ON PEXIDER DIFFERENCE FOR A PEXIDER CUBIC FUNCTIONAL EQUATION

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Let G be an abelian group and let X be a sequentially complete Hausdorff topological vector space over the field $\mathbb{Q}$ of rational numbers. We prove that the Pexiderized cubic functional equation

$$f(kx+y)+f(kx-y)=g(x+y)+g(x-y)+\frac{2}{k}g(kx)-2g(x)$$

is stable for functions f, g defined on G and taking values in X.

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1. INTRODUCTION

More than a half century ago, Ulam [8] posed the famous Ulam stability problem which was partially solved by Hyers [3] in the framework of Banach spaces. The Hyers' theorem was generalized by Aoki [2] for additive mappings. In 1978, Th.M. Rassias [6] extended the theorem of Hyers by considering the unbounded Cauchy difference inequality

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon(\|x\|^p+\|y\|^p) \quad (\varepsilon \geq 0,\, p\in[0,1)) \tag{1.1}$$

Najati and Moghimi [5] investigated the Hyers-Ulam stability of the functional equation

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+2f(2x)-2f(x)$$

in quasi-Banach spaces. The generalized Hyers-Ulam stability for a mixed additive-cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x)$$

in quasi-Banach spaces has been investigated by Najati and Eskandari [4]. Above functional equation is called the mixed additive-cubic functional equation, since the function $f(x)=ax^3+bx$ is its solution. Every solution of the mixed additive-cubic functional equation is said to be a mixed additive-cubic mapping. In [7] the problem of generalized Hyers-Ulam stability of a general mixed additive-cubic functional equation,

$$f(kx+y)+f(kx-y)=kf(x+y)+kf(x-y)+2f(kx)-2kf(x)$$

was investigated in quasi-Banach spaces.

Recently, Adam and Czerwik investigated the problem of the Hyers-Ulam stability of a generalized quadratic functional equation in linear topological spaces [1]. Let G be an abelian group and throughout this paper let X be a sequentially complete Hausdorff topological vector space over the field $\mathbb{Q}$ of rational numbers. A mapping $f : G \to X$ is said to be *quadratic* if and only if it satisfies the following functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in G$. A mapping $f : G \to X$ is said to be *additive* if and only if it satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in G$.

For a given $k \in \mathbb{N}$ and $f : G \to X$, we will use the following notation

$$Df(x,y) := f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y)-2f(kx)+2kf(x).$$

For given sets $A, B \subseteq X$ and a number $r \in \mathbb{R}$, we define the well known operations

$$A+B := \{a+b : a \in A, b \in B\}, \quad rA := \{ra : a \in A\}.$$

We denote the convex hull of a set $U \subseteq X$ by $conv(U)$ and by $\overline{U}$ the sequential closure of U. Moreover it is well know that:

(1) If $A, B \subseteq X$ are bounded sets, then $A+B$, $conv(A)$ and $\overline{A}$ are bounded subsets of X.

(2) If $A, B \subseteq X$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha\, conv(A)+\beta\, conv(B) = conv(\alpha A+\beta B)$.

(3) Let X_1 and X_2 be linear spaces over $\mathbb{R}$. If $f : X_1 \to X_2$ is a cubic function, then $f(rx) = r^3 f(x)$, for all $x \in X_1$ and all $r \in \mathbb{Q}$.

(4) Let k be a natural number and $f : G \to X$ be a mapping with $f(0) = 0$ satisfying

$$f(kx+y)+f(kx-y)=kf(x+y)+f(x-y)+2f(kx)-2kf(x)$$

then the mapping $G(x) := f(2x) - 8f(x)$ is additive and the mapping $H(x) := f(2x) - 2f(x)$ is cubic.

2. MAIN RESULTS

We start with the following lemma.

LEMMA 1. *Let $k > 1$ be a natural number, G be a k-divisible abelian group and $B \subseteq X$ be a nonempty set. If the functions $f, g : G \to X$ satisfy*

$$f(kx+y)+f(kx-y)-g(x+y)-g(x-y)-\frac{2}{k}g(kx)+2g(x)\in B \tag{2.1}$$

for all $x,y\in G$*, then*

$$Df(x,y)\in E:=(k+1)\ conv(B-B), \tag{2.2}$$

$$Dg(x,y)\in F:=k\ conv(B-B). \tag{2.3}$$

Proof. Putting $y=0$ in (2.1), we get

$$2f(kx)-\frac{2}{k}g(kx)\in B \tag{2.4}$$

for all $x\in G$. If we replace kx by x in (2.4), then we have

$$kf(x)-g(x)\in\frac{k}{2}\ B. \tag{2.5}$$

If we replace x by $x+y$ and x by $x-y$ in (2.5) we obtain

$$kf(x+y)-g(x+y)\in\frac{k}{2}\ B,\ kf(x-y)-g(x-y)\in\frac{k}{2}\ B. \tag{2.6}$$

It follows from (2.1), (2.4) and (2.6) that

$$\begin{aligned}Df(x,y)=&[f(kx+y)+f(kx-y)-g(x+y)-g(x-y)-\frac{2}{k}g(kx)\\&+2g(x)]-[2f(kx)-\frac{2}{k}g(kx)]-[kf(x+y)-g(x+y)]\\&-[kf(x-y)-g(x-y)]+[2kf(x)-2g(x)]\\&\in[B-B-\frac{k}{2}B-\frac{k}{2}B+kB]\subseteq(k+1)\ conv(B-B)=E.\end{aligned}$$

Similarly, by (2.1) and (2.5), we have

$$\begin{aligned}Dg(x,y)=&[kf(kx+y)+kf(kx-y)-kg(x+y)-kg(x-y)-2g(kx)\\&+2kg(x)]-[kf(kx+y)-g(kx+y)]-[kf(kx-y)-g(kx-y)]\\&\in kB-\frac{k}{2}B-\frac{k}{2}B\subseteq k\ conv(B-B)=F.\quad\square\end{aligned}$$

THEOREM 2. *Let* $k>1$ *be a natural number,* G *be a* $2,k$*-divisible abelian group and* $B\subseteq X$ *be a bounded set. Suppose that the functions* $f,g:G\to X$ *satisfy (2.1) for all* $x,y\in G$ *and* $f(0)=0$*. Then there exists a unique additive function* $\mathcal{A}:G\to X$ *such that*

$$\mathcal{A}(x)-h_1(x),\ \mathcal{A}(x)-h_2(x)\in\overline{conv(\Phi)},\quad(x\in G),$$

where $\Phi:=\frac{k^2+2k+4}{1-k}E+\frac{k^2+2k+8}{1-k}(-E)$*,* $h_1(x):=k(f(2x)-8f(x))$ *and* $h_2(x):=g(2x)-8g(x)$*. Moreover, the functions* $\mathcal{A}$ *is given by*

$$\mathcal{A}(x)=\lim_{n\to\infty}\frac{1}{2^{n+1}}h_1(2^{n+1}x)=\lim_{n\to\infty}\frac{1}{2^{n+1}}h_2(2^{n+1}x),\ (x\in G),$$

and the convergence of the sequences are uniform on G.

Proof. By Lemma 1, we get (2.2). Letting $x = 0$ in (2.2), we get

$$f(y) + f(-y) \in \frac{1}{1-k}E = \frac{1}{k-1}(-E) \tag{2.7}$$

for all $y \in G$. Putting $y = x$ in (2.2), we have

$$f((k+1)x) + f((k-1)x) - kf(2x) - 2f(kx) + 2kf(x) \in E. \tag{2.8}$$

Replacing x by $2x$ in (2.8), we obtain

$$f(2(k+1)x) + f(2(k-1)x) - kf(4x) - 2f(2kx) + 2kf(2x) \in E. \tag{2.9}$$

Letting $y = kx$ in (2.2), we get

$$f(2(kx)) - kf((k+1)x) - kf(-(k-1)x) - 2f(kx) + 2kf(x) \in E. \tag{2.10}$$

Letting $y = (k+1)x$ and $y = (k-1)x$ in (2.2), we have

$$f((2k+1)x) + f(-x) - kf((k+2)x) - kf(-kx) - 2f(kx) + 2kf(x) \in E,$$

and

$$f((2k-1)x) - (k+2)f(kx) - kf(-(k-2)x) + (2k+1)f(x) \in E. \tag{2.11}$$

Replacing x and y by $2x$ and x in (2.2), respectively we get

$$f((2k+1)x) + f((2k-1)x) - 2f(2kx) - kf(3x) + 2kf(2x) - kf(x) \in E. \tag{2.12}$$

Replacing x and y by $3x$ and x in (2.2), respectively we get

$$f((3k+1)x) + f((3k-1)x) - 2f(3kx) - kf(4x) - kf(2x) + 2kf(3x) \in E. \tag{2.13}$$

Replacing x and y by $2x$ and kx in (2.2), respectively we get

$$f(3kx) + f(kx) - kf((k+2)x) - kf(-(k-2)x) - 2f(2kx) + 2kf(2x) \in E. \tag{2.14}$$

Setting $y = (2k+1)x$ in (2.2), we get

$$f((3k+1)x) + f(-(k+1)x) - kf(2(k+1)x) - kf(-2kx) - 2f(kx) + 2kf(x) \in E. \tag{2.15}$$

Letting $y = (2k-1)x$ in (2.2), we have

$$f((3k-1)x) + f(-(k-1)x) - kf(-2(k-1)x) - kf(2kx) - 2f(kx) + 2kf(x) \in E. \tag{2.16}$$

Letting $y = 3kx$ in (2.2), we have

$$f(4kx) + f(-2kx) - kf((3k+1)x) - kf(-(3k-1)x) - 2f(kx) + 2kf(x) \in E. \tag{2.17}$$

Subtracting relationships (2.8),(2.15) and (2.16) from (2.13) respectively, we obtain

$$\begin{aligned}[f((3k+1)x)&+f((3k-1)x)-2f(3kx)-kf(4x)-kf(2x)+2kf(3x)]\\ &-1\times[f((k+1)x)+f((k-1)x)-kf(2x)-2f(kx)+2kf(x)]\\ &-1\times[f((3k+1)x)+f(-(k+1)x)-kf(2(k+1)x)-kf(-2kx)\\ &-2f(kx)+2kf(x)]-1\times[f((3k-1)x)+f(-(k-1)x)\\ &-kf(-2(k-1)x)-kf(2kx)-2f(kx)+2kf(x)]\in E-E-E-E.\end{aligned}$$

Hence,

$$\begin{aligned}kf(2(k+1)x)&+kf(-2(k-1)x)-2f(3kx)-kf(4x)+2kf(3x)\\ &+6f(kx)-6kf(x)-[f((k+1)x)+f(-(k+1)x)]\\ &-[f((k-1)x)+f(-(k-1)x)]+k[f(2kx)+f(-2kx)]\in E+3(-E).\end{aligned}$$

By (2.7), we get

$$\begin{aligned}kf(2(k+1)x)&+kf(-2(k-1)x)-2f(3kx)-kf(4x)+2kf(3x)\\ &+6f(kx)-6kf(x)\in E+3(-E)+\frac{2}{k-1}(-E)-\frac{k}{k-1}(-E)\\ &\subseteq\frac{2k-1}{k-1}E+\frac{3k-1}{k-1}(-E).\end{aligned}\tag{2.18}$$

By (2.11), we have

$$\begin{aligned}f((2k+1)x)&+f((2k-1)x)-kf((k+2)x)+4kf(x)-4f(kx)\\ &-kf(-(k-2)x)-k[f(kx)+f(-kx)]+[f(x)+f(-x)]\in E+E.\end{aligned}$$

Hence, by (2.7), we have

$$\begin{aligned}f((2k+1)x)&+f((2k-1)x)-kf((k+2)x)+4kf(x)-4f(kx)\\ &-kf(-(k-2)x)\in E+E+\frac{k}{k-1}(-E)-\frac{1}{k-1}(-E)\\ &\subseteq\frac{2k-1}{k-1}E+\frac{k}{k-1}(-E).\end{aligned}\tag{2.19}$$

Subtracting relationship (2.19) from (2.12), we get

$$\begin{aligned}kf((k+2)x)&+kf(-(k-2)x)-2f(2kx)+4f(kx)-kf(3x)\\ &+2kf(2x)-5kf(x)\in\frac{2k-1}{k-1}E+\frac{2k-1}{k-1}(-E).\end{aligned}\tag{2.20}$$

By (2.14) and (2.20), we have

$$f(3kx)-4f(2kx)+5f(kx)-kf(3x)+4kf(2x)-5kf(x) \in \frac{3k-2}{k-1}E+\frac{2k-1}{k-1}(-E). \tag{2.21}$$

By (2.7), (2.15), (2.16) and (2.17), we have

$$\begin{aligned} &kf(-(k+1)x)-kf(-(k-1)x)-k^2f(2(k+1)x)\\ &\quad +k^2f(-2(k-1)x)+k^2f(2kx)-(k^2-1)f(-2kx)\\ &\quad +f(4kx)-2f(kx)+2kf(x)\in(k+1)E+\frac{k^2}{k-1}(-E). \end{aligned}\tag{2.22}$$

It follows from by (2.9), (2.10) and (2.22) that

$$\begin{aligned} &f(4kx)-2f(2kx)-k^3f(4x)+2k^3f(2x)+(1-k^2)[f(2kx)+f(-2kx)]\\ &\quad +k^2[f(2(k-1)x)+f(-2(k-1)x)]+k[f((k-1)x)+f(-(k-1)x)]\\ &\quad =\{kf(-(k+1)x)-kf(-(k-1)x)-k^2f(2(k+1)x)+k^2f(2kx)\\ &\quad +k^2f(-2(k-1)x)-(k^2-1)f(-2kx)+f(4kx)-2f(kx)+2kf(x)\}\\ &\quad +k^2\{f(2(k+1)x)+f(2(k-1)x)-kf(4x)-2f(2kx)+2kf(2x)\}\\ &\quad -\{f(2(kx))-kf((k+1)x)-kf(-(k-1)x)-2f(kx)+2kf(x)\}\\ &\quad \in(k+1)E+\frac{k^2}{k-1}(-E)+k^2E+(-E)\\ &\quad \subseteq(k^2+k+1)E+\frac{k^2+k-1}{k-1}(-E). \end{aligned}$$

Hence, by (2.7), we obtain

$$\begin{aligned} f(4kx)-2f(2kx)-k^3f(4x)+2k^3f(2x)&\in(k^2-1)(\frac{1}{k-1}(-E))\\ &\quad +k^2(\frac{1}{k-1}E)+k(\frac{1}{k-1}E)+(k^2+k+1)E+\frac{k^2+k-1}{k-1}(-E)\\ &\subseteq\frac{k^3+k^2+k-1}{k-1}E+\frac{2k^2+k-2}{k-1}(-E). \end{aligned}\tag{2.23}$$

Also, we have

$$f(2kx)-2f(kx)-k^3f(2x)+2k^3f(x)\in\frac{k^3+k^2+k-1}{k-1}E+\frac{2k^2+k-2}{k-1}(-E). \tag{2.24}$$

By (2.10), we have

$$f(4kx)-kf(2(k+1)x)-kf(-2(k-1)x)-2f(2kx)+2kf(2x)\in E. \tag{2.25}$$

Similarly, from (2.23) and (2.25), we obtain

$$
\begin{aligned}
&kf(2(k+1)x) + kf(-2(k-1)x) + (2k^3-2k)f(2x) - k^3 f(4x) \\
&\in \frac{k^3+k^2+k-1}{k-1}E + \frac{2k^2+k-2}{k-1}(-E) + (-E) \\
&\subseteq \frac{k^3+k^2+k-1}{k-1}E + \frac{2k^2+2k-3}{k-1}(-E).
\end{aligned}
\tag{2.26}
$$

Also, from (2.18) and (2.26), we get

$$
\begin{aligned}
&2f(3kx) - 6f(kx) + (k-k^3)f(4x) - 2kf(3x) + (2k^3-2k)f(2x) \\
&+ 6kf(x) \in \frac{k^3+k^2+k-1}{k-1}E + \frac{2k^2+2k-3}{k-1}(-E) \\
&\qquad - (\frac{2k-1}{k-1}E + \frac{3k-1}{k-1}(-E)) \\
&\subseteq \frac{k^3+k^2+4k-2}{k-1}E + \frac{2k^2+4k-4}{k-1}(-E).
\end{aligned}
\tag{2.27}
$$

On the other hand it follows from (2.21) and (2.27) that

$$
\begin{aligned}
&8f(2kx) - 16f(kx) + (k-k^3)f(4x) + (2k^3-10k)f(2x) \\
&+ 16kf(x) \in \frac{k^3+k^2+4k-2}{k-1}E + \frac{2k^2+4k-4}{k-1}(-E) \\
&\qquad - 2(\frac{3k-2}{k-1}E + \frac{2k-1}{k-1}(-E)) \\
&\subseteq \frac{k^3+k^2+8k-4}{k-1}E + \frac{2k^2+10k-8}{k-1}(-E).
\end{aligned}
\tag{2.28}
$$

Therefore by (2.24) and (2.28), we get

$$
\begin{aligned}
&(k^3-k)(f(4x) - 10f(2x) + 16f(x)) \in 8(\frac{k^3+k^2+k-1}{k-1}E \\
&+ \frac{2k^2+k-2}{k-1}(-E)) - (\frac{k^3+k^2+8k-4}{k-1}E + \frac{2k^2+10k-8}{k-1}(-E)) \\
&\subseteq \frac{8k^3+10k^2+18k-16}{k-1}E + \frac{k^3+17k^2+16k-20}{k-1}(-E).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
k(f(4x) - 10f(2x) + 16f(x)) \in \phi :=& \frac{8k^3+10k^2+18k-16}{(k^2-1)(k-1)}E \\
&+ \frac{k^3+17k^2+16k-20}{(k^2-1)(k-1)}(-E).
\end{aligned}
$$

Let $h_1 : G \to X$ be the mapping defined by

$$h_1(x) = k(f(2x) - 8f(x)), \quad (x \in G)$$

then

$$h_1(2x) - 2h_1(x) \in \Phi, \quad (x \in G) \tag{2.29}$$

Replacing x by $2^n x$ in (2.29), we obtain

$$h_1(2^{n+1}x) - 2h_1(2^n x) \in \Phi$$

for all $x \in G$ and $n \in \mathbb{N}$. By induction we can prove

$$\frac{1}{2^{n+1}} h_1(2^{n+1}x) - h_1(x) \in \frac{2^{n+1} - 1}{2^{n+1}} conv\,(\Phi). \tag{2.30}$$

We define

$$\mathcal{A}^1_{n+1}(x) := \frac{1}{2^{n+1}} h_1(2^{n+1}x), \quad (x \in G, n \in \mathbb{N})$$

and by (2.30), we get

$$\begin{aligned}
\mathcal{A}^1_{m+n+1}(x) - \mathcal{A}^1_{n+1}(x) &= \frac{1}{2^{m+n+1}} h_1(2^{m+n+1}x) - \frac{1}{2^{n+1}} h_1(2^{n+1}x) \\
&= \frac{1}{2^{n+1}} [\frac{1}{2^m} h_1(2^m 2^{n+1}x) - h_1(2^{n+1}x)] \\
&\in \frac{1}{2^{n+1}} (\frac{2^m - 1}{2^m})\ conv(\Phi)
\end{aligned}$$

for all $m, n \in \mathbb{N}$ and $x \in G$. Since B is bounded, we conclude that $conv(\Phi)$ is bounded. It follows that the sequence $\{\mathcal{A}^1_n(x)\}$ is (uniformly) Cauchy in X for all $x \in G$. Since X is a sequential complete topological vector space, the sequence $\{\mathcal{A}^1_n(x)\}$ is convergent for all $x \in G$, and the convergence is uniform on G. Define

$$\mathcal{A}^1 : G \to X, \quad \mathcal{A}^1(x) := \lim_{n \to \infty} \mathcal{A}^1_{n+1}(x), \quad (x \in G).$$

Thus, from (2.30) and definition of the set $(\frac{2^m - 1}{2^m})\ conv(\Phi)$ we have for $n \to \infty$

$$\mathcal{A}^1(x) - h_1(x) \in \overline{conv(\Phi)}, \quad (x \in G).$$

Similarly for function g we have the results, define for g

$$\mathcal{A}^2_{n+1}(x) := \frac{1}{2^{n+1}} h_2(2^{n+1}x), \quad (x \in G, n \in \mathbb{N})$$

where $h_2(x) := g(2x) - 8g(x)$, $h_2(2x) - 2h_2(x) \in \Phi$ and

$$\mathcal{A}^2(x) := \lim_{n \to \infty} \mathcal{A}^2_{n+1}(x), \quad (x \in G).$$

Now, we prove the equality $\mathcal{A}^1 = \mathcal{A}^2$. From (2.5) we obtain

$$h_1(x) - h_2(x) = k[f(2x) - 8f(x)] - [g(2x) - 8g(x)]$$

$$\begin{aligned} &= [kf(2x) - g(2x)] - 8[kf(x) - g(x)] \\ &\in N := \frac{k}{2}\,B + 8\,\frac{k}{2}(-B), \quad (x \in G). \end{aligned}$$

Therefore

$$\frac{1}{2^{n+1}} h_1(x) - \frac{1}{2^{n+1}} h_2(x) \in \frac{1}{2^{n+1}} N, \quad (x \in G, n \in \mathbb{N}).$$

Replacing x by $2^{n+1}x$ in the above condition we get

$$\frac{1}{2^{n+1}} h_1(2^{n+1}x) - \frac{1}{2^{n+1}} h_2(2^{n+1}x) \in \frac{1}{2^{n+1}} N, \quad (x \in G, n \in \mathbb{N}).$$

Letting $n \to \infty$ we have $\mathcal{A}(x) := \mathcal{A}^1(x) = \mathcal{A}^2(x)$ for all $x \in G$.

We have from (2.29)

$$h_1(2^{n+1}x) - 2h_1(2^n x) \in \Phi$$

for all $x \in G,\ n \in \mathbb{N}$. So

$$\frac{1}{2^n} h_1(2^{n+1}x) - \frac{1}{2^{n-1}} h_1(2^n x) \in \frac{1}{2^n} \Phi.$$

Letting $n \to \infty$ we have

$$\mathcal{A}(2x) = \mathcal{A}^1(2x) = 2\mathcal{A}^1(x) = 2\mathcal{A}(x). \tag{2.31}$$

On the other hand it can be easily verified that

$$Dh_1(x, y) = k[Df(2x, 2y) - 8Df(x, y)]$$

for all $x \in G$. So, from (2.2) we obtain

$$\begin{aligned} D\mathcal{A}(x, y) &= \lim_{n\to\infty} \frac{1}{2^{n+1}} Dh_1(2^{n+1}x, 2^{n+1}y) \\ &= k \lim_{n\to\infty} \frac{1}{2^{n+1}} \{Df(2^{n+2}x, 2^{n+2}y) - 8Df(2^{n+1}x, 2^{n+1}y)\} \\ &\in k \lim_{n\to\infty} \frac{1}{2^{n+1}} (E - 8E) = 0, \qquad (x, y \in G). \end{aligned}$$

Hence, by (4), the mapping $x \to \mathcal{A}(2x) - 8\mathcal{A}(x)$ is additive. Therefore (2.29) implies that the mapping $\mathcal{A}$ is additive.

To prove the uniqueness of the additive function $\mathcal{A}$, assume that there exists another additive function $\mathcal{A}' : G \to X$ which satisfies

$$\mathcal{A}'(x) - h_1(x),\ \mathcal{A}'(x) - h_1(x) \in \overline{conv(\Phi)}, \quad (x \in G).$$

Clearly

$$\mathcal{A}'(x) - \mathcal{A}(x) = [\mathcal{A}'(x) - h_1(x)] - [\mathcal{A}(x) - h_1(x)] \in M := 2\,\overline{conv(\Phi)}.$$

Applying the same method as before we get $\mathcal{A}'(x) = \mathcal{A}(x) (x \in G)$. This completes the proof. $\square$

THEOREM 3. *Let $k > 1$ be a natural number, G be a $2, k$-divisible abelian group and $B \subseteq X$ be a bounded and balanced set. Suppose that the functions $f, g : G \to X$ satisfy (2.1) for all $x, y \in G$. Then there exists unique cubic function $\mathcal{A}' : G \to X$ such that*

$$\mathcal{A}'(x) - h_3(x),\ \mathcal{A}'(x) - h_4(x) \in \frac{1}{7}\overline{conv(\Phi')}, \quad (x \in G),$$

where $\Phi' = \frac{k^2+2k+8}{k^2-k}E + \frac{k^2+2k+4}{k^2-k}(-E)$, $h_3(x) = f(2x) - 2f(x)$ and $h_4(x) = g(2x) - 2g(x)$. Moreover, the functions $\mathcal{A}'$ is given by

$$\mathcal{A}'(x) = \lim_{n\to\infty} \frac{1}{8^{n+1}} h_3(2^{n+1}x) = \lim_{n\to\infty} \frac{1}{8^{n+1}} h_4(2^{n+1}x),\ (x \in G),$$

and the convergence of the sequences are uniform on G.

Proof. We define $h_3(x) := f(2x) - 2f(x)$, so

$$h_3(2x) - 8h_3(x) \in \Phi' := \frac{k^2+2k+8}{k^2-k}E + \frac{k^2+2k+4}{k^2-k}(-E). \tag{2.32}$$

Hence,

$$\frac{1}{8^{n+1}} h_3(2^{n+1}x) - \frac{1}{8^n} h_3(2^n x) \in \sum_{i=0}^{n} \frac{1}{8^i}\Phi'$$

for all $x \in G, n \in \mathbb{N}$. Therefore,

$$\frac{1}{8^{n+1}} h_3(2^{n+1}x) - \frac{1}{8^m} h_3(2^m x) \in \sum_{i=m}^{n} \frac{1}{8^i}\Phi' \tag{2.33}$$

for all $x \in G$ and all integers $n > m \geq 1$. Since B is bounded, we conclude that the sequence $\{\frac{1}{8^{n+1}} h_3(2^{n+1}x)\}$ is (uniformly) Cauchy in X for all $x \in G$. Since X is a sequential complete topological vector space, the sequence $\{\frac{1}{8^{n+1}} h_3(2^{n+1}x)\}$ is convergent for all $x \in G$, and the convergence is uniform on G. Define

$$\mathcal{A}'_1 : G \to X, \quad \mathcal{A}'_1(x) := \lim_{n\to\infty} \frac{1}{8^{n+1}} h_3(2^{n+1}x), \quad (x \in G).$$

We have from (2.32)

$$\frac{1}{8^{n+1}} h_3(2^{n+1}x) - \frac{1}{8^n} h_3(2^{n+1}x) \in \frac{1}{8^{n+1}}\Phi'$$

for all $x \in G,\ n \in \mathbb{N}$. Letting $n \to \infty$, we have

$$\mathcal{A}'_1(2x) = 8\mathcal{A}'_1(x). \tag{2.34}$$

On the other hand we have

$$D\mathcal{A}'_1(x, y) = \lim_{n\to\infty} \frac{1}{8^{n+1}} Dh_3(2^{n+1}x, 2^{n+1}y)$$

$$= \lim_{n\to\infty} \frac{1}{8^{n+1}}\{Df(2^{n+2}x, 2^{n+2}y) - 2Df(2^{n+1}x, 2^{n+1}y)\}$$
$$\in k \lim_{n\to\infty} \frac{1}{8^{n+1}}(E - 2E) = 0$$

for all $x, y \in G$. Hence, by (4), the mapping $x \to \mathcal{A}_1'(2x) - 2\mathcal{A}_1'(x)$ is cubic. Therefore (2.34) implies that the mapping $\mathcal{A}_1'$ is cubic. Replacing x by $2^n x$ in (2.32) we obtain

$$h_3(2^{n+1}x) - 8h_1(2^n x) \in \Phi'$$

for all $x \in G$ and $n \in \mathbb{N}$. By induction we can prove

$$\frac{1}{8^{n+1}} h_3(2^{n+1}x) - h_1(x) \in \frac{1}{7}(\frac{8^{n+1} - 1}{8^{n+1}}) conv(\Phi'). \tag{2.35}$$

Therefore

$$\mathcal{A}_1'(x) - h_1(x) \in \frac{1}{7}\overline{conv(\Phi')}.$$

Similarly for function g we have the results, define for g

$$\mathcal{A}_2'(x) := \lim_{n\to\infty} \frac{1}{8^{n+1}} h_4(2^{n+1}x), \quad (x \in G, n \in \mathbb{N})$$

where $h_4(x) := g(2x) - 2g(x)$, $h_4(2x) - 8h_4(x) \in \Phi$.

Similarly as in a previous theorem we can check that $\mathcal{A}_2'$ is a cubic function satisfying

$$\mathcal{A}_2'(x) - h_4(x) \in \frac{1}{7}\overline{conv(\Phi')}. \tag{2.36}$$

Moreover, we can prove that $\mathcal{A}'(x) := \mathcal{A}_1'(x) = \mathcal{A}_2'(x)$ for all $x \in G$.

To prove the uniqueness of the cubic function $\mathcal{A}'$, assume that there exists another cubic function $\mathcal{A}'' : G \to X$ which satisfies

$$\mathcal{A}''(x) - h_3(x),\ \ \mathcal{A}''(x) - h_4(x) \in \frac{1}{7}\overline{conv(\Phi')}, \quad (x \in G).$$

Clearly

$$\mathcal{A}''(x) - \mathcal{A}'(x) = [\mathcal{A}''(x) - h_1(x)] - [\mathcal{A}'(x) - h_1(x)] \in N := \frac{2}{7}\overline{conv(\Phi')}$$

for all $x \in G$. Hence, $\mathcal{A}'' = \mathcal{A}'$ and this completes the proof. □

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