in \mathscr{G}'(\Sigma)$ and $g \in \mathscr{G}(\Sigma)$ one has

$$(f \star g)^\star = g^\star \star f^\star, \quad (g \star f)^\star = f^\star \star g^\star\,.$$

These extensions simply rely on relations (2.5) and (2.6) and on the duality of $\mathscr{G}(\Sigma)$ and $\mathscr{G}'(\Sigma)$, setting for any $f \in \mathscr{G}'(\Sigma)$ and $g \in \mathscr{G}(\Sigma)$

$$\langle f \star g, h\rangle_{(\Sigma)} = \langle f, h \star g^\star\rangle_{(\Sigma)}\,, \quad \langle g \star f, h\rangle_{(\Sigma)} = \langle f, g^\star \star h\rangle_{(\Sigma)}\,,$$

$$\langle f^\star, h\rangle_{(\Sigma)} = \langle h^\star, f\rangle_{(\Sigma)}\,, \quad \forall\, h \in \mathscr{G}(\Sigma)\,.$$

We also introduce the subspace

$$\mathscr{G}'_\star(\Sigma) := \big\{ f \in \mathscr{G}'(\Sigma) \mid f \star \mathscr{G}(\Sigma) \subset \mathscr{G}(\Sigma),\ \ \mathscr{G}(\Sigma) \star f \subset \mathscr{G}(\Sigma) \big\}$$

and call it *the Moyal algebra associated to* $\star$. Obviously $\mathscr{G}'_\star(\Sigma)$ is invariant under the involution $f \mapsto f^\star$. We extend the composition $\star : \mathscr{G}'_\star(\Sigma) \times \mathscr{G}'_\star(\Sigma) \to \mathscr{G}'_\star(\Sigma)$ by setting

$$\langle f \star g, h\rangle_{(\Sigma)} := \langle f, g \star h\rangle_{(\Sigma)}\,, \qquad \forall\, f, g \in \mathscr{G}'_\star(\Sigma),\ h \in \mathscr{G}(\Sigma)\,.$$

Without making all the routine verification, we just state that $(\mathscr{G}'_\star(\Sigma), \star, ^\star)$ *is a *-algebra in which $\mathscr{G}(\Sigma)$ is a self-adjoint two-sided ideal.*

3. THE CANONICAL MAPPING ON SYMBOLS

Let us put $e_s := \Pi^{-1}\left[\pi(s)^*\right] \in \mathscr{G}'(\Sigma)$ for $s \in \Sigma$; thus $\phi_{u,v}(s) = \langle u, \Pi(e_s)v\rangle$ for all u, v and s and one has $\pi(s)^* = \Pi\left(e_s\right)$ and $\pi(s) = \Pi\left(e_s^\star\right)$. For every $f, g \in \mathscr{B}_1(\Sigma)$ one gets

$$\int_\Sigma \mathrm{d}\mu(s) \langle f, e_s\rangle_{(\Sigma)} \langle e_s, g\rangle_{(\Sigma)} = \langle f, g\rangle_{(\Sigma)}\,. \tag{3.1}$$

and

$$\langle f, e_s\rangle_{(\Sigma)} = f(s)\,, \quad \langle f, e_s^\star\rangle_{(\Sigma)} = \overline{f^\star(s)}\,, \qquad \mu - a.e.\ s\in\Sigma \tag{3.2}$$

Note that $f\star e_t\star g\in\mathscr{B}_1(\mathcal{H})$ for every $t\in\Sigma$ and $f,g\in\mathscr{B}_2(\Sigma)$.

Definition 3.1. The canonical map (depending implicitly on π)

$$\Upsilon : \mathscr{B}_2(\Sigma)\widehat{\otimes}\,\mathscr{B}_2(\Sigma)\to\mathscr{L}^2(\Sigma\times\Sigma)\cong\mathscr{L}^2(\Sigma)\widehat{\otimes}\,\mathscr{L}^2(\Sigma)$$

is defined by

$$[\Upsilon(f\otimes h)](s,t) := \langle f\star e_t\star h, e_s\rangle_{(\Sigma)} = (f\star e_t\star h)(s)\,. \tag{3.3}$$

Remark 3.2. It is easy to check that (3.3) is equivalent to

$$\langle\Upsilon(f\otimes h), g\otimes k\rangle_{(\Sigma\times\Sigma)} = \big\langle f\star\overline{k}\star h, g\big\rangle_{(\Sigma)} = \big\langle f\star\overline{k}, g\star h^\star\big\rangle_{(\Sigma)}\,. \tag{3.4}$$

This offers a good starting point for extensions to larger spaces.

A very useful fact is that $\mathscr{L}^2(\Sigma\times\Sigma)$ is a *-algebra, in a way that does not depend on our starting point, the map $\pi:\Sigma\to\mathbb{B}(\mathcal{H})$. Besides the canonical scalar product, one also has the composition of kernels

$$(F\bullet G)(s,t) := \int_\Sigma d\mu(r)F(s,r)G(r,t) \tag{3.5}$$

and the involution $F^\bullet(s,t) := \overline{F(t,s)}$. On the other hand, on the Hilbert tensor product $\mathscr{B}_2(\Sigma)\widehat{\otimes}\,\mathscr{B}_2(\Sigma)$ we can consider the law $\bigstar := \star\otimes\overline{\star}$ uniquely defined by

$$(f_1\otimes f_2)\bigstar(g_1\otimes g_2) := (f_1\star g_1)\otimes(g_2\star f_2) \tag{3.6}$$

and the tensor product involution $(f_1\otimes f_2)^\bigstar := f_1^\star\otimes f_2^\star$.

Theorem 3.3. *The canonical mapping* $\Upsilon : (\mathscr{B}_2(\Sigma)\widehat{\otimes}\,\mathscr{B}_2(\Sigma)\,,\bigstar\,,^\bigstar\,)\to(\mathscr{L}^2(\Sigma\times\Sigma)\,,\bullet\,,^\bullet\,)$ *is a well-defined isometric morphism of* **-algebras with range* $\mathscr{B}_2(\Sigma)\widehat{\otimes}\,\mathscr{B}_2(\Sigma)$.

Proof. Notice the relation

$$e_s^\star\star\phi_{u,v}\star e_t = \phi_{\pi(s)u,\pi(t)v}\,, \qquad \forall\, s,t\in\Sigma,\ u,v\in\mathcal{H}\,, \tag{3.7}$$

which is a direct consequence of the definitions. Using (3.3), (2.6), (3.7) and (2.7), for any $u,u',v,v'\in\mathcal{H}$ and any $s,t\in\Sigma$ one has

$$\begin{aligned}\big[\Upsilon\big(\phi_{u,v}\otimes\phi_{u',v'}\big)\big](s,t) &= \big\langle e_s^\star\star\phi_{u,v}\star e_t, \phi_{u',v'}^\star\big\rangle_{(\Sigma)} = \big\langle\phi_{\pi(s)u,\pi(t)v},\phi_{v',u'}\big\rangle_{(\Sigma)}\\ &= \big\langle\pi(s)u,v'\big\rangle\big\langle\pi(t)v,u'\big\rangle = \phi_{u,v'}(s)\,\phi_{v,u'}(t)\,.\end{aligned}$$

Thus $\Upsilon(\phi_{u,v}\otimes\phi_{u',v'})=\phi_{u,v'}\otimes\phi_{v,u'}$ which, together with the orthogonality relations verified by the functions $\phi_{u,v}$, implies easily the isometry property and the identification of the range.

Let us check the algebraic properties. For $f,g,h,k\in\mathscr{B}_2(\Sigma)$ one has

$$\begin{aligned}\Upsilon[(f\otimes h)\bigstar(g\otimes k)](s,t)&=\Upsilon\left[(f\star g)\otimes(h\,\overline{\star}\,k)\right](s,t)\\&=\langle(f\star g)\star e_t,e_s\star(k\star h)^\star\rangle_{(\Sigma)}\,.\end{aligned}$$

On the other hand, using (2.6) and relation (3.1)

$$\begin{aligned}[\Upsilon(f\otimes h)\bullet\Upsilon(g\otimes k)](s,t)&=\int_\Sigma d\mu(r)\,[\Upsilon(f\otimes h)](s,r)\,[\Upsilon(g\otimes k)](r,t)\\&=\int_\Sigma d\mu(r)\,\langle f\star e_r,e_s\star h^\star\rangle_{(\Sigma)}\,\langle g\star e_t,e_r\star k^\star\rangle_{(\Sigma)}\\&=\int_\Sigma d\mu(r)\,\langle e_r,f^\star\star e_s\star h^\star\rangle_{(\Sigma)}\,\langle g\star e_t\star k,e_r\rangle_{(\Sigma)}\\&=\langle g\star e_t\star k,f^\star\star e_s\star h^\star\rangle_{(\Sigma)}\\&=\langle f\star g\star e_t,e_s\star h^\star\star k^\star\rangle_{(\Sigma)}\,.\end{aligned}$$

The fact that Υ intertwines the two involutions also follows from (2.5) and (2.6):

$$\begin{aligned}\Upsilon\Big[(f\otimes h)^\bigstar\Big](s,t)&=\langle f^\star\star e_t,e_s\star h\rangle_{(\Sigma)}=\langle(e_s\star h)^\star,(f^\star\star e_t)^\star\rangle_{(\Sigma)}\\&=\langle h^\star\star e_s^\star,e_t^\star\star f\rangle_{(\Sigma)}=\overline{\langle e_t^\star\star f,h^\star\star e_s^\star\rangle_{(\Sigma)}}\\&=\overline{\langle f\star e_s,e_t\star h^\star\rangle_{(\Sigma)}}=\overline{[\Upsilon(f\otimes h)]\,(t,s)}\,.\quad\square\end{aligned}$$

In order to establish the connection between the mappings Φ, Π and Υ, let us consider the representation of $\left(\mathscr{L}^2(\Sigma\times\Sigma)\,,\bullet\,,^\bullet\right)$ on the Hilbert space $\mathscr{L}^2(\Sigma)$ by integral operators

$$[\mathfrak{Int}(F)h](s):=\int_\Sigma d\mu(t)F(s,t)h(t)\,;\tag{3.8}$$

the range of the representation is composed of all the Hilbert-Schmidt operators on $\mathscr{L}^2(\Sigma)$. By composing with Υ, one gets a representation

$$\mathfrak{Int}\circ\Upsilon:\left(\mathscr{B}_2(\Sigma)\widehat{\otimes}\mathscr{B}_2(\Sigma)\,,\bigstar\,,^\bigstar\right)\to\mathbb{B}\big[\mathscr{L}^2(\Sigma)\big]\,.\tag{3.9}$$

Since $\mathfrak{Int}(f\otimes g)=\langle\cdot,\overline{g}\rangle_{(\Sigma)}f$, we see easily that $\mathfrak{Int}(F)$ leaves the closed subspace $\mathscr{B}_2(\Sigma)$ invariant for any element F of $\mathscr{B}_2(\Sigma)\widehat{\otimes}\,\mathscr{B}_2(\Sigma)$; then, with restrictions, one clearly has $\mathfrak{Int}\left[\mathscr{B}_2(\Sigma)\widehat{\otimes}\,\mathscr{B}_2(\Sigma)\right]=\mathbb{B}_2[\mathscr{B}_2(\Sigma)]$. With this interpretation we are going to show that $\mathfrak{Int}\circ\Upsilon$ is unitarily equivalent with the tensor product of the representation Π with its opposite $\overline{\Pi}$ (defined by $\overline{\Pi}(f) := \Pi(f)^*$), via the unitary operator $\Phi : \mathcal{H}\widehat{\otimes}\,\overline{\mathcal{H}} \to \mathscr{B}_2(\Sigma)$. For $f, g, h \in \mathscr{B}_2(\Sigma)$ we set

$$[\{\mathsf{L}, \mathsf{R}\}(f \otimes g)]\, h := f \star h \star g = (\mathsf{L}_f \circ \mathsf{R}_g)\, h = (\mathsf{R}_g \circ \mathsf{L}_f)\, h\,.$$

PROPOSITION 3.4. *Setting* $\mathbf{U}_\Phi(\mathbf{S}) := \Phi \circ \mathbf{S} \circ \Phi^{-1}$, $\mathbf{S} \in \mathbb{B}(\mathcal{H}\widehat{\otimes}\,\overline{\mathcal{H}})$ *one has*

$$\mathfrak{Int} \circ \Upsilon = \mathbf{U}_\Phi \circ (\Pi \otimes \overline{\Pi}) = \{\mathsf{L}, \mathsf{R}\}. \tag{3.10}$$

Proof. Let $f, g, k \in \mathscr{B}_2(\Sigma)$ and $s \in \Sigma$. Using the fact that the products of three elements is a trace-class symbol one gets

$$\begin{aligned}
\{[(\mathfrak{Int} \circ \Upsilon)(f \otimes g)]\, k\}\, (s) &= \int_\Sigma \mathrm{d}\mu(t)\, [\Upsilon(f \otimes g)]\, (s,t)\, k(t) \\
&= \int_\Sigma \mathrm{d}\mu(t)\, \langle e_t, f^\star \star e_s \star g^\star \rangle_{(\Sigma)}\, k(t) \\
&= \left\langle \int_\Sigma \mathrm{d}\mu(t) k(t) e_t, f^\star \star e_s \star g^\star \right\rangle_{(\Sigma)} \\
&= \langle f \star k \star g, e_s \rangle_{(\Sigma)} = (f \star k \star g)(s)\,.
\end{aligned}$$

On the other hand, computing on $k = \phi_{u,v}$ (such that $\Phi^{-1}(h) = u \otimes v$) and using (2.9) we have

$$\begin{aligned}
\left\{ \left[\mathbf{U}_\Phi \circ (\Pi \otimes \overline{\Pi}) \right] (f \otimes g) \right\} \phi_{u,v} &= \Phi \left\{ [\Pi(f) \otimes \Pi(g)^*]\, (u \otimes v) \right\} \\
&= \phi_{\Pi(f)u, \Pi(g)^* v} = f \star \phi_{u,v} \star g \\
&= [\{\mathsf{L}, \mathsf{R}\}(f \otimes g)]\, \phi_{u,v}\,,
\end{aligned}$$

and the proof is finished since the elements $f \otimes g$ form a total set. $\square$

As in the second part of Section 2, let $\mathcal{G}$ be a Fréchet space continuously and densely contained in $\mathcal{H}$ and set $\mathscr{G}(\Sigma) := \Phi[\mathcal{G}\widehat{\otimes}_p\overline{\mathcal{G}}]$. We are going to use the abbereviations

$$\begin{gathered}
\mathscr{G}(\Sigma \times \Sigma) := \mathscr{G}(\Sigma)\widehat{\otimes}_p \mathscr{G}(\Sigma)\,, \\
\mathscr{B}_2(\Sigma \times \Sigma) := \mathscr{B}_2(\Sigma)\widehat{\otimes}\, \mathscr{B}_2(\Sigma)
\end{gathered}$$

and

$$\mathscr{G}'(\Sigma \times \Sigma) := \mathscr{G}(\Sigma \times \Sigma)'\,.$$

Recall that the family $\{\phi_{u_1,v_1} \otimes \phi_{u_2,v_2} \mid u_1, v_1, u_2, v_2 \in \mathcal{G}\}$ is a total subset of $\mathscr{G}(\Sigma \times \Sigma)$. Since we have $\Upsilon\left(\phi_{u_1,v_1} \otimes \phi_{u_2,v_2}\right) = \phi_{u_1,v_2} \otimes \phi_{v_1,u_2}$ (see the proof of Theorem 3.3), it follows that $\Upsilon\left[\mathscr{G}(\Sigma \times \Sigma)\right] = \mathscr{G}(\Sigma \times \Sigma)$, so we get by restriction an isomorphism of Fréchet *-algebras

$$\left(\mathscr{G}(\Sigma \times \Sigma)\,, \bigstar\,,^{\bigstar}\right) \xrightarrow{\Upsilon} \left(\mathscr{G}(\Sigma \times \Sigma)\,, \bullet\,,^{\bullet}\,\right). \tag{3.11}$$

Continuity follows from the Closed Graph Theorem or directly from a careful examination of the projective topologies. Therefore, as an extension of Theorem 3.3, one gets

Corollary 3.5. *Assume that the Fréchet space $\mathcal{G}$ is continuously and densely embedded in $\mathcal{H}$. Then there is an isomorphism of Gelfand triples*

$$\big(\mathscr{G}(\Sigma\times\Sigma),\mathscr{B}_2(\Sigma\times\Sigma),\mathscr{G}'(\Sigma\times\Sigma)\big)\xrightarrow{\Upsilon}\big(\mathscr{G}(\Sigma\times\Sigma),\mathscr{B}_2(\Sigma\times\Sigma),\mathscr{G}'(\Sigma\times\Sigma)\big)$$

that is unitary at the level of the $\mathscr{B}_2$-spaces and respects the *-algebraic structures.

The two Gelfand triples above are identical as locally convex spaces but very different as *-algebras; see (3.11) for instance.

The procedure of extension of the algebraic structure described at the end of Section 2 can be applied to the *-algebra $(\mathscr{G}(\Sigma\times\Sigma),\bigstar,^\bigstar)$, getting a Moyal algebra $\mathscr{G}'_\bigstar(\Sigma\times\Sigma)$, and to $(\mathscr{G}(\Sigma\times\Sigma),\bullet,^\bullet)$, getting a Moyal algebra $\mathscr{G}'_\bullet(\Sigma\times\Sigma)$. One gets easily

Corollary 3.6. *The map Υ restricts (or extends, depending on the starting point) to an isomorphism of *-algebras $\Upsilon:\mathscr{G}'_\bigstar(\Sigma\times\Sigma)\to\mathscr{G}'_\bullet(\Sigma\times\Sigma)$.*

Remark 3.7. Later on we are going to need the fact that the unitary integral operator $\mathfrak{Int}:\mathscr{B}_2(\Sigma)\widehat{\otimes}\,\mathscr{B}_2(\Sigma)\to\mathbb{B}_2[\mathscr{B}_2(\Sigma)]$ extends to a linear isomorphism $\mathfrak{Int}:\mathscr{G}'(\Sigma\times\Sigma)\to\mathbb{B}[\mathscr{G}(\Sigma),\mathscr{G}'(\Sigma)_\sigma]$. This is actually the isomorphism of Remark 2.2 with $\mathcal{G}$ replaced by the Fréchet space $\mathscr{G}(\Sigma)$ (and $\mathcal{H}$ replaced by the Hilbert space $\mathscr{B}_2(\Sigma)$). We set $\mathfrak{Int}(f\otimes g)h=\langle h,\overline{g}\rangle_{(\Sigma)}f$ for every $f,g\in\mathscr{G}'(\Sigma)$ and $h\in\mathscr{G}(\Sigma)$ and this mapping defined on the algebraic tensor product $\mathscr{G}'(\Sigma)\otimes\mathscr{G}'(\Sigma)$, compatible with the initial $\mathfrak{Int}$, extends to the final isomorphism.

We now perform *localization* of the canonical mapping on symbols; it amounts essentially to regarding Υ as a "function of two variables" and then fixing the first one in a convenient way. For $h\in\mathscr{B}_2(\Sigma)\setminus\{0\}$ (often called *window* or *analyzing vector*) we set

$$\Upsilon_h:\mathscr{B}_2(\Sigma)\to\mathscr{B}_2(\Sigma\times\Sigma)\,,\qquad\Upsilon_h(f):=\Upsilon(f\otimes h)\,.$$

Defining

$$J_h:\mathscr{B}_2(\Sigma)\to\mathscr{B}_2(\Sigma)\widehat{\otimes}\,\mathscr{B}_2(\Sigma)\ \text{ by }\ J_h(f):=f\otimes h\,,$$

then in terms of the adjoint $J_h^\dagger:\mathscr{B}_2(\Sigma)\widehat{\otimes}\,\mathscr{B}_2(\Sigma)\to\mathscr{B}_2(\Sigma)$ given by

$$\left[J_h^\dagger(F)\right](s)=\langle F(s,\cdot),h\rangle_{(\Sigma)}\,,\qquad\mu-\text{a.e. }s\in\Sigma\,,$$

one has the relations

$$J_h^\dagger\circ J_k=\langle k,h\rangle_{(\Sigma)}\,\mathrm{id}\,,\qquad J_k\circ J_h^\dagger=\mathrm{id}\otimes\mathfrak{Int}(k\otimes\overline{h})\,.$$

Then clearly $\Upsilon_h=\Upsilon\circ J_h$ and the following formulae hold:

$$\Upsilon_h^\dagger\circ\Upsilon_k=J_h^\dagger\circ J_k=\langle k,h\rangle_{(\Sigma)}\,\mathrm{id}\,,\tag{3.12}$$

$$\Upsilon_k \circ \Upsilon_h^\dagger = \Upsilon \circ \big[\mathrm{id} \otimes \mathfrak{Int}(k \otimes \overline{h})\big] \circ \Upsilon^{-1}.$$

In particular $\| h \|_{(\Sigma)}^{-1} \Upsilon_h$ is an isometry with range $\Upsilon[\mathscr{B}_2(\Sigma) \otimes h]$.

If h and k are not orthogonal, we get from (3.12) for any $f \in \mathscr{B}_2(\Sigma)$

$$f = \frac{1}{\langle h, k \rangle_{(\Sigma)}} \Upsilon_h^\dagger \left[\Upsilon_k(f) \right] , \tag{3.13}$$

which is referred to as *the inversion (or the reconstruction) formula.*

An easy computation leads to the explicit formula for the adjoint

$$\Upsilon_h^\dagger(F) = \int_\Sigma \int_\Sigma \mathrm{d}\mu(s)\, \mathrm{d}\mu(t)\, F(s,t)\; e_s \star h^\star \star e_t^\star , \qquad \forall\, F \in \mathscr{B}_2(\Sigma \times \Sigma) ,$$

which should be interpreted weakly, applied by duality on $f \in \mathscr{B}_2(\Sigma)$.

Remark 3.8. By using Theorem 3.3, one gets for $f, g, h, k \in \mathscr{B}_2(\Sigma)$

$$\Upsilon_h(f) \bullet \Upsilon_k(g) = \Upsilon_{k \star h}(f \star g) , \qquad \Upsilon_h(f)^\bullet = \Upsilon_{h^\star}(f^\star) . \tag{3.14}$$

Therefore, very often, we are going to use self-adjoint idempotent windows $h = h^\star = h \star h$, for which *the localized canonical map Υ_h will be a monomorphism of *-algebras.* To give a useful example, one gets a self-adjoint projection $h = \phi_{w,w}$ for any unit vector w.

Let us use now the opportunities offered by the Gelfand triples constructed above. Most of the time we fix a window h in $\mathscr{G}(\Sigma) \setminus \{0\}$ and, besides the initial $\Upsilon_h = \Upsilon \circ J_h$, we work both with the restriction $\Upsilon_h : \mathscr{G}(\Sigma) \to \mathscr{G}(\Sigma \times \Sigma)$ and (especially) with the extension $\Upsilon_h : \mathscr{G}'(\Sigma)_\sigma \to \mathscr{G}'(\Sigma \times \Sigma)_\sigma$. They are all well-defined linear injective and continuous because of the isomorphism properties of Υ specified in Corollary 3.5 and the obvious mapping properties of

$$J_h : \mathscr{G}'(\Sigma) \to \mathscr{G}'(\Sigma) \otimes \mathscr{G}(\Sigma) \subset \mathscr{G}'(\Sigma) \otimes \mathscr{G}'(\Sigma) \subset \mathscr{G}'(\Sigma \times \Sigma) .$$

On the other hand, $\Upsilon_h : \mathscr{G}(\Sigma) \to \mathscr{G}(\Sigma \times \Sigma)$ possesses an adjoint $\Upsilon_h^\dagger : \mathscr{G}'(\Sigma \times \Sigma)_\nu \to \mathscr{G}'(\Sigma)_\nu$ which is linear and continuous for any of the topologies $\nu = \sigma, \beta$ and which extends the previous $\Upsilon_h^\dagger : \mathscr{B}_2(\Sigma \times \Sigma) \to \mathscr{B}_2(\Sigma)$. *The inversion formula (3.13) still holds for $f \in \mathscr{G}'(\Sigma)$ and $h, k \in \mathscr{G}(\Sigma)$ with $\langle h, k \rangle_{(\Sigma)} \neq 0$*, by the continuity of the ingredients of the expression $\Upsilon_h^\dagger \circ \Upsilon_k : \mathscr{G}'(\Sigma)_\nu \to \mathscr{G}'(\Sigma)_\nu$ and the fact that $\mathscr{B}_2(\Sigma)$ is dense in $\mathscr{G}'(\Sigma)_\nu$ for $\nu = \sigma$. One gets the same conclusion for $\nu = \beta$ by analyzing in detail the way these ingredients were defined.

Remark 3.9. We indicate now briefly another approach to extending the mapping Υ_h, based on Remark 3.2. Formula (3.4) suggests defining $\Upsilon_h : \mathscr{G}'(\Sigma) \to \mathscr{G}'(\Sigma \times \Sigma)$ by

$$\langle \Upsilon_h(f), g \otimes k \rangle_{(\Sigma \times \Sigma)} := \big\langle f, g \star h^\star \star \overline{k}^{\,\star} \big\rangle_{(\Sigma)} = \big\langle f \star \overline{k}, g \star h^\star \big\rangle_{(\Sigma)} , \tag{3.15}$$

where $h \in \mathscr{G}(\Sigma) \setminus \{0\}$ is the fixed window and g, k are arbitrary elements of $\mathscr{G}(\Sigma)$. One can also write the quantities in (3.15) as $\langle \mathfrak{Int}[\Upsilon_h(f)]\overline{g}, k\rangle_{(\Sigma)}$. Rather straightforward arguments justifies this definition and extracts from it the useful properties of Υ_h; the reader will easily check the compatibility between the two points of view.

4. COORBIT SPACES OF SYMBOLS

We keep the same setting as above, consisting of the data $(\Sigma, \mu, \mathcal{G}, \mathcal{H}, \pi)$, assuming Hypohesis 2.1 and the fact that $\mathcal{G}$ is a Fréchet space embedded densely and continuously in the Hilbert space $\mathcal{H}$.

Definition 4.1. Let $\mathscr{M}$ be a vector subspace of $\mathscr{G}'(\Sigma \times \Sigma)$. Its coorbit space associated to the window $h \in \mathscr{G}(\Sigma) \setminus \{0\})$ is

$$\mathsf{CO}_h(\mathscr{M}) := \{f \in \mathscr{G}'(\Sigma) \mid \Upsilon_h(f) \in \mathscr{M}\}\,. \tag{4.1}$$

Clearly $\mathsf{CO}_h[\mathscr{G}'(\Sigma\times\Sigma)] = \mathscr{G}'(\Sigma)$ and $\mathsf{CO}_h[\mathscr{B}_2(\Sigma\times\Sigma)] = \mathscr{B}_2(\Sigma)$. The next result gives other examples of coorbit spaces.

Proposition 4.2. (1) *Let $h \in \mathscr{G}(\Sigma) \setminus \{0\}$. Then $f \in \mathscr{G}'(\Sigma)$ belongs to $\mathscr{G}(\Sigma)$ if and only if $\Upsilon_h(f) \in \mathscr{G}(\Sigma \times \Sigma)$. In other words one has $\mathsf{CO}_h[\mathscr{G}(\Sigma \times \Sigma)] = \mathscr{G}(\Sigma)$.*

(2) *If $h \star h = h^\star = h$, then $\Upsilon_h : \mathscr{G}'(\Sigma) \to \mathscr{G}'(\Sigma \times \Sigma)$ restricts to a monomorphism of *-algebras $\Upsilon_h : \mathscr{G}'_\star(\Sigma) \to \mathscr{G}'_\bullet(\Sigma \times \Sigma)$ and one has $\mathsf{CO}_h[\mathscr{G}'_\bullet(\Sigma \times \Sigma)] = \mathscr{G}'_\star(\Sigma)$.*

(3) *If $\mathscr{M}$ is a *-subalgebra of $\mathscr{G}'_\bullet(\Sigma \times \Sigma)$, then $\mathsf{CO}_h(\mathscr{M})$ is a *-subalgebra of $\mathscr{G}'_\star(\Sigma)$.*

Proof. 1. The "only if part" follows directly from the way Υ has been constructed in the previous section. For the "if part", assume that $\Upsilon_h(f) \in \mathscr{G}(\Sigma \times \Sigma)$. By the extended inversion formula and by the obvious mapping property $\Upsilon_h^\dagger : \mathscr{G}(\Sigma \times \Sigma) \to \mathscr{G}(\Sigma)$ one has

$$f = \| h \|_{(\Sigma)}^{-2}\, \Upsilon_h^\dagger\, [\Upsilon_h(f)] \in \mathscr{G}(\Sigma)\,.$$

2 and 3. The extensions of the algebraic structures and of the isomorphism Υ (Corollary 3.5) allow us to state useful versions of (3.14); for example, this relation also hold for $g, h, k \in \mathscr{G}(\Sigma)$ and $f \in \mathscr{G}'(\Sigma)$. Then 2 and 3 follow easily from the definitions and from 1; see also Corollary 3.6. $\square$

If $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ is a normed space (usually taken to be continuously embedded in $\mathscr{G}'(\Sigma \times \Sigma)$ endowed with the weak*-topology), we endow $\mathsf{CO}_h(\mathscr{M})$ with the norm

$$\| f \|_{\mathsf{CO}_h(\mathscr{M})} := \| \Upsilon_h(f) \|_{\mathscr{M}}\,. \tag{4.2}$$

Proposition 4.3. *Let $\mathscr{M}$ be continuously embedded in $\mathscr{G}'(\Sigma\times\Sigma)_\sigma$.*

(1) *For $h\in\mathscr{G}(\Sigma)\setminus\{0\}$, $\mathsf{CO}_h(\mathscr{M})$ is a normed space continuously embedded in $\mathscr{G}'(\Sigma)_\sigma$.*

(2) *If $\mathscr{M}$ is a Banach space, $\mathsf{CO}_h(\mathscr{M})$ is also a Banach space.*

Proof. 1. It is obvious that $\|\cdot\|_{\mathsf{CO}_h(\mathscr{M})}$ is a norm; recall that $\Upsilon_h:\mathscr{G}'(\Sigma)\to\mathscr{G}'(\Sigma\times\Sigma)$ is injective. If f_n converges to f in $\mathsf{CO}_h(\mathscr{M})$, then $\Upsilon_h(f_n)$ converges to $\Upsilon_h(f)$ in $\mathscr{M}$, hence also *-weakly in $\mathscr{G}'(\Sigma\times\Sigma)$. But $\Upsilon:\mathscr{G}'(\Sigma\times\Sigma)_\sigma\to\mathscr{G}'(\Sigma\times\Sigma)_\sigma$ is a topological isomorphism and thus $f_n\otimes h$ converges to $f\otimes h$ *-weakly in $\mathscr{G}'(\Sigma\times\Sigma)$. It follows that f_n converges to f *-weakly in $\mathscr{G}'(\Sigma)$.

2. Let us show that the inducing process preserves completeness. Clearly $\mathsf{CO}_h(\mathscr{M})$ is isometrically isomorphic to $\mathscr{M}(h):=\mathscr{M}\cap\Upsilon_h[\mathscr{G}'(\Sigma)]$. So we have to show that $\mathscr{M}(h)$ is closed in $\mathscr{M}$. Let $\big(\Upsilon_h(f_n)\big)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{M}(h)$, converging to $G\in\mathscr{M}$. Due to the continuity of the embedding of $\mathscr{M}$ in $\mathscr{G}'(\Sigma\times\Sigma)$ we also have

$$\Upsilon_h(f_n)\underset{n\to\infty}{\longrightarrow}G\ \text{ in }\mathscr{G}'(\Sigma\times\Sigma)\,. \tag{4.3}$$

Set $f:=\frac{1}{\|h\|^2}\Upsilon_h^\dagger(G)\in\mathscr{G}'(\Sigma)$. From the extended inversion formula

$$f-f_n=\frac{1}{\|\,h\,\|^2_{(\Sigma)}}\Upsilon_h^\dagger[G-\Upsilon_h(f_n)]\underset{n\to\infty}{\longrightarrow}0\ \text{ in }\ \mathscr{G}'(\Sigma)$$

which, together with (4.3), imply $G=\Upsilon_h(f)\in\mathscr{M}(h)$. $\square$

Plainly, in Proposition 4.3 the weak* topology can be replaced by others, as $\nu=\beta$ for instance.

If $\|\cdot\|_{\mathscr{M}}$ is a C^*-norm then $\|\cdot\|_{\mathsf{CO}_h(\mathscr{M})}$ will also be a C^*-norm:

$$\begin{aligned}\|\,f^\star\star f\,\|_{\mathsf{CO}_h(\mathscr{M})}&=\|\,\Upsilon_h(f^\star\star f)\,\|_{\mathscr{M}}=\|\,\Upsilon_h(f)^\bullet\bullet\Upsilon_h(f)\,\|_{\mathscr{M}}\\&=\|\,\Upsilon_h(f)\,\|^2_{\mathscr{M}}=\|\,f\,\|^2_{\mathsf{CO}_h(\mathscr{M})}\,.\end{aligned}$$

We study now the dependence of the spaces on the window h .

Proposition 4.4. *Assume that for some $h,k\in\mathscr{G}(\Sigma)\setminus\{0\}$ one has $\big(\Upsilon_k\circ\Upsilon_h^\dagger\big)\mathscr{M}\subset\mathscr{M}$. Then $\mathsf{CO}_h(\mathscr{M})\subset\mathsf{CO}_k(\mathscr{M})$. In addition, if $\mathscr{M}$ is a Banach space continuously embedded in $\mathscr{G}'(\Sigma\times\Sigma)$, the embedding of $\mathsf{CO}_h(\mathscr{M})$ in $\mathsf{CO}_k(\mathscr{M})$ is continuous.*

Proof. By the extended inversion formula, if $f\in\mathsf{CO}_h(\mathscr{M})$ then

$$f=\frac{1}{\|\,h\,\|^2_{(\Sigma)}}\Upsilon_h^\dagger[\Upsilon_h(f)]\in\Upsilon_h^\dagger(\mathscr{M})\,,$$

which implies $\mathsf{CO}_h(\mathscr{M}) \subset \Upsilon_h^\dagger(\mathscr{M})$. So we need to show that $\Upsilon_h^\dagger(\mathscr{M}) \subset \mathsf{CO}_k(\mathscr{M})$. But

$$f = \Upsilon_h^\dagger(G), \text{ with } G \in \mathscr{M} \quad \Longrightarrow \quad \Upsilon_k(f) = [\Upsilon_k \circ \Upsilon_h^\dagger](G) \in \mathscr{M}.$$

To prove the topological embedding, note that $\Upsilon_k \circ \Upsilon_h^\dagger \in \mathbb{B}(\mathscr{M})$ by the Closed Graph Theorem. This and the inversion formula give

$$\| f \|_{\mathsf{CO}_k(\mathscr{M})} = \frac{\left\| \Upsilon_k \left[\left(\Upsilon_h^\dagger \circ \Upsilon_h \right) f \right] \right\|_{\mathscr{M}}}{\| h \|_{(\Sigma)}^2} \le \frac{\| \Upsilon_k \circ \Upsilon_h^\dagger \|_{\mathbb{B}(\mathscr{M})}}{\| h \|_{(\Sigma)}^2} \| f \|_{\mathsf{CO}_h(\mathscr{M})} . \quad \square$$

Let us say that the subspace $\mathscr{M}$ is *admissible* if one has $\left(\Upsilon_k \circ \Upsilon_h^\dagger\right) \mathscr{M} \subset \mathscr{M}$ for every $h, k \in \mathscr{G}(\Sigma) \setminus \{0\}$. By the result above, *if a Banach space $\mathscr{M}$ continuously embedded in $\mathscr{G}'(\Sigma \times \Sigma)_\sigma$ is admissible, we could speak of the Banachizable space* $\mathsf{CO}(\mathscr{M}) := \Upsilon_h^{-1}(\mathscr{M})$, which is continuously embedded in $\mathscr{G}'(\Sigma)_\sigma$; the vector space and the topology do not depend on $h \in \mathscr{G}(\Sigma) \setminus \{0\}$.

Recall that $\Upsilon_k \circ \Upsilon_h^\dagger = \Upsilon \circ J_k \circ J_h^\dagger \circ \Upsilon^{-1}$. Set $\mathscr{N} := \Upsilon^{-1}(\mathscr{M})$; for admissibility, it is enough to prove that

$$J_{k,h} := J_k \circ J_h^\dagger = \mathrm{id} \otimes \mathfrak{Int}(k \otimes \overline{h}) \,:\, \mathscr{G}'(\Sigma \times \Sigma) \to \mathscr{G}'(\Sigma \times \Sigma)$$

restricts to an operator $J_{k,h} : \mathscr{N} \to \mathscr{N}$. This holds for $\mathscr{M} = \mathscr{G}(\Sigma \times \Sigma)$, case in which $\mathscr{N} = \mathscr{G}(\Sigma \times \Sigma) \equiv \mathscr{G}(\Sigma) \widehat{\otimes}_p \mathscr{G}(\Sigma)$. In general, it may fail even for $h = k$. If, however, $(\Upsilon_h \circ \Upsilon_h^\dagger) \mathscr{M} \subset \mathscr{M}$ for every $h \in \mathscr{G}(\Sigma)$, we say that $\mathscr{M}$ is *diagonally admissible*; this is weaker than admissible.

PROPOSITION 4.5. (1) *If $\mathscr{M}$ is a diagonally admissible Banach space continuously embedded in $\mathscr{G}'(\Sigma \times \Sigma)$, for $h \in \mathscr{G}(\Sigma) \setminus \{0\}$ the map $\Upsilon_h^\dagger : \mathscr{G}'(\Sigma \times \Sigma) \to \mathscr{G}'(\Sigma)$ restricts to a retract $\Upsilon_h^\dagger : \mathscr{M} \to \mathsf{CO}_h(\mathscr{M})$.*

(2) *If $\mathscr{G}(\Sigma \times \Sigma)$ is dense in $\mathscr{M}$, then $\mathscr{G}(\Sigma)$ is dense in $\mathsf{CO}_h(\mathscr{M})$.*

(3) *For any $h \in \mathscr{G}(\Sigma) \setminus \{0\}$, the family of coorbit spaces $\mathsf{CO}_h(\mathscr{M})$ of all diagonally admissible Banach spaces $\mathscr{M}$ continuously embedded in $\mathscr{G}'(\Sigma \times \Sigma)$ is stable by any interpolation method.*

Proof. 1. We know from the proof of Proposition 4.4 that $\mathsf{CO}_h(\mathscr{M}) \subset \Upsilon_h^\dagger(\mathscr{M})$, as a consequence of the inversion formula; the opposite inclusion follows from diagonal admissibility. Then apply the Closed Graph Theorem to conclude that $\Upsilon_h^\dagger \in \mathbb{B}[\mathsf{CO}_h(\mathscr{M}), \mathscr{M}]$. The inversion formula shows that this mapping is a retract.

2. The inclusion $\mathscr{G}(\Sigma) \subset \mathsf{CO}_h(\mathscr{M})$ follows from Lemma 4.2 and from the monotony of the inducing process, so we only need to prove density.

Let $f \in \mathsf{CO}_h(\mathscr{M})$, *i.e.* $F := \Upsilon_h(f) \in \mathscr{M}$. One has $\| F - F_n \|_{\mathscr{M}} \to 0$ for a sequence $\{F_n\}_{n \in \mathbb{N}} \subset \mathscr{G}(\Sigma \times \Sigma)$. Then $f_n := \| h \|_{(\Sigma)}^{-2} \Upsilon_h^\dagger(F_n) \in \mathscr{G}(\Sigma)$, since

$\mathscr{G}(\Sigma) = \mathsf{CO}[\mathscr{G}(\Sigma\times\Sigma)]$ and $\mathscr{G}(\Sigma\times\Sigma)$ is admissible. By the point 1

$$\| f - f_n \|_{\mathsf{CO}_h(\mathscr{M})} = \| h \|_{(\Sigma)}^{-2} \| \Upsilon_h^\dagger [\Upsilon_h(f) - F_n] \|_{\mathsf{CO}_h(\mathscr{M})} \le c \, \| F - F_n \|_{\mathscr{M}} \to 0$$

which proves that $\mathscr{G}(\Sigma)$ is dense in $\mathsf{CO}_h(\mathscr{M})$.

3. We need to know that diagonal admissibility is preserved by interpolation; this is obvious from the definitions [35]. It is well-known that the interpolation functors $\mathbf{F}$ commute with retracts so, by 1, we get

$$\mathbf{F}\,[\mathsf{CO}_h(\mathscr{M}_0), \mathsf{CO}_h(\mathscr{M}_1)\,] = \mathsf{CO}_h\,[\,\mathbf{F}(\mathscr{M}_0), \mathbf{F}(\mathscr{M}_1)\,]$$

for any diagonally admissible interpolation couple $(\mathscr{M}_0, \mathscr{M}_1)$. $\square$

We treat now the problem of duality.

Proposition 4.6. *Let $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ be a Banach space continuously embedded in $\mathscr{G}'(\Sigma\times\Sigma)_\sigma$ and containing $\mathscr{G}(\Sigma\times\Sigma)$ continuously and densely. Assume also that $\mathscr{M}$ is diagonally admissible and that $h \in \mathscr{G}(\Sigma)\setminus\{0\}$. Then one has a canonical identification between $\mathsf{CO}_h(\mathscr{M})'$ and $\mathsf{CO}_h(\mathscr{M}')$.*

Proof. Let us denote by $i : \mathscr{G}(\Sigma\times\Sigma) \to \mathscr{M}$ the canonical injection. Then the transpose $i' : \mathscr{M}'_\beta \to \mathscr{G}'(\Sigma\times\Sigma)_\beta \hookrightarrow \mathscr{G}'(\Sigma\times\Sigma)_\sigma$ is continuous, where each dual has been endowed with its own strong topology; note that $\mathscr{M}'_\beta$ is a Banach space. The map i' is also injective, since i was assumed to have a dense range; thus we can apply to $\mathscr{M}' = \mathscr{M}'_\beta$ the coorbit procedure.

Let us set $\varphi : \mathsf{CO}_h(\mathscr{M}') \to \mathsf{CO}_h(\mathscr{M})'$ by

$$\langle \varphi(f), g\rangle_{(\Sigma)} := \langle \Upsilon_h(f), \Upsilon_h(g)\rangle_{(\Sigma\times\Sigma)}, \qquad \forall\, f \in \mathsf{CO}_h(\mathscr{M}'),\ g \in \mathsf{CO}_h(\mathscr{M}).$$

Actually the duality of the r.h.s. is the one between $\mathscr{M}'$ and $\mathscr{M}$ but, by compatibility, we use the notation $\langle\cdot,\cdot\rangle_{(\Sigma\times\Sigma)}$. One has for $g \in \mathsf{CO}_h(\mathscr{M})$

$$\begin{aligned} |\,\langle \varphi(f), g\rangle_{(\Sigma)}\,| &= |\,\langle \Upsilon_h(f), \Upsilon_h(g)\rangle_{(\Sigma\times\Sigma)}\,| \\ &\le \|\,\Upsilon_h(f)\,\|_{\mathscr{M}'} \,\|\,\Upsilon_h(g)\,\|_{\mathscr{M}} \\ &= \|\,f\,\|_{\mathsf{CO}_h(\mathscr{M}')} \,\|\,g\,\|_{\mathsf{CO}_h(\mathscr{M})}, \end{aligned}$$

which shows that the functional $\varphi(f)$ is continuous and the correspondence $f \mapsto \varphi(f)$ is contractive.

We show now surjectivity. For $f' \in \mathsf{CO}_h(\mathscr{M})'$, we denote by f'_0 its restriction to $\mathscr{G}(\Sigma)$; then $f'_0 \in \mathscr{G}'(\Sigma)$. We must show that $f'_0 \in \mathsf{CO}_h(\mathscr{M}')$, *i.e.* that $\Upsilon_h(f'_0) \in \mathscr{M}'$. For $G \in \mathscr{G}(\Sigma\times\Sigma)$, by the choice of f' and by the first assertion in Proposition 4.5 (using diagonal admissibility)

$$\left|\langle \Upsilon_h(f'_0)\,, G\rangle_{(\Sigma\times\Sigma)}\right| = \left|\left\langle f'_0, \Upsilon_h^\dagger(G)\right\rangle_{(\Sigma)}\right| \le c_1 \,\|\,\Upsilon_h^\dagger(G)\,\|_{\mathsf{CO}_h(\mathscr{M})} \le c_2 \,\|\,G\,\|_{\mathscr{M}}.$$

This and the density of $\mathscr{G}(\Sigma\times\Sigma)$ in $\mathscr{M}$ shows indeed that $\Upsilon_h(f_0')\in\mathscr{M}'$. By the extended inversion formula it follows that $\varphi(\|h\|_{(\Sigma)}^{-2}\,f_0')=f'$.

Being a continuous linear bijection between two Banach spaces, φ is an isomorphism by the Open Mapping Theorem. $\square$

5. COORBIT SPACES OF VECTORS

Let us fix an element $w\in\mathcal{H}$. The family $\{w(s):=\pi(s)^*w\mid s\in\Sigma\}$ is a tight continuous frame [11], as a consequence of Hypothesis 2.1. The constant of the frame is $C=\|w\|^2$; we assume it to be 1 by normalizing w. So we have in weak sense

$$1=\int_\Sigma \mathrm{d}\mu(s)|w(s)\rangle\langle w(s)|\,.$$

We define $\phi_w:\mathcal{H}\to\mathscr{B}_2(\Sigma)\subset\mathscr{L}^2(\Sigma)$ by

$$\phi_w(u):=\phi_{u,w}=\langle\pi(\cdot)u,w\rangle=\langle u,w(\cdot)\rangle\,,$$

with adjoint $\phi_w^\dagger:\mathscr{B}_2(\Sigma)\to\mathcal{H}$ given by

$$\phi_w^\dagger(f)=\int_\Sigma \mathrm{d}\mu(s)f(s)\pi(s)^*w=\Pi(f)w\,. \tag{5.1}$$

To show the analogy with the localized mappings $\Upsilon_h=\Upsilon\circ J_h$ of the previous section, we could use $\mathcal{H}\ni u\mapsto j_w(u):=u\otimes w\in\mathscr{B}_2(\Sigma)$ to write $\phi_w=\Phi\circ j_w$. The kernel associated to the frame is

$$p_w(s,t):=\langle w(t),w(s)\rangle=\langle\pi(t)^*w,\pi(s)^*w\rangle=\phi_{w(t),w}(s)\,,$$

defining a self-adjoint integral operator $P_w=\mathfrak{Int}(p_w)$ in $\mathscr{B}_2(\Sigma)$. One checks easily that $P_w=\phi_w\phi_w^\dagger$ is the final projection of the isometry ϕ_w, so $P_w\left[\mathscr{B}_2(\Sigma)\right]$ is a closed subspace of $\mathscr{B}_2(\Sigma)$. Since $\phi_w^\dagger\phi_w=1$, one has the inversion formula

$$u=\int_\Sigma \mathrm{d}\mu(t)\left[\phi_w(u)\right](t)\,w(t)$$

and the reproducing formula $\phi_w(u)=P_w\left[\phi_w(u)\right]$, *i.e.*

$$\left[\phi_w(u)\right](s)=\int_\Sigma \mathrm{d}\mu(t)\,\langle w(t),w(s)\rangle\left[\phi_w(u)\right](t)\,.$$

Thus $\mathscr{P}_w(\Sigma):=P_w[\mathscr{B}_2(\Sigma)]$ is a reproducing kernel Hilbert space with reproducing kernel p_w, composed of bounded continuous functions on Σ.

Let us assume, in addition, that the “window” w belongs to the Fréchet space $\mathcal{G}$ continuously and densely embedded in $\mathcal{H}$. These results above justify the linear mapping $\phi_w:\mathcal{G}'\to\mathscr{G}'(\Sigma)$ which will be used to pull back algebraic and topological structure.

Definition 5.1. Let $(\mathcal{M}, \|\cdot\|_{\mathcal{M}})$ be a Banach space continuously embedded in $\mathscr{G}'(\Sigma)$. Its coorbit space (associated to the couple (π, w)) is

$$\mathsf{co}_w(\mathcal{M}) := \{u \in \mathcal{G}' \mid \phi_w(u) \in \mathcal{M}\} \tag{5.2}$$

with the norm $\| u \|_{\mathsf{co}_w(\mathcal{M})} := \| \phi_w(u) \|_{\mathcal{M}}$.

Remark 5.2. Since coorbit spaces of vectors are not our main concern, we are not going to develop their theory. After suitable adaptations, a lot can be said just by specializing results from [11]. It is also possible to make convenient modifications in the preceding section on coorbit spaces symbols, by making the replacement $\Upsilon_h \mapsto \phi_w$. So let us just state the rather obvious fact, that will be used below, that $\mathsf{co}_w(\mathcal{M})$ is a Banach space continuously embedded in $\mathcal{G}'$. Simple arguments based on the inversion formula and the mapping properties of $\phi_w^\dagger$ show that $\mathsf{co}_w[\mathscr{B}_2(\Sigma)] = \mathcal{H}$, $\mathsf{co}_w[\mathscr{G}(\Sigma)] = \mathcal{G}$ and $\mathsf{co}_w[\mathscr{G}'(\Sigma)] = \mathcal{G}'$ (if $\mathcal{G}$ is not Banach, the second and the third examples can be taken in the category of vector spaces).

The next result will imply a general result on boundedness of operators $\Pi(f)$ between modulation spaces. It is a consequence of Proposition 3.4, but we give a direct proof.

Proposition 5.3. (1) *For any $f \in \mathscr{B}_2(\Sigma)$ and $w_1, w_2 \in \mathcal{H}$, one has*

$$\phi_{w_2}\Pi(f)\phi_{w_1}^\dagger = \mathfrak{Int}\left[\Upsilon_{\phi_{w_2,w_1}}(f)\right]. \tag{5.3}$$

(2) *If $f \in \mathscr{G}'(\Sigma)$ and $w_1, w_2 \in \mathcal{G}$, the same identity holds in the space $\mathbb{B}[\mathscr{G}(\Sigma), \mathscr{G}'(\Sigma)]$.*

Proof. 1. We compute for $k \in \mathscr{B}_2(\Sigma)$ and $s \in \Sigma$, using (5.1), (2.4) and the symbolic calculus

$$\begin{aligned}\left[\left(\phi_{w_2}\Pi(f)\phi_{w_1}^\dagger\right) k\right](s) &= \left\langle \pi(s)\Pi(f)\left[\phi_{w_1}^\dagger(k)\right], w_2\right\rangle \\ &= \langle \Pi(k)w_1, \Pi(f)^*\pi(s)^*w_2\rangle \\ &= \int_\Sigma d\mu(t)\, k(t)\, \langle \pi(t)^*w_1, \Pi(f)^*\pi(s)^*w_2\rangle \\ &= \int_\Sigma d\mu(t)\, k(t)\, \langle \Pi\left(e_s^\star \star f \star e_t\right) w_1, w_2\rangle \\ &= \int_\Sigma d\mu(t)\, k(t)\, \langle e_s^\star \star f \star e_t, \phi_{w_2,w_1}\rangle_{(\Sigma)} \\ &= \left(\mathfrak{Int}\left[\Upsilon_{\phi_{w_2,w_1}}(f)\right] k\right)(s).\end{aligned}$$

2. This can be justified repeating the computation above in weak sense, applied to a "test function" $l \in \mathscr{G}(\Sigma)$, using our previous information about

the extended objects. One has, with suitable interpretations and also using Remark 3.9

$$\begin{aligned}\left\langle \left(\phi_{w_2}\Pi(f)\phi^\dagger_{w_1}\right) k, l\right\rangle_{(\Sigma)} &= \left\langle \Pi(f)\,\phi^\dagger_{w_1} k, \phi^\dagger_{w_2} l\right\rangle_{(\Sigma)} = \langle \Pi(f)\Pi(k)w_1, \Pi(l)w_2\rangle \\ &= \langle \Pi(l^\star \star f \star k)w_1, w_2\rangle = \langle l^\star \star f \star k, \phi_{w_1,w_2}\rangle_{(\Sigma)} \\ &= \langle f, l \star \phi_{w_1,w_2} \star k^\star\rangle_{(\Sigma)} = \left\langle \Upsilon_{\phi_{w_2,w_1}}(f), l \otimes \overline{k}\right\rangle_{(\Sigma\times\Sigma)} \\ &= \left\langle \mathfrak{Int}\left[\Upsilon_{\phi_{w_2,w_1}}(f)\right] k, l\right\rangle_{(\Sigma)}. \quad \square\end{aligned}$$

Corollary 5.4. *Let* $(\mathcal{M}_1, \|\cdot\|_{\mathcal{M}_1})$ *and* $(\mathcal{M}_2, \|\cdot\|_{\mathcal{M}_2})$ *be two Banach spaces continuously embedded in* $\mathscr{G}'(\Sigma)$ *and* $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ *be a Banach space continuously embedded in* $\mathscr{G}'(\Sigma\times\Sigma)$. *If* $\mathfrak{Int}(\mathscr{M}) \subset \mathbb{B}(\mathcal{M}_1,\mathcal{M}_2)$, *then* $\Pi[\mathsf{CO}_{\phi_{w_2,w_1}}(\mathscr{M})] \subset \mathbb{B}[\mathsf{co}_{w_1}(\mathcal{M}_1), \mathsf{co}_{w_2}(\mathcal{M}_2)]$.

Proof. Assume that $f \in \mathsf{CO}_{\phi_{w_2,w_1}}(\mathscr{M})$, meaning that $\Upsilon_{\phi_{w_2,w_1}}(f) \in \mathscr{M}$. Let also $u \in \mathsf{co}_{w_1}(\mathcal{M}_1)$, thus $\phi_{w_1}(u) \in \mathcal{M}_1$. Applying the assumption, we get

$$v := \mathfrak{Int}\left[\Upsilon_{\phi_{w_2,w_1}}(f)\right]\phi_{w_1}(u) \in \mathcal{M}_2 .$$

By (5.3), this is written $v = \phi_{w_2}[\Pi(f)]\,u \in \mathcal{M}_2$, *i.e.* $\Pi(f)u \in \mathsf{co}_{w_2}(\mathcal{M}_2)$. On the other hand

$$\begin{aligned}\|\Pi(f)u\|_{\mathsf{co}_{w_2}(\mathcal{M}_2)} &= \|\phi_{w_2}[\Pi(f)u]\|_{\mathcal{M}_2} = \|\mathfrak{Int}\left[\Upsilon_{\phi_{w_2,w_1}}(f)\right]\phi_{w_1}(u)\|_{\mathcal{M}_1} \\ &\le C\,\|\phi_{w_1}(u)\|_{\mathcal{M}_1} = C\,\|u\|_{\mathsf{co}_{w_1}(\mathcal{M}_1)} . \quad \square\end{aligned}$$

The formalism benefits from a good choice of a space $\mathcal{G}$. We follow now the approach of [10] and [11] to show that such choices are possible. Consider continuous *admissible weights* $A : \Sigma\times\Sigma \to [1,\infty)$ which are

- bounded along the diagonal: $A(s,s) \le C < \infty$ for all $s \in \Sigma$,
- symmetric: $A(s,t) = A(t,s)$ for all $s,t \in \Sigma$,
- satisfying $A(s,t) \le A(s,r)A(r,t)$ for all $r,s,t \in \Sigma$.

It is easy to see that

$$\mathscr{A}_A := \{K : \Sigma\times\Sigma \to \mathbb{C} \text{ measurable} \mid \|K\|_{\mathscr{A}_A} < \infty\}$$

is a Banach *-algebra of kernels with the norm

$$\|K\|_{\mathscr{A}_A} := \max\left\{\operatorname*{ess\,sup}_{s\in\Sigma}\int_\Sigma d\mu(t)(AK)(s,t)\,,\ \operatorname*{ess\,sup}_{t\in\Sigma}\int_\Sigma d\mu(s)(AK)(s,t)\right\}.$$

Picking some (inessential) point $r \in \Sigma$ one defines the weight $a \equiv a_r : \Sigma \to [1,\infty)$ by $a(s) := A(s,r)$. Fixing also a unit vector $w_0 \in \mathcal{H}$, we require that the kernel p_{w_0} given by (5) be an element of $\mathscr{A}_A$. Then set $\mathcal{G} \equiv \mathcal{G}_{a,w_0} := \{w \in \mathcal{H} \mid \phi_{w_0}(w) \in \mathscr{L}^1_a(\Sigma)\}$ with norm

$$\| w \|_{\mathcal{G}_{a,w_0}} := \| \phi_{w_0}(w) \|_{\mathscr{L}^1_a(\Sigma)} = \int_\Sigma d\mu(s)\, a(s)\, | [\phi_{w_0}(w)](s) | .$$

PROPOSITION 5.5. (1) *$\mathcal{G}_{a,w_0}$ is a Banach space continuously and densely embedded in $\mathcal{H}$.*

(2) *Setting $\mathcal{G} = \mathcal{G}_{a,w_0}$, then the space $\mathscr{G}(\Sigma) := \Phi\left(\mathcal{G}\widehat{\otimes}_p \overline{\mathcal{G}}\right)$ is given by*

$$\mathscr{G}(\Sigma) = \left\{ h \in \mathscr{B}_2(\Sigma) \mid \Upsilon_{\phi_{w_0,w_0}}(h) \in \mathscr{L}^1_{a\otimes a}(\Sigma \times \Sigma) \right\}.$$

Proof. 1. For the Banach space property we send to [11, Prop. 3.1]. The embedding follows from Hypothesis 2.1, from $a(\cdot) \geq 1$ and from the bound $[\phi_{w_0}(w)](\cdot) \leq D < \infty$:

$$\| w \|^2_{\mathcal{H}} = \int_\Sigma d\mu(s)\, | [\phi_{w_0}(w)](s) |^2 \leq D \int_\Sigma d\mu(s)\, a(s)\, | [\phi_{w_0}(w)](s) | .$$

To prove density, it is enough to check that the total family $\{w_0(s) = \pi(s)^* w_0 \mid s \in \Sigma\}$ is contained in $\mathcal{G}_{a,w_0}$. As in [11, (3.9)], it follows easily from the requirement $p_{w_0} \in \mathscr{A}_A$ that

$$\| w_0(s) \|_{\mathcal{G}_{a,w_0}} \leq a(s)\, \| p_{w_0} \|_{\mathscr{A}_A} . \tag{5.4}$$

If we are not interested in the dependence of s of the “constant” in (5.4), much less can be required on p_{w_0}. For other possible developments we refer to Ch. 3 in [11].

2. The weight can be absorbed in the measure: one has $\mathscr{L}^1_a(\Sigma\,;\mu) = \mathscr{L}^1(\Sigma\,;a\mu)$ and $\mathscr{L}^1_{a\otimes a}(\Sigma\times\Sigma\,;\mu\otimes\mu) = \mathscr{L}^1(\Sigma\times\Sigma\,;(a\mu)\otimes(a\mu))$, so it is enough to consider the case $A(r,s) = 1$.

By the first computation of the proof of Theorem 3.3 we have

$$\Upsilon_{\phi_{w_0,w_0}}[\Phi(u\otimes v)] := \Upsilon(\phi_{u,v}\otimes\phi_{w_0,w_0}) = \phi_{u,w_0}\otimes\phi_{v,w_0} = \phi_{w_0}(u)\otimes\phi_{w_0}(v),$$

which implies that $\Upsilon_{\phi_{w_0,w_0}} \circ \Phi = \phi_{w_0}\otimes\phi_{w_0}$. Since $\mathscr{L}^1(\Sigma\times\Sigma)$ can be seen as the completed projective tensor product $\mathscr{L}^1(\Sigma)\widehat{\otimes}_p\mathscr{L}^1(\Sigma)$, the result follows easily. $\square$

So $\mathcal{G}_{a,w_0}$ is a coorbit space of vectors $\mathsf{co}_{w_0}[\mathscr{L}^1_a(\Sigma)]$ and the reservoir of symbol-windows associated to $\mathcal{G}_{a,w_0}$ is also a coorbit space.

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Universidad de Chile,
Departamento de Matemáticas,
Facultad de Ciencias
Las Palmeras 3425, Casilla 653,
Santiago, Chile mantoiu@uchile.cl