COHOMOLOGY OF THE LIE SUPERALGEBRA OF CONTACT VECTOR FIELDS ON WEIGHTED DENSITIES ON THE SUPERSPACE $\mathbb{K}^{1|n}$

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Over the $(1,n)$-dimensional real superspace, we compute the first differential cohomology of the Lie superalgebra $\mathcal{K}(n)$ with coefficients in the superspace $\mathbb{F}_\lambda^n$ of λ-densities. Following Feigin and Fuchs, we explicitly give 1-cocycles spanning these cohomology spaces.

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1. INTRODUCTION

Let $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$, we define the superspace $\mathbb{K}^{1|n}$ in terms of its superalgebra of functions, denoted by $C^\infty(\mathbb{K}^{1|n})$ and which is the space spanned by the elementary functions

$$f_{i_1,\ldots,i_n}(x)\theta_1^{i_1}\cdots\theta_n^{i_n}$$

where x is an arbitrary even variable, $f_{i_1,\ldots,i_n} \in C^\infty(\mathbb{K})$ and $\theta_1, \ldots, \theta_n$ are the odd variables, that is, $\theta_i\theta_j = -\theta_j\theta_i$, therefore $i_1,\ldots,i_n \in \{0, 1\}$. The superspace $\mathbb{K}^{1|n}$ is naturally equipped with a contact structure given by the standard 1-form:

$$\alpha_n = dx + \sum_{i=1}^{n} \theta_i d\theta_i. \tag{1.1}$$

In this paper we restrict ourselves to the space of polynomial functions $\mathbb{K}[x,\theta]$ instead of $C^\infty(\mathbb{K}^{1|n})$, that is, the functions $f_{i_1,\ldots,i_n}$ are elements of $\mathbb{K}[x]$.

Consider the contact vector fields

$$\bar{\eta}_i = \frac{\partial}{\partial\theta_i} - \theta_i\frac{\partial}{\partial x}$$

and consider the Lie superalgebra $\mathcal{K}(n)$ of polynomial contact vector fields on $\mathbb{K}^{1|n}$, that is, $\mathcal{K}(n)$ is the superspace of polynomial vector fields on $\mathbb{K}^{1|n}$

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preserving the 1-form α_n. It is well known that the Lie superalgebra $\mathcal{K}(n)$ is spanned by the fields of the form:

$$X_F = F\partial_x - \frac{1}{2}\sum_{i=1}^{n}(-1)^{|F|}\bar{\eta}_i(F)\bar{\eta}_i,$$

where $F \in \mathbb{K}[x,\theta]$ and $|F|$ is the parity of F.

Now, for any $\lambda \in \mathbb{K}$, we define a structure of $\mathcal{K}(n)$-module over $\mathbb{K}[x,\theta]$ by

$$\mathbb{L}^{\lambda}_{X_F} = X_F + \lambda F' \quad \text{where} \quad F' = \frac{\partial F}{\partial x}. \tag{1.2}$$

The corresponding $\mathcal{K}(n)$-module is the space of polynomial weighted densities on $\mathbb{K}^{1|n}$ of weight λ with respect to α_n which we denote

$$\mathbb{F}^n_\lambda = \left\{ F\alpha_n^\lambda \mid F \in \mathbb{K}[x,\theta] \right\}. \tag{1.3}$$

The Lie superalgebra $\mathcal{K}(n-1)$ can be realized as a subalgebra of $\mathcal{K}(n)$:

$$\mathcal{K}(n-1) = \left\{ X_F \in \mathcal{K}(n) \mid \partial_n F = 0 \right\} \quad \text{where} \quad \partial_i = \frac{\partial}{\partial\theta_i}.$$

Therefore, the spaces $\mathbb{F}^n_\lambda$ are also $\mathcal{K}(n-1)$-modules. Note that, the Lie superalgebra $\mathcal{K}(n-1)$ is also isomorphic to

$$\mathcal{K}(n-1)^i = \left\{ X_F \in \mathcal{K}(n) \mid \partial_i F = 0 \right\}.$$

Let $\mathbb{K}[x,\theta]^i = \{F \in \mathbb{K}[x,\theta] \mid \partial_i F = 0\}$, thus, we have

$$\mathcal{K}(n-1)^i = \left\{ X_F \in \mathcal{K}(n) \mid F \in \mathbb{K}[x,\theta]^i \right\}.$$

Our purpose in this paper is to compute the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n), \mathbb{F}^n_\lambda)$, where $\mathrm{H}^*_{\mathrm{diff}}$ denotes the differential cohomology, that is, only cochains given by differential operators are considered. Of course, the case $n = 0$ corresponds to the classical setting and it was studied by Feigin and Fuchs [3], while the case $n = 1$ was studied by Agrebaoui and Ben Fraj [1].

2. THE SPACE $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n), \mathbb{F}^n_\lambda)$

We consider the Lie superalgebra $\mathcal{K}(n)$ acting on $\mathbb{F}^n_\lambda$ and we compute the first cohomology space $\mathcal{K}(n)$ with coefficients in $\mathbb{F}^n_\lambda$. Consider a 1-cocycle $\Upsilon \in \mathrm{Z}^1(\mathcal{K}(n); \mathbb{F}^n_\lambda)$. The cocycle relation reads (see, *e.g.*, [4]):

(2.4)
$$(-1)^{|g||\Upsilon|} g\cdot\Upsilon(h) - (-1)^{|h|(|g|+|\Upsilon|)} h\cdot\Upsilon(g) - \Upsilon([g,\ h]) = 0 \quad \text{for any } g, h \in \mathcal{K}(n).$$

Our main result is the following:

THEOREM 2.1. *The space* $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n);\mathbb{F}^n_\lambda)$ *has the following structure:*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n);\mathbb{F}^n_\lambda) \simeq \begin{cases} \mathbb{K}^2 & \textit{if} \quad n=2 \textit{ and } \lambda=0, \\ \mathbb{K} & \textit{if} \quad \begin{cases} n=0 \textit{ and } \lambda=0,\,1,\,2, \\ n=1 \textit{ and } \lambda=0,\,\frac{1}{2},\,\frac{3}{2}, \\ n=2 \textit{ and } \lambda=1, \\ n=3 \textit{ and } \lambda=0,\frac{1}{2}, \\ n\geq 4 \textit{ and } \lambda=0, \end{cases} \\ 0 & \textit{otherwise.} \end{cases} \tag{2.5}$$

A base for the nontrivial $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n);\mathbb{F}^n_\lambda)$ *is given by the cohomology classes of the 1-cocycles which are collected in the following table:*

TABLE 1

(n,λ)	1-cocycles
$(n,0)$	$\Upsilon^n_\lambda(X_F)=F'$
$(0,1)$	$\Upsilon^0_1(X_F)=F''dx^1$
$(0,2)$	$\widetilde{\Upsilon}^0_2(X_F)=F'''dx^2$
$(1,\frac{1}{2})$	$\Upsilon^1_{\frac{1}{2}}(X_F)=\bar{\eta}_1(F')\alpha_1^{\frac{1}{2}}$
$(1,\frac{3}{2})$	$\Upsilon^1_{\frac{3}{2}}(X_F)=\bar{\eta}_1(F'')\alpha_1^{\frac{3}{2}}$
$(2,0)$	$\Upsilon^2_0(X_F)=(-1)^{\lvert F\rvert}\bar{\eta}_1\bar{\eta}_2(F)$
$(2,1)$	$\Upsilon^2_1(X_F)=(-1)^{\lvert F\rvert}\bar{\eta}_1\bar{\eta}_2(F')\alpha_2$
$(3,\frac{1}{2})$	$\Upsilon^3_{\frac{1}{2}}(X_F)=\bar{\eta}_1\bar{\eta}_2\bar{\eta}_3(F)\alpha_3^{\frac{1}{2}}.$

The proof of Theorem 2.1 will be the subject of Section 3. In fact, we need first the description of $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n-1),\mathbb{F}^n_\lambda)$ and $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n),\mathcal{K}(n-1)^i,\mathbb{F}^n_\lambda)$.

2.1. THE SPACE $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n-1);\mathbb{F}^n_\lambda)$

The space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n);\mathbb{F}^n_\lambda)$ is closely related to $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n-1);\mathbb{F}^n_\lambda)$. Thus, we first recall the description of $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(0),\mathbb{F}^0_\lambda)$ and $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1),\mathbb{F}^1_\lambda)$. The spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(0),\mathbb{F}^0_\lambda)$ were computed in [3]:

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(0),\mathbb{F}^0_\lambda) \simeq \begin{cases} \mathbb{K} & \text{if} \quad \lambda=0,1,2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

The following 1-cocycles span the corresponding cohomology spaces:

$$\Upsilon^0_\lambda(X_F)=F^{(\lambda+1)}dx^\lambda, \quad \text{where } \lambda=0,1,2\ . \tag{2.7}$$

The spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1);\mathbb{F}^1_\lambda)$ were computed in [1]:

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1);\mathbb{F}^1_\lambda) \simeq \begin{cases} \mathbb{K} & \text{if} \quad \lambda = 0, \frac{1}{2}, \frac{3}{2}, \\ 0 & \text{otherwise}. \end{cases} \tag{2.8}$$

The following 1-cocycles span the corresponding cohomology spaces:

$$\Upsilon^1_\lambda(X_F) = \bar{\eta}_1^{2\lambda}(F')\alpha_1^\lambda, \quad \text{where } \lambda = 0, \tfrac{1}{2}, \tfrac{3}{2}\ . \tag{2.9}$$

Proposition 2.2. *As a $\mathcal{K}(n-1)$-module, we have*

$$\mathbb{F}^n_\lambda \simeq \mathbb{F}^{n-1}_\lambda \oplus \Pi\left(\mathbb{F}^{n-1}_{\lambda+\frac{1}{2}}\right) \tag{2.10}$$

where Π stands for the parity change map.

Proof. Any element $F \in \mathbb{K}[x,\theta]$ can be uniquely expressed as

$$F = F_1 + F_2\theta_n \quad \text{with} \quad \partial_n F_1 = \partial_n F_2 = 0.$$

As in [2], we easily show that the map

$$\begin{array}{rcl} \Phi_\lambda : \mathbb{F}^n_\lambda & \longrightarrow & \mathbb{F}^{n-1}_\lambda \oplus \Pi\left(\mathbb{F}^{n-1}_{\lambda+\frac{1}{2}}\right) \\ F\alpha_n^\lambda & \longmapsto & \left(F_1\alpha^\lambda_{n-1}, \Pi\left(F_2\alpha_{n-1}^{\lambda+\frac{1}{2}}\right)\right), \end{array}$$

is $\mathcal{K}(n-1)$-isomorphism. In fact, we easily check that

$$\mathbb{L}^\lambda_{X_H}(F)\alpha^\lambda = (\mathbb{L}^\lambda_{X_H}(F_1) + \mathbb{L}^{\lambda+\frac{1}{2}}_{X_H}(F_2)\theta_n)\alpha^\lambda_n. \quad \square$$

Proposition 2.3. *The space $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1),\mathbb{F}^2_\lambda)$ has the following structure:*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1);\mathbb{F}^2_\lambda) \simeq \begin{cases} \mathbb{K}^2 & \textit{if} \quad \lambda = 0, \\ \mathbb{K} & \textit{if} \quad \lambda = -\frac{1}{2},\ \frac{1}{2},\ 1,\ \frac{3}{2}, \\ 0 & \textit{otherwise}. \end{cases} \tag{2.11}$$

The spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1);\mathbb{F}^2_\lambda)$ are spanned by the cohomology classes of the 1-cocycles $\Theta^2_{j,\ell}$ defined by

$$\Theta^2_{j,\ell}(X_F) = \bar{\eta}_1^j(F')\theta_2^\ell\alpha_2^{\frac{j-\ell}{2}} \tag{2.12}$$

where $\ell = 0, 1,\ j = 0,\ 1,\ 3$ and $\lambda = \frac{j-\ell}{2}$.

Proof. First, it is easy to see that we can deduce the structure of $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n);\Pi(\mathbb{F}^n_\lambda))$ from $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n);\mathbb{F}^n_\lambda)$. Indeed, to any $\Upsilon \in Z^1_{\mathrm{diff}}(\mathcal{K}(n);\mathbb{F}^n_\lambda)$ corresponds $\Pi\circ\Upsilon \in Z^1_{\mathrm{diff}}(\mathcal{K}(n);\Pi(\mathbb{F}^n_\lambda))$. Obviously, Υ is a coboundary if and only if $\Pi\circ\Upsilon$ is a coboundary.

Second, according to Proposition 2.2, we obtain the following isomorphism between cohomology spaces:

(2.13)
$$\mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(n-1),\mathbb{F}^n_\lambda\right)\simeq\ \mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(n-1),\mathbb{F}^{n-1}_\lambda\right)\oplus\ \mathrm{H}^1_{\mathrm{diff}}\left(\mathcal{K}(n-1),\Pi(\mathbb{F}^{n-1}_{\lambda+\frac{1}{2}})\right).$$

Thus, we deduce the structure of $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(1),\mathbb{F}^2_\lambda)$ from the knowledge of the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(0),\mathbb{F}^0_\lambda)$ which are given by Feigen and Fuchs [3]. □

2.2. THE SPACES $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n),\mathcal{K}(n-1)^i;\mathbb{F}^n_\lambda)$

As a first step towards the proof of Theorem 2.1, we shall need to study the $\mathcal{K}(n-1)^i$-relative cohomology $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n),\mathcal{K}(n-1)^i;\mathbb{F}^n_\lambda)$. From the cocycle relation (2.4) we deduce the following equations for all g, $h\in\mathbb{K}[x,\theta]^i$:

(2.14)
$$(-1)^{|g||\Upsilon|}X_g\cdot\Upsilon(X_{h\theta_i})-(-1)^{(|h|+1)(|g|+|\Upsilon|)}X_{h\theta_i}\cdot\Upsilon(X_g)-\Upsilon([X_g,X_{h\theta_i}])=0,$$

(2.15)
$$(-1)^{(|g|+1)|\Upsilon|}X_{g\theta_i}\cdot\Upsilon(X_{h\theta_i})-(-1)^{(|h|+1)(|g|+|\Upsilon|+1)}X_{h\theta_i}\cdot\Upsilon(X_{g\theta_i})-\Upsilon([X_{g\theta_i},X_{h\theta_i}])=0.$$

PROPOSITION 2.4. *Up to a coboundary, any 1-cocycle $\Upsilon\in Z^1_{\mathrm{diff}}(\mathcal{K}(n);\mathbb{F}^n_\lambda)$ has the following general form:*

(2.16)
$$\Upsilon(X_F)=\sum a_{\ell_1\ell_2\cdots\ell_n}\bar\eta_1^{\ell_1}\bar\eta_2^{\ell_2}\cdots\bar\eta_n^{\ell_n}(F),$$

where the coefficients $a_{\ell_1\ell_2\cdots\ell_n}$ are functions of θ_i, not depending on x.

Proof. A priori $\Upsilon(X_F)$ has the following general form:

$$\Upsilon(X_F)=\sum a_{k,\epsilon}(x,\theta)\partial_x^k\partial_1^{\varepsilon_1}\cdots\partial_n^{\varepsilon_n}(F)\alpha_n^\lambda;\ \varepsilon_i=0,1.$$

But, since $-\eta_i^2=\partial_x$, and $\partial_i=\eta_i-\theta_i\eta_i^2$, then $\Upsilon(X_F)$ can be expressed as:

$$\Upsilon(X_F)=\sum a_{\ell_1\ell_2\cdots\ell_n}(x,\theta)\bar\eta_1^{\ell_1}\bar\eta_2^{\ell_2}\cdots\bar\eta_n^{\ell_n}(F)\alpha_n^\lambda,$$

where the coefficients $a_\ell(x,\theta)$ are arbitrary functions.

Now, from the cocycle relation (2.4), we deduce that Υ is $\mathcal{K}(n-1)$-invariant since we have $\Upsilon(X_F)=0$ for all $F\in\mathcal{K}(n-1)$. Especially Υ is invariant with respect X_1, therefore the functions $a_{\ell_1\ell_2\cdots\ell_n}$ are not depending on x. □

We shall need the following description of $\mathcal{K}(n)$-invariant mappings.

Lemma 2.5. *Let*

$$A : \mathbb{F}^n_{-\frac{1}{2}} \to \mathbb{F}^n_\lambda, \quad F\alpha_n^{-\frac{1}{2}} \mapsto A(F)\alpha_n^\lambda$$

be a linear differential operator. If A is $\mathcal{K}(n)$-invariant then $A(F) = F\alpha_n^\lambda$.

Proof. A straightforward computation. □

Theorem 2.6. *For all $n \in \mathbb{N}$ and for all $i = 1, \ldots, n$, we have*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n), \mathcal{K}(n-1)^i; \mathbb{F}^n_\lambda) \simeq 0. \tag{2.17}$$

Proof. Let $\Upsilon \in Z^1_{\mathrm{diff}}(\mathcal{K}(n); \mathfrak{F}^n_\lambda)$ and assume that the restriction of Υ to some $\mathcal{K}(n-1)^i$ is a coboundary, that is, there exists $b \in \mathbb{F}^n_\lambda$ such that

$$\Upsilon(X_F) = \delta(b)(X_F) = (-1)^{|F||b|} X_F \cdot b \quad \text{for all} \quad X_F \in \mathcal{K}(n-1)^i.$$

By replacing Υ by $\Upsilon - \delta b$, we can suppose that $\Upsilon|_{\mathcal{K}(n-1)^i} = 0$. Thus, the map Υ is $\mathcal{K}(n-1)^i$-invariant and therefore the equation (2.15) becomes:

$$(-1)^{(|g|+1)|\Upsilon|} X_{g\theta_i} \cdot \Upsilon(X_{h\theta_i}) - (-1)^{(|h|+1)(|g|+|\Upsilon|+1)} X_{h\theta_i} \cdot \Upsilon(X_{g\theta_i}) = 0. \tag{2.18}$$

According to the isomorphism (2.10), the map Υ is decomposed into two components

$$\Pi(\mathbb{F}^{n-1,i}_{-\frac{1}{2}}) \;\to\; \mathbb{F}^{n-1,i}_{\lambda}, \qquad \Pi(\mathbb{F}^{n-1,i}_{-\frac{1}{2}}) \;\to\; \Pi(\mathbb{F}^{n-1,i}_{\lambda+\frac{1}{2}}). \tag{2.19}$$

So, each of these linear maps is $\mathcal{K}(n-1)^i$-invariant. More precisely, using equation (2.18), we get, up to a scalar factor:

$$\Upsilon(F_2\theta_i\alpha_n^{-1}) \simeq \begin{cases} \nu_1 F_2\theta_i\alpha_n^{-1} & if \quad \lambda = -1, \\ \nu_2 F_2\alpha_n^{-\frac{1}{2}} & if \quad \lambda = -\frac{1}{2}, \\ 0 & otherwise. \end{cases}$$

Using (2.15), we prove that $\nu_1 = 0$, therefore, for $\lambda = -1$, Υ is identically zero.

But, in the case $\lambda = -\frac{1}{2}$, we show that

$$\Upsilon(X_F) = \delta(\theta_n). \quad \square$$

Corollary 2.7. *Any 1-cocycle $\Upsilon \in Z^1_{\mathrm{diff}}(\mathcal{K}(n); \mathbb{F}^n_\lambda)$ is a coboundary if and only its restriction to $\mathcal{K}(n-1)^i$, for some $i \in \{1, \ldots n\}$, is a coboundary.*

This description will be useful in the proof of Theorem 2.1.

3. PROOF OF THEOREM 2.1

Consider a 1-cocycle Υ of $\mathcal{K}(n)$ with coefficients in $\mathfrak{F}^n_\lambda$. According to Theorem (2.6, if $\Upsilon|_{\mathcal{K}(n-1)^i}$, for $i=1,2,\cdots,n$ is trivial then the 1-cocycle Υ is trivial. Thus, assume that $\Upsilon|_{\mathcal{K}(n-1)^i}$, is nontrivial. Of course, up to coboundary, the general form of $\Upsilon|_{\mathcal{K}(n-1)^i}$, is given by the structure of the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n-1);\mathbb{F}^{n-1}_\lambda)$ together with the isomorphism (2.13), while $\Upsilon|_{\Pi(\mathbb{F}^{(n-1),\ i}_{\lambda+\frac{1}{2}})}$, can be essentially described by equations (2.14), (2.15) and Proposition 2.4. Thus, we have to distinguish all these cases:

(i) The case where $n=2$

Considering the Proposition (2.3), $\Upsilon|_{\mathcal{K}(1)^i}$, for $i=1,2$ is nontrivial for $\lambda=-\frac{1}{2},\ 0,\ \frac{1}{2},\ 1,\ \frac{3}{2}$, we deduce that, up to a coboundary, the non-zero restrictions of the cocycle Υ on $\mathcal{K}(1)^i$ can be expressed as:
For $\lambda=-\frac{1}{2},\ 0,\ \frac{1}{2},\ 1,\ \frac{3}{2}$,

$$\Upsilon|_{\mathcal{K}(1)^i}=\begin{cases}\varepsilon(-1)^iF_1'\alpha_2^{-\frac{1}{2}} & \text{if } \lambda=-\frac{1}{2},\\ \left(\varepsilon_1F_1'+\varepsilon_2(-1)^i\bar{\eta}_{3-i}(F_1')\theta_i\right) & \text{if } \lambda=0,\\ \varepsilon\bar{\eta}_{3-i}(F_1')\alpha_2^{\frac{1}{2}} & \text{if } \lambda=\frac{1}{2},\\ \varepsilon(-1)^i\bar{\eta}^3_{3-i}(F_1')\theta_i\alpha_2 & \text{if } \lambda=1,\\ \varepsilon\bar{\eta}^3_{3-i}(F_1')\alpha_2^{\frac{3}{2}} & \text{if } \lambda=\frac{3}{2}.\end{cases}$$

where $i\in\{1,\ 2\}$ and the coefficients $\varepsilon,\ \varepsilon_i$ are constants.

Now, by Proposition 2.4, we can write

$$\Upsilon(X_{h\theta_1\theta_2})=\sum_k\left(a_{0,k}+\sum_{j=1}^{2}\sum_{1\le i_1<\cdots<i_j\le 2}a_{i_1\cdots i_j,k}\theta_{i_1}\cdots\theta_{i_j}\right)h^{(k)}\alpha_2^\lambda.$$

For each case, we solve the equations (2.14) and (2.15) for $\varepsilon,\ \varepsilon_i,\ a_{0,k}$, $a_{i_1\cdots i_j,k}$.

We obtain

a) For $2\lambda=-1,\ 1,\ 3$, the coefficient ε vanishes; so, by Theorem 2.6, Υ is a coboundary. Hence $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(2);\mathbb{F}^2_\lambda)=0$.

b) For $\lambda=0$, the coefficients $\varepsilon_i\neq 0$ and, up to a coboundary, Υ is equal $\varepsilon_1\Upsilon^2_0+\varepsilon_2\widetilde{\Upsilon}^2_0$, see Theorem 2.1. Hence $\dim\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(2);\mathbb{F}^2_0)=2$.

c) For $\lambda=1$, the coefficient $\varepsilon\neq 0$ and, up to a coboundary, Υ is a multiple of Υ^2_1, see Theorem 2.1. Hence $\dim\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(2);\mathbb{F}^2_1)=1$.

(ii) Note that, by Theorem (2.6), the restriction of any nontrivial differential 1-cocycle Υ of $\mathcal{K}(3)$ with coefficients in $\mathbb{F}^3_\lambda$ to $\mathcal{K}(2)^i$, for $i=1,2,3$, is a nontrivial 1-cocycle. Furthermore, using arguments similar to those of the proof of Proposition 2.3 together with the above result, we deduce that $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(2)^i;\mathbb{F}^3_{\lambda,\mu}) = 0$ if $2\lambda \neq -1, 0, 1, 2$. Then, we consider only the cases where $2\lambda = -1, 0, 1, 2$ and, as before, we get the result for $n = 3$. Hence $\mathrm{dim}\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(3);\mathbb{F}^3_{\lambda}) = 2$ and span by the cohomology classes of the 1-cocycles Υ defined by $\Upsilon^3_0(X_F) = F'$ and $\Upsilon^3_{\frac{1}{2}}(X_F) = \bar{\eta}_1\bar{\eta}_2\bar{\eta}_3(F)\alpha_3^{\frac{1}{2}}$.

(iii) Note that, by Theorem (2.6), the restriction of any nontrivial differential 1-cocycle Υ of $\mathcal{K}(4)$ with coefficients in $\mathbb{F}^4_{\lambda}$ to $\mathcal{K}(3)^i$, for $i = 1, \dots, 4$, is a nontrivial 1-cocycle. Furthermore, using arguments similar to those of the proof of Proposition 2.3 together with the above result, we deduce that $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(3)^i;\mathbb{F}^4_{\lambda}) = 0$ if $2\lambda \neq -1, 0, 1$. Then, we consider only the cases where $2\lambda = -1, 0, 1$ and, as before, we get the result for $n = 4$. Hence $\mathrm{dim}\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(4);\mathbb{F}^4_0) = 1$ and $\mathrm{dim}\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(4);\mathbb{F}^4_{\lambda}) = 0$ for $\lambda \neq 0$, span by the divergence.

(iv) We proceed by recurrence over n. In a similar way as in (iii), we get the result for $n = 5$. Now, we assume that it holds for some $n \geq 5$. Again, the same arguments recurrence assumption show that $\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n)^i;\mathbb{F}^{n+1}_{\lambda}) = 0$ if $2\lambda \neq -1, 0$. So, we consider only the cases where $2\lambda = -1, 0$, we proceed as in (i) and we get the result for $n+1$. Finally, $\mathrm{dim}\mathrm{H}^1_{\mathrm{diff}}(\mathcal{K}(n);\mathbb{F}^n_{\lambda}) = 1$, for $n \geq 4$ and span by the divergence. □

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